The Remarkable Mathematical Structure of Scattering Amplitudes

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CERN & Brown University

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Who is the 'Scattering Amplitude Community'

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The past few years have seen growing interest in the study of the mathematical structure of scattering amplitudes,

as evidenced in part by dedicated annual conferences including

- 2009 Durham, UK
- 2010 Queen Mary, UK
- 2011 University of Michigan, US

and ...

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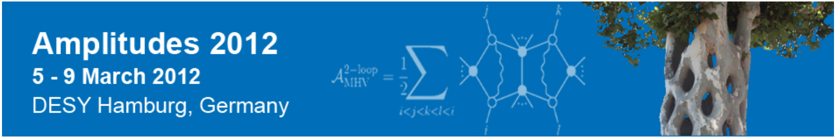
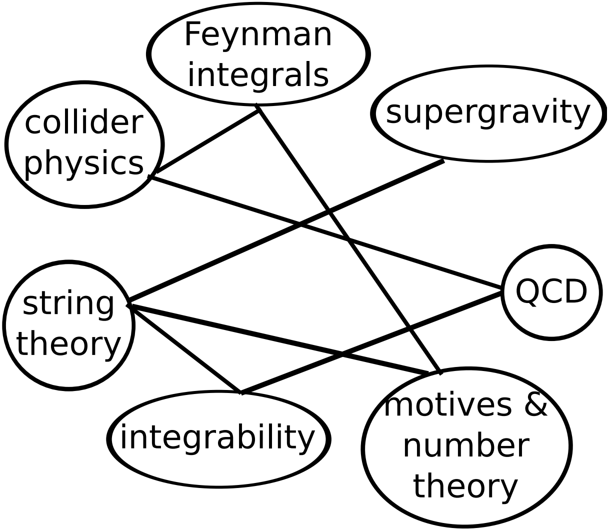


Rutger H. Boels (chair, Hamburg)
Gudrun Heinrich (MPI Munich)
Johannes Henn (IAS Princeton)
Pierpaolo Mastrolia (MPI Munich)
Jan Plefka (HU Berlin)
Volker Schomerus (DESY)

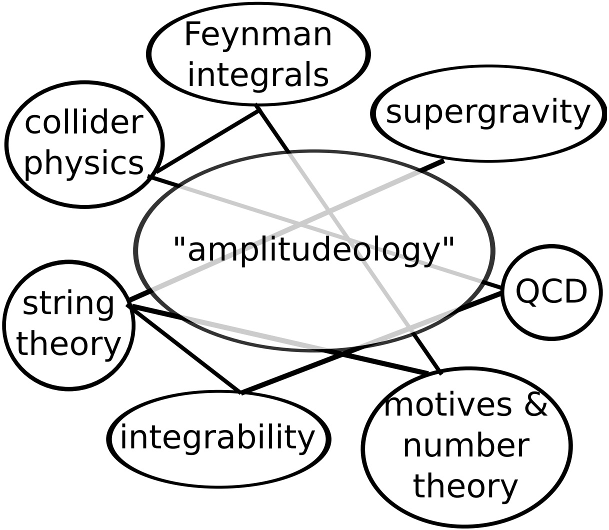
Who is the 'Scattering Amplitude Community'

A diverse collection of interrelated subjects ...



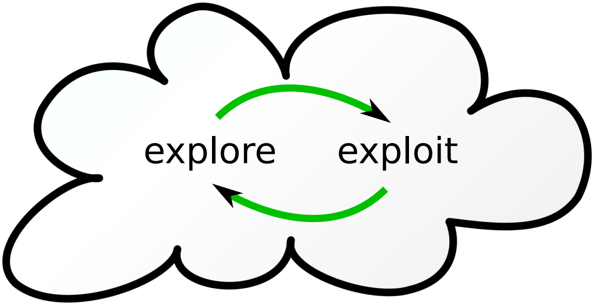
Who is the 'Scattering Amplitude Community'

... bringing together a community ...



Who is the ‘Scattering Amplitude Community’

... with the common goal to



the remarkable, powerful, and long-hidden mathematical structure in scattering amplitudes, particularly in gauge theory.

Why Yang-Mills Theory?

Why so much attention on gauge theory?

Why not a 'simpler' quantum field theory, like scalar ϕ^3 theory?

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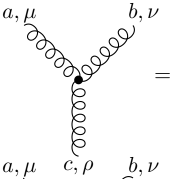
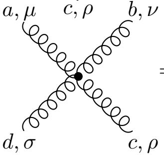
- is known to have deep connections to mathematics,
- and (perhaps unexpectedly) has simpler amplitudes!

Why Yang-Mills Theory?

Feynman rules for gauge fields have been in textbooks for decades



$$= g f^{abc} \left[g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu \right]$$



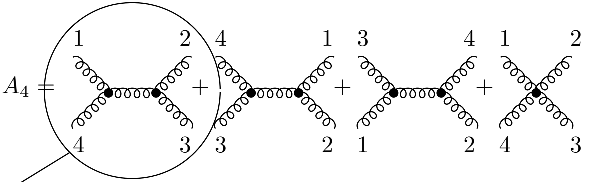
$$= -i g^2 f^{abc} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + \text{permutations}$$

They don't look very simple, but today we have very powerful computers to help us...

Why Yang-Mills Theory?

Unfortunately the number of diagrams grows very rapidly with the number of particles,

and each individual diagram gives quite a messy expression...



$$g^2 f^{abe} f^{cde} \left[g_{\mu\nu}(k-p_1)_\rho + g_{\nu\rho}(p_1-p_4)_\mu + g_{\rho\mu}(p_4-k)_\nu \right] \epsilon_1^\mu \epsilon_2^\lambda \epsilon_3^\sigma \epsilon_4^\nu$$
$$\times \frac{i}{k^2} \left[g_{\sigma\lambda}(k-p_2)^\rho + g_\lambda{}^\rho (p_2-p_3)_\sigma + g^\rho{}_\sigma (p_3-k)_\lambda \right], \; k = p_1 + p_4$$

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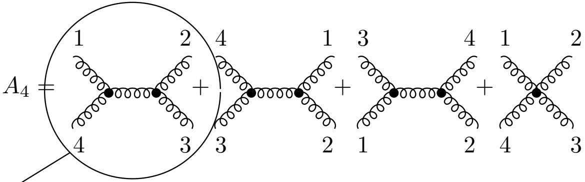

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Already for five gluons, writing out all terms takes dozens of pages.

For eight gluons there are over 34,000 Feynman diagrams!

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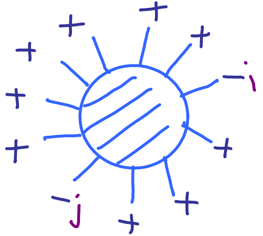
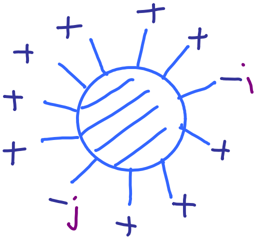
by showing you one of the most important formulas in our field



$$= \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

due to Parke & Taylor (1984) and Berends & Giele (1986), for the tree-level n-gluon MHV amplitude (2 negative and $n-2$ positive helicity).

Spinor Helicity Variables

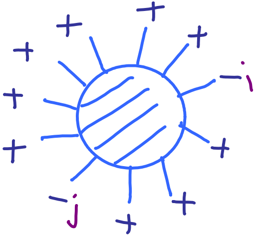


$$= \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Here we express the amplitude not in terms of the momenta and polarizations of the gluons,
but rather in terms of spinor helicity variables defined by

$$p = \begin{pmatrix} p_0 + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} \lambda^1 \tilde{\lambda}^1 & \lambda^1 \tilde{\lambda}^2 \\ \lambda^2 \tilde{\lambda}^1 & \lambda^2 \tilde{\lambda}^2 \end{pmatrix} \qquad \langle ij \rangle = \lambda_i^1 \lambda_j^2 - \lambda_i^2 \lambda_j^1$$

Simple Amplitudes vs. Simple Lagrangians



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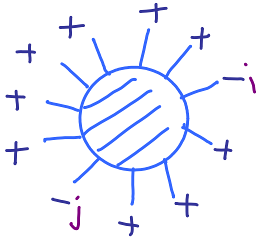

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The importance of this formula goes well beyond simply knowing the value for certain amplitudes (after all, few of us will ever need to know the 100-gluon amplitude, for example);

rather it is important because it clearly reveals that the Feynman rules (i.e. the Lagrangian) are completely the wrong way to think about even perturbative QCD.

Im particular, simple amplitudes do not require simple Lagrangians...

The New S-Matrix Theory

Today there exist several simple and efficient algorithms for computing (analytically) any desired tree-level amplitude of gluons, something which would have seemed impossible not long ago

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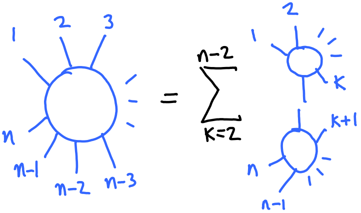
(and, with not much extra work, also charged fermions, electroweak bosons, Higgs, etc.).

These methods make no reference to spacetime, Lagrangians, or gauge symmetry–particles are physical and on-shell at every step.

For example the BCFW recursion relation



New versus Old S-Matrix Theory

This sounds, in spirit, identical to “old” S-matrix theory.

Then, the idea was to

1. enumerate the principles that the S-matrix should obey (locality, analyticity, etc.), and then
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Often the answer has properties which would have been impossible to guess in advance.— I’ll give two examples.

It is typical to have several equivalent formulas for a given amplitude, which appear to be completely unrelated.

For example, the four-point amplitude in scalar ϕ^3 theory is

$$A = -\frac{1}{s} - \frac{1}{t} - \frac{1}{u}$$

or

$$A = \frac{10s^2}{tu} + \frac{20s}{t} - \frac{9s}{tu} + \frac{32s}{u} + \frac{10u}{t} - \frac{15}{t} + \frac{34t}{u} - \frac{15}{u} + 42 + \frac{22t}{s} - \frac{5u}{ts} + \frac{10u}{s} + \frac{12t^2}{us} - \frac{5t}{us} - \frac{11}{s}$$

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These formulas are secretly equal to each other in light of $s + t + u = 0$.

Amplitude Representations

A much less trivial example involves the $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ gluon amplitude, which has the following two representations:

$$\frac{[4|5+6|1\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle[2|3+4|5\rangle\, s_{234}} + \frac{[6|1+2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3+4|5\rangle\, s_{62}}$$

$$\frac{s_{123}^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|4+5|6\rangle} + \frac{\langle 12\rangle^3[45]^3}{[34]\langle 61\rangle[3|4+5|6\rangle[5|6+1|2\rangle\, s_{612}}$$

$$+ \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[61][1|2+3|4\rangle[5|6+1|2\rangle\, s_{234}}$$

the equality of which is essentially impossible to prove by hand (instead one can use numerical experimentation).

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Andrew Hodges showed that this amplitude is equal to the volume of a certain polytope in complex projective space, and the formulas arise from triangulating the polytope in two different ways.

An Experimental Science

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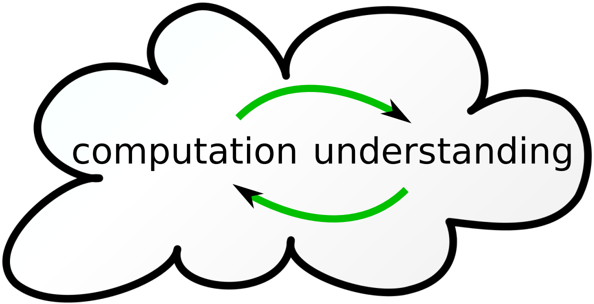

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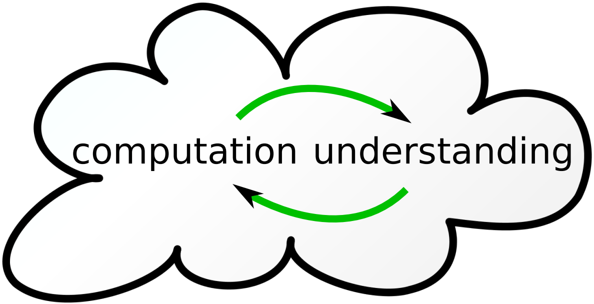

Another example is the discovery of dual conformal symmetry by Drummond, Henn, Korchemsky & Sokatchev.

Dual Conformal Symmetry

The on-shell condition $p^2 = 0$ for the momenta of massless particles was solved by switching to spinor helicity variables.

Now, every scattering amplitude has an overall momentum conserving

$$\delta^4(p_1 + p_2 + \cdots + p_n)$$

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We can similarly solve momentum conservation by using dual variables x_i defined by

$$p_1 = x_1 - x_2, \qquad p_2 = x_2 - x_3, \qquad p_n = x_n - x_1$$

so that $p_1 + p_2 + \cdots + p_n = 0$ is automatic.

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It has also been proven to be an exact symmetry of supersymmetric Yang-Mills theory.

Why Supersymmetric Yang-Mills Theory?

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Much of the interest in this theory stems from strong indications that it is exactly solvable (in the planar limit).

Indeed some important aspects of the theory have already been solved (Beisert, Eden & Stadaucher; Gromov, Kazakov & Vieira; and many others).

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Solving a nontrivial, yet very relevant, quantum field theory in four dimensions would be a tremendous theoretical accomplishment.

It is the 'harmonic oscillator' of quantum field theories.

Solving Supersymmetric Yang-Mills Theory

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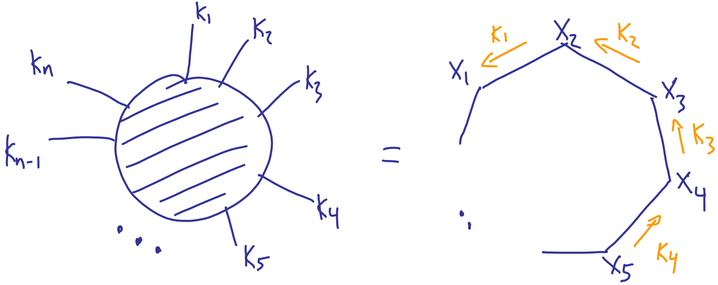
In supersymmetric Yang-Mills they have been shown to be exactly equivalent:



Moving Past Tree-Level

Moving forward to loop (quantum) level, I would argue that we have seen one revolution come to fruition, but are still in desperate need of another:

$$A = \int d^4p_1 \cdots d^4p_L \underbrace{\sum \text{feynman diagrams}}$$

$$\Downarrow$$

$$= \int d^4p_1 \cdots d^4p_L \left(\text{relatively simple integrand}\right)$$

Methods have been developed for efficiently processing the integrand, but there is no practical general algorithm for writing down the results for such integrals; usually they must still be done on a case-by-case basis.

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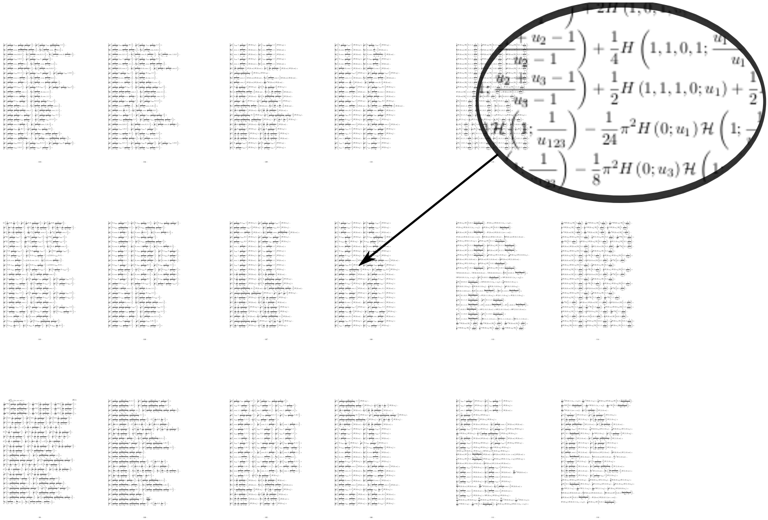
For one-loop QCD this process has been automated, with great success, by the BlackHat code of Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower & Maître.

The Simplest Multi-Loop Amplitude in SYM Theory

In supersymmetric Yang-Mills theory the simplest nontrivial (i.e., not fixed by dual conformal symmetry) scattering is the 2-loop 6-gluon amplitude.

In a heroic effort it was computed analytically, in terms of generalized polylogarithm functions, by Del Duca, Duhr & Smirnov (2009).

The Simplest Multi-Loop Amplitude in SYM Theory



Simplicity has to be Believed to be Seen

My collaborators Goncharov, Vergu & Volovich 'knew' (i.e., hoped), that this couldn't be the end of the story.

Using the concept of the symbol of a transcendental function, related to the theory of motives, we simplified it down to

$$R(u_1, u_2, u_3) = \sum_{i=1}^{3} L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

where L_4 and J are simple polynomials in $\log$ and Li_n and

$$x_i^{\pm} = u_i \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{(u_1+u_2+u_3-1)^2 - 4u_1u_2u_3}}{2u_1u_2u_3}$$

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Moreover, nobody cares about the particular value of this scattering amplitude...

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Let me emphasize that the value of our result is not that it computes the 2-loop 6-gluon amplitude — it had already been computed by DDS!

Moreover, nobody cares about the particular value of this scattering amplitude...

Rather, our result is of value because, like the Parke-Taylor formula at tree-level, it gives serious hope to the idea that we might be able to unlock the secrets of Yang-Mills theory at higher loops.

The Symbol

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It trivializes functional identities; for example the rules

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trivialize the well-known identity

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(x) + \log(x)\log(1-x) - \frac{\pi^2}{6} = 0$$

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It is a general mathematical technique which can be (and has been) used wherever polylogarithm functions appear, including QCD.

Looking Forward

We know the symbol won't take us all the way to the end of the journey ... since sufficiently complicated Feynman integrals (even in supersymmetric Yang-Mills) are not expressible in terms of generalized polylogarithm functions alone.

It is just an example of one of the many steps our community has taken, each with the goal of reaching the top of the next hill and letting us see across the following valley.

The Philosophy of Amplitudeology

1. Simplifications do not happen by accident.
2. This is an experimental science. (Get the answer first, by any means necessary, then analyze it.)
3. Simplicity has to be believed to be seen.