Geometry for 2-Form Gauge Fields

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Before we come to geometry for 2-form gauge fields:

What is a 1-form gauge field?

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What is geometry for a 1-form gauge field?

- **A hermitian line bundle with connection.**

Electrodynamics on $\mathbb{R}^n$

Relevant:

- a metric
- a field strength F (2-form) satisfying Maxwell's equations

$$\mathrm{d}F = 0 \quad \text{and} \quad \mathrm{d} \star F = J.$$

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Auxiliary structure:

- gauge potential A (1-form) satisfying $\mathrm{d}A = F$.
- different choices of A are related by a gauge transformation,

$$A' = A + \frac{1}{\mathrm{i}} \mathrm{d}g g^{-1}$$

for a function $g : \mathbb{R}^n \to U(1)$.

Example: Charged Particle

- We describe the particle by a curve
$$\phi : [0,1] \to \mathbb{R}^n.$$
For simplicity, we assume $\phi(0) = \phi(1)$.
- The particle gathers a contribution of
$$S_F(\phi) = \oint \phi^* A$$
to its action.
- By Stokes' Theorem, this contribution is gauge invariant.

Electrodynamics on Curved Spacetime

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We can still work locally:

- Cover the manifold by open sets,

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

- The sets U_α can be chosen topologically so good that there exist **local** gauge potentials A_α with $\mathrm{d}A_\alpha = F|_{U_\alpha}$.

- On two-fold intersections $U_\alpha \cap U_\beta$ **two** local gauge potentials are present: A_α and A_β. They differ by a gauge transformation

$$A_\beta = A_\alpha + \frac{1}{\mathrm{i}} \, \mathrm{d}g_{\alpha\beta} g_{\alpha\beta}^{-1}.$$

- On three-fold intersections, we demand a consistency condition:

$$g_{\alpha\gamma} = g_{\beta\gamma} \cdot g_{\alpha\beta}.$$

The Charged Particle on a Curved Spacetime

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- What we **can** define is the **exponential** of this contribution:

$$\exp\left(\mathrm{i}S_L(\phi)\right) := \prod_{i=1}^{N} \exp\left(\mathrm{i}\int_{t_i-1}^{t_i} \phi^* A_{\alpha(i)}\right) \cdot g_{\alpha(i)\alpha(i+1)}(\phi(t_i))$$

This is still enough to derive the **equations of motion**!

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Is this contribution still gauge invariant?

- It is invariant under **local** gauge transformations

$$A'_\alpha = A_\alpha + \frac{1}{\mathrm{i}} \mathrm{d} h_\alpha h_\alpha^{-1} \qquad g'_{\alpha\beta} = g_{\alpha\beta} h_\beta^{-1} h_\alpha$$

Geometry: Line Bundles with Connection

Definition:

1. *The collection* $L := \{A_\alpha, g_{\alpha\beta}\}$ *is a* **hermitian line bundle with connection** *of curvature* F.
2. *The collection* $\{h_\alpha\}$ *is an* **equivalence** $L \to L'$.

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Upshot:

- **A 1-form gauge field is an equivalence class of hermitian line bundles with connection.**
- The curvature of the connection is the field strength.
- The holonomy of the connection describes the coupling of charged particles to the field.

For a fixed a field strength F, are there non-equivalent choices of line bundles with connection of curvature F?

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If F is any field strength, is there a line bundle with curvature F?

- In general: no.
- **Theorem:** *There exists a line bundle with connection of curvature F if and only if*

$$\int_B F \in \mathbb{Z}$$

for any 2-dimensional submanifold $B \subset M$.

Relevance of Line Bundles

- **Dirac's magnetic Monopoles:**

  Question: Why quantizes the existence of a sole magnetic monopole the electric charge?

  Answer: For the field F of a monopole, no global gauge potential can be chosen. We thus need an hermitian line bundle with connection of curvature F. The existence of such line bundles quantizes F.

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- **Aharonov-Bohm effect:**

  Question: Electrons are affected by an „infinitely long and thin“ solenoid although the field strength is zero. Why?

  Answer: The line bundle is, though flat, non-trivial.

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Simplified Situation:

- there is a global gauge potential B (2-form) with $\mathrm{d}B = H$.
- a charged string $\phi : \Sigma \to M$ couples to the gauge field by

$$S_H(\phi) := \int_\Sigma \phi^* B$$

In general, however, global gauge potentials do **not** exist.

If no **global** gauge potential B can be chosen, we work locally:

- We cover M with open sets U_α with good topology. Then, we can choose local gauge potentials B_α.
- On two-fold intersections, there are two potentials present: B_α and B_β. They differ by a (1-form) gauge potential $A_{\alpha\beta}$:

$$B_\beta = B_\alpha + \mathrm{d}A_{\alpha\beta}.$$

- On three-fold intersections, three gauge potentials are present: $A_{\alpha\beta}$, $A_{\beta\gamma}$ and $A_{\alpha\gamma}$: they differ by a gauge transformation

$$A_{\alpha\gamma} = A_{\beta\gamma} + A_{\alpha\beta} + \frac{1}{\mathrm{i}}\,\mathrm{d}g_{\alpha\beta\gamma}g_{\alpha\beta\gamma}^{-1}$$

- On four-fold intersections, we demand that these gauge transformations satisfy the consistence condition

$$g_{\beta\gamma\delta} \cdot g_{\alpha\beta\delta} = g_{\alpha\gamma\delta} \cdot g_{\alpha\beta\gamma}.$$

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Example: Wess-Zumino-Witten Models

- $M = G$ is a compact Lie group, and the field strength H is given by
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- **Question**: Do gerbes with this curvature exist?
Answer: Depends on k:
  - if G is simple and simply-connected for all $k \in \mathbb{Z}$.
  - if $G = SO(3)$ only for $k \in 2\mathbb{Z}$.

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- **Question**: Are there inequivalent choices?
  Answer: Depends on the topology of the group G:
  - if G is simple and simply-connected, no.
  - if $G = SO(4n)/\mathbb{Z}_2$, yes: two.

Recent Results that use the Geometry of Gerbes

- D-branes:
  - twisted vector bundles (Kapustin, hep-th/9909089)
  - gerbe modules (Gawędzki, hep-th/0701071)
- Unoriented string theories:
  - Jandl structures (Schreiber-Schweigert-KW, hep-th/0512283)
  - Classification of unoriented WZW models (Gawędzki-Suszek-KW, hep-th/0701071)
- Topological defect lines:
  - Gerbe bimodules (Fuchs-Schweigert-KW, hep-th/0703145)
- (in progress:) D-branes in unoriented string theories (Gawędzki-Suszek-KW).