Generalised Geometry, Supergravity and D-branes

Collaborative Research Centre 676

A6: Mathematical Aspects of String Compactification

Paul Smyth

II. Institut für Theoretische Physik
Universität Hamburg

The aim of the project is the study of generalised supersymmetric string compactifications. In particular the investigation of manifolds with SU(3)- and SU(3)×SU(3)-structure as possible string backgrounds is proposed. The consistent embedding of D-branes, orientifold-planes and background fluxes is also planned.

Aim of this research

To further develop the relations between the mathematics and physics of manifolds with $SU(3) \times SU(3)$ structure

Methods of investigation:

- Extend the class of supersymmetric compactifications that have been studied to date.
- Consider configurations of supergravity solitons (D-branes) that give rise to domain walls in four-dimensional spacetimes.
- Our goal is to understand the constraints placed on the internal geometry and their consequences for low-energy physics.

What are the benefits of studying domain walls in flux compactifications for physics and mathematics?

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What are the benefits of studying domain walls in flux compactifications for physics and mathematics?

We will consider string theory compactifications at the level of the low energy effective field theory - 10D supergravity.

To have interesting 'beyond standard model' phenomenology we would like to have some supersymmetry remaining the 4D:

$$\delta\psi_M = \nabla_M \varepsilon = 0 \,,$$

where ψ_M is the gravitino (graviton superpartner), ε is an infinitesimal spinor parameter and ∇_M is the usual covariant derivative on spinors $(M = 0, \ldots, 9)$.

Compactifying amounts to saying the 10D spacetime has a compact, 6D component:

$$\mathcal{M}_{10} = M_4 \times Y \,,$$

M_4 is a 4D maximally symmetric spacetime x^μ, $(\mu = 0, \ldots, 3)$ and Y is a 6D compact Riemannian manifold y^m, $(m = 4, \ldots, 9)$.

Introduction continued - constraints from supersymmetry

The differential equations $\delta\psi_M = 0$ imply integrability conditions on the spinors:

$$[\nabla_M, \nabla_N]\varepsilon = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\varepsilon = 0$$

where R_{MNPQ} & Γ^{PQ} are the 10D Riemann tensor and gamma matrices. The external $(M, N = \mu, \nu)$ parts of this imply that

- M_4 is flat Minkowski spacetime and $\varepsilon = \varepsilon(y)$.

The remaining internal parts give conditions on Y, in particular on its *structure group* which tells you how to patch together local frames over all of Y.

- Globally $\exists\ \varepsilon$ on $Y \Rightarrow Y$ has a reduced structure group $SU(3) \subset SO(6)$ and a complex structure $J : T_Y \to T_Y\ ,\ J^2 = -1$, formed from a bilinear of ε.
- Also, $\nabla_m\varepsilon(y) = 0\ \Rightarrow Y$ is Ricci-flat. It is a (complex) Calabi-Yau manifold. See e.g. V. Cortes SFB Lectures '07

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Introduction contd.

The 4D field theory which appears from the Calabi-Yau compactification of Type II supergravity has $\mathcal{N} = 2$ supersymmetry i.e. 8 supercharges.

The goal for theorists is to find a compactification of string theory to 4D which has spontaneously broken $\mathcal{N} = 1$ supersymmetry and a Standard Model sector (One of string theory's 'other' problems).

One way to extend to $\mathcal{N} = 1$ supersymmetry in 4D is to include fluxes (i.e. form fields) on the internal manifold Y. (see e.g. Graña '05, Gurrieri et al '02)

$\Rightarrow \nabla_M \varepsilon \neq 0$, and Y is no longer Ricci-flat - what is the interplay between these new geometries and the physics of $\mathcal{N} = 1$ compactifications?

We will consider warped compactifications of Type II supergravity which preserve 4D Poincaré invariance and $\mathcal{N} = 1$ supersymmetry,

10D metric ansatz

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n$$

$A(y)$ is known as a *warp factor*, and all other fields which appear later will only have $y-$dependence e.g. the scalar dilaton Φ, and the higher degree form fields $F_{(n)}$ & $H_{(3)}$.

- The appearance of the internal form fields, or *fluxes*, has a major effect on the geometry:

Internal fluxes $\Rightarrow$ $R_{mn} \neq 0$ and Y is a manifold with torsion i.e. $\nabla_M \varepsilon \neq 0$.

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$\mathcal{N} = 1$ constraints on Y - $\exists$ two $\mathrm{SU}(3)$ structures

As in $\mathcal{N} = 2$ Calabi-Yau compactifications supersymmetry constrains the geometry, but now $\delta\psi_M = 0$ is more complicated.

For $\mathcal{N} = 1$ supersymmetry in 4D we decompose the 10D spinors as

$$\varepsilon_{(i)} \quad = \quad \zeta_+(x) \otimes \eta_\pm^{(i)}(y) + \text{c.c.} \quad , \quad (i = 1, 2)$$

- One can easily see again that $\varepsilon = \varepsilon(y)$, i.e. $\zeta(x) = \zeta$ is constant.
- Also, $\exists$ 2 globally defined internal spinors $\eta^{(1)}, \eta^{(2)}$, with chirality $\pm$, each defining a reduced structure group $\mathrm{G} = \mathrm{SU}(3) \subset \mathrm{SO}(6)$ associated to the tangent bundle T_Y.

In order to study this pair of structure groups and the differential conditions on the spinors $\eta_\pm^{(i)}$ it is convenient to use Generalised Complex Geometry.

Generalised complex geometry and $T_Y \oplus T_Y^\star$ Hitchin '02, Gualtieri, Witt '04

Consider the formal sum of the tangent and cotangent bundles over Y: $T_Y \oplus T_Y^\star$.

- Sections of $T_Y \oplus T_Y^\star$ are a sum of a vector field plus a one-form: $X + \xi$.
- $T_Y \oplus T_Y^\star$ has $SO(6,6)$ structure, and $\eta^{(1)}, \eta^{(2)} \sim \mathcal{J} \Rightarrow SU(3) \times SU(3)$.

A generalised almost complex structure $\mathcal{J} : T_Y \oplus T_Y^\star \to T_Y \oplus T_Y^\star$ with $\mathcal{J}^2 = -\mathbf{1}$.

- Describes complex and symplectic manifolds, but more generally manifolds which locally have a product form (Generalised Darboux Theorem)

complex k−fold $\times$ $(6-2k)$ symplectic manifold

The important point is that $\eta^{(1)}, \eta^{(2)}$ spinor bilinears are described using $T_Y \oplus T_Y^\star$.

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Supersymmetry conditions for $\mathcal{N} = 1$ Minkowski compactifications

It turns out that it is possible to write the complete set of supersymmetry conditions $\delta\psi_M = 0$ in an elegant way:

$$\begin{aligned} e^{-2A+\Phi}(d + H\wedge)\left[e^{2A-\Phi}\Psi_-\right] &= dA \wedge \bar{\Psi}_- + \frac{e^{\Phi}}{8} i\tilde{F} \\ (d + H\wedge)\left[e^{2A-\Phi}\Psi_+\right] &= 0 \end{aligned}$$

with warp factor A and form fields $F,\ H$ & scalar Φ (c.f. $\mathcal{N} = 2 : \nabla_m \eta = 0$ on C.Y.)

The spinors $\eta_{\pm}^{(i)}$ have been combined into $SU(3) \times SU(3)$ bispinors

$$\Psi^{\pm} = \eta_{+}^{(1)} \otimes \eta_{\pm}^{(2)\dagger}$$

which can be understood as sums of bilinears or forms via a Fierz identity.

What does this all mean?

$SU(3) \times SU(3)$ compactifications describe $D = 4, \mathcal{N} = 1$ SUSY theories.

- If $\eta^{(1)} = \eta^{(2)}$ and $d\Psi^{\pm} = 0$ then Y is a Calabi-Yau manifold, e.g. $\Psi^{-} = \frac{\mathrm{i}}{8}\Omega$ - the holomorphic volume form.
- In general the differential conditions for $\mathcal{N} = 1$ supersymmetry in 4D imply the $SU(3) \times SU(3)$ structure manifold Y is a twisted generalised Calabi-Yau manifold.
- The resulting low-energy effective theories are actively studied, both for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry, but explicit realisations of smooth compact $SU(3) \times SU(3)$ geometries are hard to find.
- However, the formalism provides a compact way to deal with known supergravity cases (e.g. $SU(3)$ structure), as well as possible $SU(3) \times SU(3)$ backgrounds that include *stringy* effects.

Current goal:

Extend our understanding of $SU(3) \times SU(3)$ backgrounds beyond compactifications to 4D Minkowski and Anti-de Sitter spacetimes

- Apply previous tools to study $\frac{1}{2}$-supersymmetric solutions of the $\mathcal{N} = 1$ theory – i.e. the solitons of the supergravities $\sim$ D-branes in 10D.
- We focus on *domain walls* as they have applications in physics† and mathematics*.

† Dual gravity description of domain walls in supersymmetric gauge theories.

* Embedding 6D $SU(3) \times SU(3)$-structure into 7D $G_2 \times G_2$-structure manifolds.

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$D = 4,\ \mathcal{N} = 1$ Domain Walls from 10D

Consider a $D = 4$ domain wall metric ansatz in 10D ($\beta, \delta = 0, \ldots, 2$):

$$ds^2 = e^{2A(y,r)}\left(e^{-2U(r)}\eta_{\beta\delta}dx^\beta dx^\delta + dr^2\right) + g_{mn}(r,y)dy^m dy^n$$

determined by a profile function $\mathcal{W}(r) = \frac{3A}{2} + \frac{5U}{2}$ and a charge f_{dw}

$$F = \tilde{F} + dr \wedge f_{dw}$$

Guided by the projection condition for a half-supersymmetric solution in 10D we make an ansatz for the 4D projection

$$\boxed{\gamma^r \zeta_- = \alpha^* \zeta_+}$$

where α is a phase and γ is a 4D gamma matrix.

Supersymmetry constraints for domain walls

Using the tools discussed above for $\mathcal{N}=1$ compactifications, we find the following constraints on supersymmetric domain walls in terms of the spinor bilinears $\Psi^{\pm}$

$$(d+H\wedge)\left[e^{2A-\Phi}\Psi_+\right] = \frac{e^{\Phi}}{8} f_{dw} + e^{-A}\, e^{\mathcal{W}(r)}\,\mathrm{Re}\,\partial_r\left(e^{-\mathcal{W}(r)}\Psi^-\right) \quad \left(\xrightarrow{Mink.} 0\right)$$

$$e^{-2A+\Phi}(d+H\wedge)\left[e^{2A-\Phi}\Psi_-\right] = dA\wedge\bar{\Psi}_- + \frac{e^{\Phi}}{8} i\tilde{F} + e^{-A}\, e^{\mathcal{W}(r)}\,\mathrm{Re}\,\partial_r\left(e^{-\mathcal{W}(r)}\Psi^+\right) \quad \left(\xrightarrow{Mink.} 0\right)$$

- This correctly reproduces the Minkowski and AdS constraints.
- The ∂_r-term describes how Y can be understood as a manifold $X_7 = Y \times \mathbb{I}_r$ of $G_2 \times G_2$ structure - i.e. it describes a Hitchin flow (Witt; ... Louis & Vaula).

Domain walls in $\mathcal{N} = 1$ supergravity

- We have constructed the supersymmetry constraints on $\frac{1}{2}$-supersymmetric domain wall solutions of $\mathcal{N} = 1$ 4D supergravity arising from $SU(3) \times SU(3)$ compactifications.

- One can show that these are consistent with the corresponding field equations.

What makes this physically interesting?

- Such domain walls interpolate between different vacua in the string landscape. Our constraints (∂_r-terms) provide a starting point for studying the transitions between vacua in $SU(3) \times SU(3)$ compactifications.

- They are gravity duals of domain walls in $\mathcal{N} = 1$ super-Yang Mills theory, about which there remain open questions e.g. dynamical description.

Conclusions and future directions

- The interplay between $\mathcal{N} = 2$ compactifications of supergravity and the mathematics of Calabi-Yau manifolds can be extended to more 'phenomenologically' interesting cases.
- $\mathcal{N} = 1$ compactifications are described by $SU(3) \times SU(3)$ structure manifolds, harnessing the tools of generalised complex geometry. Spinor bilinears play an important role.
- We have extended the analysis of $SU(3) \times SU(3)$ structure compactifications to $\frac{1}{2}$-supersymmetric domain wall solutions of the resulting $\mathcal{N} = 1$ theory and constructed the corresponding supersymmetry constraints.
- Our goal is to use these constraints to understand the properties of supergravity and super-Yang Mills domain walls more thoroughly.