Evolution and Dynamics of Cusped Light-Like Wilson Loops in Loop Space

3rd Workshop on the QCD Structure of the Nucleon
Bilbao, Spain

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Universiteit Antwerpen

Outline

1. Introduction: TMDs
2. Wilson Loops and their Evolution
3. Geometrical Approach
4. Conclusion

TMDs

Transverse Momentum Dependent PDFs

$$f(x, \mathbf{k}_\perp) = \frac{1}{2} \int \frac{\mathrm{d}z^- \, \mathrm{d}^2\mathbf{z}_\perp}{2\pi(2\pi)^2} \, e^{ik\cdot z} \, \langle P, S | \, \overline{\psi}(z) \, \mathcal{U}^\dagger_{(z;\infty)} \mathcal{U}_{(\infty;0)} \, \psi(0) \, | P, S \rangle \Big|_{z^+=0}$$

Gauge link: $\mathcal{U}(z;\infty) = e^{\, ig \int_\infty^z \mathrm{d}x^\mu \, A_\mu(x)}$

Singularities

- **UV poles** $\sim \frac{1}{\epsilon}$ — removed by standard renormalisation
- **Rapidity divergencies** $\sim \ln\theta$ — resummed by use of Collins-Soper
- **Overlapping divergencies** $\sim \frac{1}{\epsilon}\ln\theta$ — generalised renormalisation
- **Self-energy divergencies** — treated by modification of soft factors

See talk by I.O. Cherednikov on friday.

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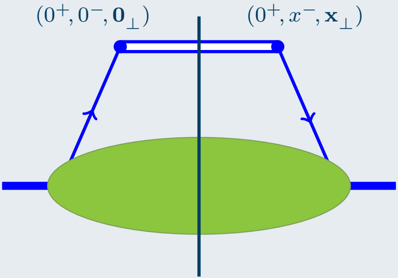
TMDs

Cusp Anomalous Dimension



- "Hidden cusp" contribution to Γ_{cusp}
- $\Gamma_{cusp} = \frac{\alpha_s C_F}{\pi} (\chi \coth \chi - 1)$
- $\cosh \chi = \frac{p \cdot k}{|p||k|} \overset{\text{on-LC}}{\longrightarrow} \infty$
- $\Gamma_{cusp} \overset{\text{on-LC}}{\longrightarrow} \frac{\alpha_s C_F}{\pi}$

Nucl. Phys. B283 (1987) 342-364, *G.P. Korchemsky and A.V. Radyushkin*
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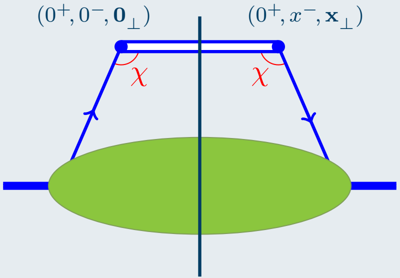

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Definition of Wilson Loops

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$$\mathcal{W}[C] = \frac{1}{N_c} \mathrm{tr} \langle 0 | \mathcal{P} e^{\mathrm{i}g \oint_{\mathcal{C}} \mathrm{d}z^\mu \, A^a_\mu(z) t_a} | 0 \rangle$$

$$\mathcal{C} : \quad z^\mu(s) \quad s = 0 \ldots 1 \quad \text{where} \quad z^\mu(0) \equiv z^\mu(1)$$

Wilson Loops...

- are characterised by their geometry
- can be used as elementary objects defining QCD (in loop space)
- can exhibit dualities to objects in standard QCD, depending on the structure of the loop

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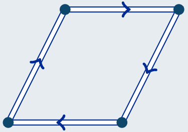
Motivation

Wilson loops $\Leftrightarrow$ $\mathcal{N} = 4$ Super Yang-Mills



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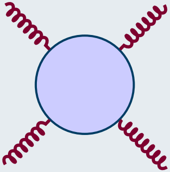

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- Duality between a planar Wilson loop made up from N light-like segments and the N-gluon planar scattering amplitude in $\mathcal{N} = 4$ SYM.
- Momenta p_i of external gluons in SYM are equal to light-like segment lengths $p_i \equiv x_i - x_{i+1}$ of the loop

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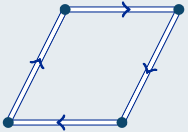

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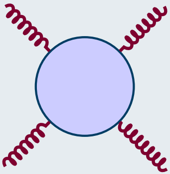

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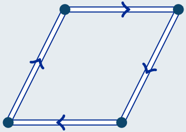

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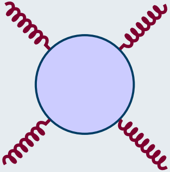

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- The UV singularities of the Wilson loop are related to the IR singularities of the $\mathcal{N} = 4$ SYM scattering amplitude.
- The evolution equation for the amplitude as function of IR-cutoff is governed by the cusp anomalous dimension of the Wilson loop.

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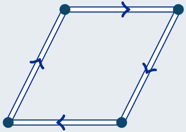

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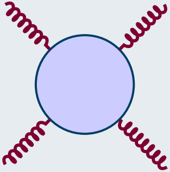

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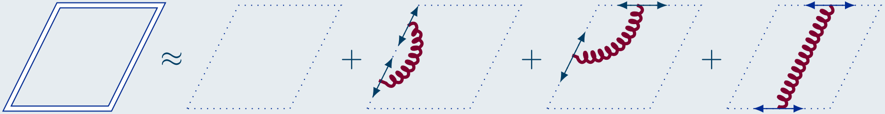
- It is thus instructive to investigate further how the Wilson loop behaves in function of (changes of) its geometry and derive its evolution equations.
- These might lead to evolution equations for TMDs, as the singular parts of the latter are related to those of Wilson loops.

Leading Order Calculation

Expansion and Calculation



$$\mathcal{W}_{LO} = 1 - \frac{\alpha_s C_F}{\pi}(2\pi\mu^2)^\epsilon\,\Gamma(1-\epsilon)\left[\frac{1}{\epsilon^2}\left(-\frac{s}{2}\right)^\epsilon + \frac{1}{\epsilon^2}\left(-\frac{t}{2}\right)^\epsilon - \frac{1}{2}\ln^2\frac{s}{t}\right]$$

Korchemskaya, Korchemsky (1992); Bassetto, Korchemskaya, Korchemsky, Nardelli (1993)

- Mandelstam Energy/rapidity variables: $s = (p_1 + p_2)^2, t = (p_2 + p_3)^2$
- $s = -t$, but watch out for pole structure: $(-s + i\varepsilon)^\epsilon + (-t + i\varepsilon)^\epsilon$
- Absorb π and expansion of $\Gamma(1-\epsilon)$ in μ: $\overline{\mu}^2 = \pi\mu^2 e^{\gamma_E}$

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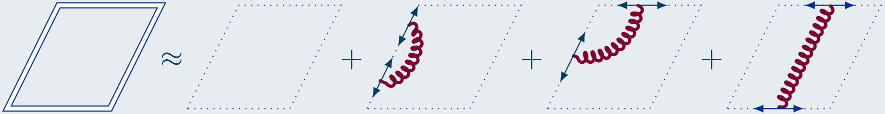

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Energy Evolution at Leading Order

One-Loop Evolution

$$\mathcal{W}_{LO} = 1 - \frac{\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon^2}\left((-\overline{s}+\mathrm{i}\varepsilon)^{\epsilon} + (\overline{s}+\mathrm{i}\varepsilon)^{\epsilon}\right) + \frac{\pi^2}{2} - 2\zeta(2) + \mathcal{O}(\epsilon)\right]$$

- Not multiplicatively renormalisable due to extra light-cone divergencies
- Dual to the TMD case

Energy logarithmic derivative gives an evolution equation ($\overline{s} = \overline{\mu}^2 s$):

$$\frac{\mathrm{d}\ln\mathcal{W}}{\mathrm{d}\ln s} = -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon}\left[(\overline{s}+\mathrm{i}\varepsilon)^{\epsilon} - (-\overline{s}+\mathrm{i}\varepsilon)^{\epsilon}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\frac{\mathrm{d}\ln\mathcal{W}}{\mathrm{d}\ln s} = -2\frac{\alpha_s C_F}{\pi}\left[(\overline{s}+\mathrm{i}\varepsilon)^{\epsilon} + (-\overline{s}+\mathrm{i}\varepsilon)^{\epsilon}\right] \to -2\frac{\alpha_s C_F}{\pi} = -2\Gamma_{cusp}$$

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Makeenko-Migdal Approach

Wilson Loops as Fundamental Gauge-Invariant Degrees of Freedom

$$\mathcal{W}_n(\Gamma_1, \ldots, \Gamma_n) = \mathrm{tr}\, \langle 0 | \, \mathcal{T} \frac{1}{N_c^n} \Phi(\Gamma_1) \cdots \Phi(\Gamma_n) \, | 0 \rangle$$

$$\Phi(\Gamma_i) = \mathcal{P} e^{ig \int_{\Gamma_i} \mathrm{d}z^\mu \, A_\mu(z)}$$

Formalism

Area derivative: $$\frac{\delta}{\delta \sigma_{\mu\nu}(x)} \Phi(\Gamma) = \lim_{|\delta\sigma|} \frac{\Phi(\Gamma \delta \Gamma_x) - \Phi(\Gamma)}{|\delta \sigma_{\mu\nu}(x)|}$$

Path derivative: $$\partial_\mu \Phi(\Gamma) = \lim_{|\delta x_\mu|} \frac{\Phi(\delta x_\mu^{-1} \Gamma \delta x_\mu) - \Phi(\Gamma)}{|\delta x_\mu|}$$

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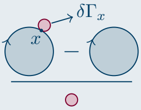

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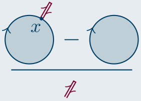

Makeenko-Migdal Approach

Mandelstam Formula

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\,\mathrm{tr}\,\Phi(\Gamma) = \mathrm{i}g\,\mathrm{tr}\left\{F^{\mu\nu}(x)\Phi(\Gamma)\right\}$$

- Relates geometry evolution of a loop with its gauge content
- But area functional derivative is not well-defined for arbitrary contours..

  $\Rightarrow$ no information about cusps, divergencies, etc...!

Schwinger approach

- Fundamental quantum dynamical principle: $\delta\,\langle{}'|{}''\rangle = \frac{\mathrm{i}}{\hbar}\,\langle{}'|\,\delta S\,|{}''\rangle$
- Applied to $\mathcal{W}_1$, this gives the Makeenko-Migdal equation:

$$\partial^\nu \frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathcal{W}_1(\Gamma) = g^2 N_c \oint_\Gamma \mathrm{d}z^\mu\,\delta^{(4)}(x-z)\mathcal{W}_2(\Gamma_{xz}\Gamma_{zx})$$

Migdal (1980); Makeenko, Migdal (1981); Brandt et al. (1982)

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Makeenko-Migdal Approach

Remarks:

- MM equation is exact and non-perturbative, but not closed and difficult.
- Most interesting loops are divergent and have obstructions. Of particular interest are cusped loops, but in that case we lack a renormalised version of the MM equation.
- The area functional derivative is not a well-defined operation. In particular, the area differentiation of cusped loops is (at least) not straightforward.
- Problems when doing continuous deformations of loops in Minkowski space: a consistent definition of the derivatives is missing.
- Connecting loop functionals to observables is highly non-trivial.
- No known solution to the MM equations in four-dimensional Mikowskian space-time.

Makeenko-Migdal Approach

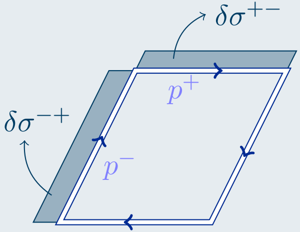
Simplifications:

- Large-N_c limit: factorisation property $\mathcal{W}_2(\Gamma_1, \Gamma_2) \approx \mathcal{W}_1(\Gamma_1)\mathcal{W}_1(\Gamma_2)$.
- Null-plane light-cone rectangular contours are essentially two-dimensional.
- Light-like polygons with conserved angles: no angle-dependent contributions (which might break the MM equation).
- By using area differentiation the power of divergencies decreases.

Can we combine the geometric approach of the MM equations with the evolution equations governed by the cusp anomalous dimension?

Angle-Conserving Deformations of the Null Plane

Deformations



- Null-plane defined by $\mathbf{z}_\perp \equiv 0$
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- $\delta\sigma^{+-} = \oint \mathrm{d}z^- \, z^+ = p^+ \delta p^-$
  $\delta\sigma^{-+} = \oint \mathrm{d}z^+ \, z^- = p^- \delta p^+$
- Area variable: $\Sigma = p_1 \cdot p_2 = \frac{1}{2} s = p^+ p^-$

Use energy evolution to find the geometrical evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \frac{\mathrm{d}\ln\mathcal{W}}{\mathrm{d}\ln s} = -2\Gamma_{cusp}$$

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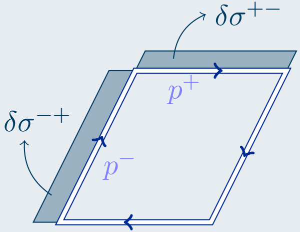

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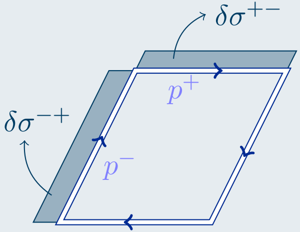

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Use energy evolution to find the geometrical evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \frac{\mathrm{d}\ln\mathcal{W}}{\mathrm{d}\ln\Sigma} = -4\Gamma_{cusp}$$

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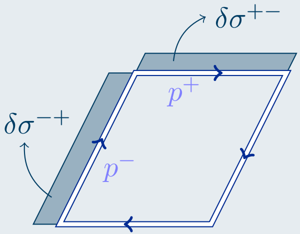

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Motivated by this, we conjecture a more general evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \left[\sigma_{\mu\nu} \frac{\delta}{\delta\sigma_{\mu\nu}} \ln \mathcal{W} \right] = - \sum_i \Gamma_{cusp}$$

Phys. Rev. D86 (2012) 085035, *I.O. Cherednikov, T. Mertens and F.F. VdV.*

Relation to TMDs

Relation to TMDs

- Related by overlapping divergencies ($\sim \frac{1}{\epsilon} \ln \theta$).
- Those are the only divergencies that contribute after $\frac{\mathrm{d}}{\mathrm{d}\ln \mu} \frac{\mathrm{d}}{\mathrm{d}\ln \theta}$.
- $\theta = \frac{\eta}{p \cdot N^-}$ is the rapidity cut-off
- $p \sim N^+$, so $\theta \sim (N^+ N^-)^{-1}$

Cherednikov, Stefanis (2008)

Evolution Equation for TMD On the Light-Cone

$$\frac{\mathrm{d}}{\mathrm{d}\ln \mu} \left[\frac{\mathrm{d}}{\mathrm{d}\ln \theta} \ln f(x, \mathbf{k}_\perp) \right] = 2\Gamma_{cusp}$$

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Conclusion

- Loop space:
  - Wilson loops as fundamental degrees of freedom
  - Makeenko-Migdal approach gives good description of geometrical properties
  - But system of MM eqs is not closed and cannot be applied trivially...
- For  : $\delta_{\sigma_{\mu\nu}} \sim \partial_s$ and MM eqs $\sim$ energy/rapidity evolution eqs.
- Geometrical properties of loop space $\sim$ dynamics in cusps
- Conjecture: MM approach can be applied to construct energy/rapidity evolution equations in many interesting situations. Specific properties are determined by contours with cusps.