A Generalization of States of Low Energy on Globally Hyperbolic Spacetimes

Marcos Brum[1,2], Klaus Fredenhagen[2]
marcos.brum@desy.de

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[1]Physics Institute, Federal University of Rio de Janeiro, Brazil
[2]II Institute for Theoretical Physics, Hamburg University, Germany

Summary

Introduction

- There is no general, covariant definition of vacuum state on Globally Hyperbolic Spacetimes;
- The renormalized energy density of quantum fields may assume negative values. This permits the creation of exotic spacetimes and violations of the Second Law of Thermodynamics.
- On the other hand, **the expectation value of energy density on a Hadamard state averaged along a timelike curve** – or even, over a timelike hypersurface –, with compactly supported averaging function, is bounded from below!
  - This does not guarantee the existence of a state at which the energy density is minimal!
- The construction of states for which the energy density is minimal is achieved by smearing the expectation value of the energy density of a free, minimally coupled massive scalar field on a quasifree state along a timelike curve on Robertson-Waker spacetime!
- We intend to show how far one can go and which obstacles prevent one to complete this construction for more general spacetimes.

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AQFT on Curved Spacetime

- The KG differential operator, being hyperbolic, possesses unique **Advanced** and **Retarded Fundamental Solutions**,

$$(\Box + m^2)E^{\pm} = E^{\pm}(\Box + m^2) = \mathbb{1}$$

- We set $E := E^{+} - E^{-}$ and define the hermitean form

$$\gamma(f, g) := -i \int_{M} d^4x \sqrt{|g|}(fEg)$$

- The space of test functions together with this form constitute a sympletic space. The Weyl representation of the algebra over this space is generated by $W(f)$ and satisfies

$$W(f) = \exp\left(-i\phi(f)\right)$$
$$W(f)W(g) = \exp\left(-\frac{i}{2}\gamma(f,g)\right)W(f+g)$$
$$W(f)^{*} = W(-f)$$

- The Cauchy problem for the KG operator is well-posed, i.e., the algebra of operators on any spacelike hypersurface is related by an isomorphism to the algebra on a spacelike Cauchy hypersurface, on which initial conditions are defined.

Definition of States

- A **State** is a functional on the space of field operators over the complex numbers.
  - By means of the GNS construction, to any state ω is associated a *Hilbert Space* $\mathscr{H}_\omega$, a *Representation* π_ω of the fields as operators on this Hilbert space and a *Cyclic Vector* Ω_ω in this Hilbert space.
  - States are required to satisfy the **Hadamard Condition**, i. e., they are required to possess the same singularity structure as the states on Minkowski Spacetime.
  - We will restrict attention to states which are completely described by their two-point function, named *Quasifree States*.

QEI

The QEI may be obtained in two forms:

Difference QEI:

$$\int_{\mathbb{R}} \mathrm{d}t f^2(t) \left(\langle T^{\mathrm{ren}}_{ab} u^a u^b \rangle_\omega (\gamma(t)) - \langle T^{\mathrm{ren}}_{ab} u^a u^b \rangle_{\omega_0} (\gamma(t)) \right) \geq -\mathcal{B}_D$$

$$\mathcal{B}_D = \int_0^\infty \frac{d\xi}{\pi} \left[f \otimes f \vartheta^* \langle T^{\mathrm{split}}_{ab'} u^a u^{b'} \rangle_{\omega_0} \right]^\wedge (-\xi, \xi)$$

ϑ^* is the pullback: $\mathbb{R}^2 \to \mathcal{M} \otimes \mathcal{M}$, $\langle \ldots \rangle_{\omega_0}$ is defined as the action of a diff. op. on the two point function of the state ω_0.

Microlocal Analysis assures the finiteness of this bound!

Absolute QEI:

$$\int_{\mathbb{R}} \mathrm{d}t f^2(t) \langle T^{\mathrm{ren}}_{ab} u^a u^b \rangle_\omega (\gamma(t)) \geq -\mathcal{B}_A$$

$$\mathcal{B}_A = \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \left[f \otimes f \vartheta^* T^{\mathrm{split}} \widetilde{H} \right]^\wedge (-\xi, \xi) + \text{“curvature terms”}$$

$T^{\mathrm{split}}\widetilde{H}$ is the same as $\langle \ldots \rangle_{\omega_0}$, but with the first terms of the Hadamard series replacing the two point function.

Examination of the Sobolev wavefront set of $T^{\mathrm{split}}\widetilde{H}$ assures the finiteness of this bound!

SLE on RW spacetime

- States of Low Energy are states on which the renormalized expectation value of the energy density is minimal!
  - They are of the Hadamard form and are more general than Adiabatic States because SLE do not assume that spacetime is asymptotically flat!
- We note that a quasifree state possessing the same symmetries of the RW spacetime (homogeneity and isotropy) can be easily decomposed, in Fourier space, as an orthogonal sum of the fourier modes of solutions of the KG equation:

$$\omega_2(x,x') = \int d\mu(k) \left[T^*_k(x^0) Y_k(\vec{x}) T_k(x'^0) Y^*_k(\vec{x'}) \right] =: \int d\mu(k) \omega_{2_k}(x,x')$$

- Since the KG operator is a 2nd order Diff. Op. with real coefficients,

$$T_k(x^0) = \alpha_k S_k(x^0) + \beta_k S^*_k(x^0) \qquad |\alpha_k|^2 - \beta_k^2 = 1$$

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$$\omega_2(x,x') = \int d\mu(k)\left[T_k^*(x^0)Y_k(\vec{x})T_k(x'^0)Y_k^*(\vec{x'})\right] =: \int d\mu(k)\omega_{2_k}(x,x')$$

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SLE on RW spacetime

- The energy density

$$W = \lim_{x' \to x} \int d\mu(k) W_{2_k} = \lim_{x' \to x} \int dt f^2(t) \int d\mu(k) \langle T_{ab} \delta_0^a \delta_0^b \rangle_{\omega_{2_k}}(x, x')$$

may be minimized by choosing

$$\beta_k = \sqrt{\frac{c_1}{\sqrt{c_1^2 - |c_2|^2}} - \frac{1}{2}} \qquad \alpha_k = e^{i\varphi} \sqrt{\frac{c_1}{\sqrt{c_1^2 - |c_2|^2}} + \frac{1}{2}}$$

Where

$$c_1 = \int dt f^2(t) \left(|\dot{S}_k(t)|^2 + \omega_k^2 |S_k(t)|^2 \right) \qquad c_2 = \int dt f^2(t) \left((\dot{S}_k(t))^2 + \omega_k^2 S_k(t)^2 \right)$$

$$\omega_k^2 = \lambda_k^2 / a^2(t) + m^2$$

And

$$\mathrm{Arg}\alpha_k = -\mathrm{Arg}c_2 + \pi =: \varphi$$

SLE on expansion cosmologies

We will consider a globally hyperbolic spacetime without spatial symmetries

$$ds^2 = dt^2 - a^2(t) \sum_{i,j=1}^{3} h^{ij} dx^i dx^j$$

- Now we cannot claim invariance of the state because we do not have a group of symmetries. Therefore there is no reason why the two-point function would be decomposable as an orthogonal sum of modes!

WORK IN PROGRESS:

- We are in the process of proving that, in the static case, i. e., $a(t)$ =constant, the orthogonal decomposition can be made.
- This class of spacetimes can be obtained as solutions to the Einstein Equations for the cases when:
  - there is a perfect fluid AND constant, negative spatial curvature;
  - there is a nonperfect fluid and nonconstant spatial curvature.

Conclusions and Outlook

- The Algebraic Approach to QFT allows us to define a concept of state without recourse to a Hilbert space of states!
- States of Low Energy on RW spacetime are the closest one has got in defining a Hadamard state on which the energy density is minimal;
- Although it is a simple concept, its construction in more general spacetimes is only possible under very simplifying assumtions;
- Further questions: non-minimal coupling, non-diagonal Bogolubov Transformations, interacting fields, different types of fields, application to backreaction...

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