Sigma model for pions

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1 Introduction and Motivation

Suppose we have just built an *MeV-collider*: the first hadronic particles we would see are pseudo-scalar particles ($J^P = 0^-$, i.e. particles with total spin 0 and odd parity), of which some are electrically charged, while some others are electrically neutral, with masses roughly between 130-550 MeV. With increasing energy, we would then produce and observe a vector particle ($J^P = 1^-$) of mass 770 MeV, i.e. the ρ meson[1], and then the first spin 1/2 particles, i.e. the nucleons p and n ($J^P = 1/2^+$) with masses of 940 MeV. In the following table we summarize these states, also with their composition in terms of quarks:

	mass [MeV]	J^P	QCD
π^0	135.0	0^-	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$
$\pi^\pm$	139.6	0^-	$u\bar{d}$
$K^\pm$	493.7	0^-	$u\bar{s}$
K^0, $\bar{K}^0$	497.6	0^-	$d\bar{s}$
η	547.8	0^-	$\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}$
$\rho^\pm$	775.5	1^-	$u\bar{d}$
ρ^0	775.1	1^-	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$
p	938.3	$\frac{1}{2}^+$	uud
n	939.6	$\frac{1}{2}^+$	udd

The aim of this notes is to try to develop a low energy description for the pseudo-scalars, and the language which will be used is the language of Effective Field Theories (*EFT*): the first question is how we can build an action for these degrees of freedom, and the second question will be, what is the scale at which our theory breaks down.

Looking at the spectrum of the light mesons, one could notice that the three pions are nearly degenerate besides being light. We are familiar with Goldstone's theorem, even if we will derive a classical proof of the theorem in the next section, which states that for every broken internal symmetry generator there exists a massless boson. Our guess is thus that the three pions, in the limit in which we consider them as massless, are Goldstone's Bosons of a spontaneously broken internal symmetry. To achieve this, an internal symmetry group G should be broken to a subgroup H, where G should have at least three more generators than H. The degeneracy of the three pions suggests also that the pions form a degenerate triplet under the unbroken H. The same reasoning could be applied for the whole set of eight light mesons.

Looking at the rest of the spectrum, we see that at energies $\sim$1 GeV new types of particles arise, and they could not be identified as Goldstone's Bosons. But as long as we consider energies much smaller than $\sim$1 GeV, these resonances could not be exited and therefore produced on-shell. In other words, if we consider the nucleons as *UV* degrees of freedom, they are irrelevant in the *IR*, and we can construct a consistent theory of purely Goldstone's Bosons. The *UV* degrees of freedom enter in principle into the loops, but never on-shell: their action is therefore just a renormalization of the couplings. We expect therefore that our *EFT* of light mesons breaks down at the scale $\sim$1 GeV.

As already mentioned, our guess is that the light mesons are Goldstone's Bosons of a spontaneously broken internal symmetry. It is thus worth to start our notes with a review of the Classical proof of Goldstone's theorem.

[1]actually, we would produce a triplet of nearly degenerate vector particles, with electric charges 0 and ± 1 respectively

2 Classical proof of Goldstone's theorem

Consider a classical field theory with n scalar fields ϕ^A, $A = 1\dots n$, with a Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm kin} - V(\phi^A)$$

invariant under a Lie group G (internal symmetry). Let $\langle\phi^A\rangle$ be the minimum-energy configuration of the potential, and assume that there is a subgroup H of G under which the vacuum configuration is invariant, i.e. $h\langle\phi^A\rangle = \langle\phi^A\rangle\ \forall\ h \in H$. We will show now that there is a zero eigenvalue of the scalar mass matrix for each generator of the coset G/H.

Under a small G-transformation, the variation of the Lagrangian is zero:

$$0 = \delta\mathcal{L} = \frac{\delta V}{\delta\phi^A}\delta\phi^A \tag{1}$$

where $\delta\phi^A$ is defined as the infinitesimal variation of ϕ^A under G, i.e.

$$g:\ \phi^A \ \rightarrow\ \left[e^{i\alpha^a T^a}\right]^A{}_C\,\phi^C = \left[\mathbb{1} + i\alpha^a T^a + \dots\right]^A{}_C\,\phi^C \quad\Rightarrow\quad \delta\phi^A = \left[i\alpha^a T^a\right]^A{}_C\,\phi^C.$$

Let us differentiate Eq. (1) w.r.t. a different direction ϕ^B

$$0 = \frac{\delta}{\delta\phi^B}\left[\frac{\delta V}{\delta\phi^A}\delta\phi^A\right] = \frac{\delta^2 V}{\delta\phi^A\delta\phi^B}\delta\phi^A + \frac{\delta V}{\delta\phi^A}\delta^{AB}$$

and evaluating it at the vacuum configuration, we obtain

$$\begin{aligned} 0 &= \frac{\delta^2 V}{\delta\phi^A\delta\phi^B}_{\big|\langle\phi\rangle}\delta\langle\phi^A\rangle + 0 \\ \delta\langle\phi^A\rangle &= i\left(\alpha^a T^a\right)_{AC}\langle\phi^C\rangle \begin{cases} = 0, & \text{if } T^a \in H \\ \neq 0, & \text{if } T^a \in G/H \end{cases} \end{aligned} \tag{2}$$

where α^a are arbitrary parameters which define the transformation under G, and T^a $a = 1, \dots, \dim(G)$ are the generators of G. Therefore

$$0 = \frac{\delta^2 V}{\delta\phi^A\delta\phi^B}_{\big|\langle\phi\rangle}\delta\langle\phi^A\rangle = M^2_{AB}\left(\alpha^a T^a\right)^A{}_C\langle\phi^C\rangle$$

so that $T^a\langle\phi\rangle$ are $\dim(G)$ eigenvectors of M^2_{AB} with vanishing eigenvalue. However the multiplicity of the eigenvalue 0 is only $\dim(G)$-$\dim(H)$, since the generators of the subgroup H annihilate the vacuum, cfr. Eq. (2), and therefore cannot be eigenvectors of M^2_{AB}.

M^2_{AB} is nothing but the mass matrix for the scalars, and we conclude that the number of zero eigenvalues of M^2_{AB} is equal to the number of generators that do not annihilate the vacuum.

The flat directions of the potential define a manifold which we call *vacuum manifold* $\mathbb{V}$, made of physically equivalent vacua: the exitations along this flat directions are called *Goldstone bosons* (*GB*), and in a quantum interpretation we identify these exitations as massless scalar particles $\pi(x)$. This proof suggests thus a geometrical interpretation of the *GB*, which will guide us in the rest of the discussion: the *GB* are maps from the space-time to the vacuum manifold

$$\pi(x) : \mathbb{R}^4 \rightarrow \mathbb{V}.$$

3 *CCWZ* formalism

3.1 A (simple) example

Consider the $U(1)$-invariant theory of a scalar field ϕ, with *SSB* of the $U(1)$ symmetry:

$$\begin{aligned}\mathcal{L} &= |\partial_\mu \phi|^2 - V(|\phi|) \\ V(|\phi|) &= \frac{\lambda}{4}\left[|\phi|^2 - \frac{\mu^2}{\lambda}\right]^2\end{aligned}$$

$\mathcal{L}$ is manifestly invariant under the $U(1)$ transformation $\phi \to e^{i\alpha}\phi$.

The configuration $|\phi|^2 = \mu^2/\lambda = v^2$ minimizes the potential, and the vacuum manifold is the 1-dimensional sphere $\mathcal{S}^1$ with radius $|v|$ in the Re[ϕ]-Im[ϕ] plane. The $U(1)$-breaking will generate a *GB*: now we ask ourself how to parametrize the two degrees of freedom of the complex scalar ϕ.

Remembering that $U(1)$ is isomorphic to the group of 2-dimensional rotations $O(2)$, a naïve attempt would be to parametrize ϕ as a two-components vector

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

transforming *linearly* under the global symmetry $U(1)$:

$$U(1) \sim O(2) : \phi \to \phi' = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

One then usually expands around one of the degenerate vacuum expectation values v of the vacuum manifold $\mathcal{S}^1$, e.g.

$$\tilde{\phi}(x) = \phi(x) - \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

and by substituting back $\tilde{\phi}$ into the Lagrangian, we would obtain a mass term only for $\phi_2(x)$, plus some polynomial interactions between $\phi_1(x)$ and $\phi_2(x)$. However, this parametrization does not capture the geometric interpretation of the *GB* we had before, since the massless $\phi_1(x)$ *is not* an exitation along the vacuum manifold $\mathcal{S}^1$. In principle one could recover this geometric interpretation by integrating out the massive $\phi_2(x)$ field.

Is there a clever (and simpler) way to capture our geometrical interpretation of the *GB*? The answer is yes, with the following *non-linear* exponential representation

$$\phi(x) = \frac{1}{\sqrt{2}}\left[\sigma(x) + v\right] e^{i\pi(x)/v}$$

where now $\sigma(x)$ and $\pi(x)$ are radial and angular exitations respectively, and v is defined as before. The physics should be independent of the chosen parametrization of the fields of the theory: field redefinitions cannot modify S-matrix elements[2] (*Equivalence Theorem of the S-matrix*). Now the *GB*, which will be indeed the $\pi(x)$ field, is manifestly a map on the vacuum manifold:

$$\pi(x) : \mathbb{R}^4 \to \mathcal{S}^1.$$

[2]if two fields are related nonlinearly, e.g. $\phi = \chi F(\chi)$ with $F(0) = 1$, then the same experimental observables result if one calculates with the field ϕ using $\mathcal{L}(\phi)$ or instead with χ using $\mathcal{L}(\chi F(\chi))$. Indeed, since $F(0) = 1$, ϕ and χ have the same free field behavior and single particle singularities, and therefore the S-matrices are equivalent.

Substituting back into the Lagrangian, we obtain the following terms:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \sigma(x)\right]^2 + \frac{1}{2}\left[\partial_\mu \pi(x)\right]^2 + v\,\sigma(x)\left(\frac{\partial_\mu \pi(x)}{v}\right)^2 + \frac{1}{2}\sigma(x)^2\left(\frac{\partial_\mu \pi(x)}{v}\right)^2 - V\left(\sigma(x)\right). \qquad (3)$$

From (3), we can extrapolate some general properties of the *GB*:

- there is no mass terms for the pion
- the pion is only derivatively coupled: therefore, if we restrict ourself to low momentum regime, we have a weakly coupled theory for the pion
- the Lagrangian is not manifestly $U(1)$-invariant. However, one could notice that the $U(1)$ transformation on ϕ acts just a *shift*-transformation on the pion

$$e^{i\alpha}\phi = \frac{1}{\sqrt{2}}\left(\sigma + v\right)e^{i(\pi+\alpha\cdot v)/v} \quad \Rightarrow \quad U(1):\ \pi \to \pi + \alpha\cdot v$$

  and indeed (3) is *shift-invariant*. Thus the $U(1)$ is said to be *hidden* and realized *non-linearly* as a shift-symmetry on the pion.
- if one integrates out the massive radial exitation $\sigma(x)$, the resulting low-energy Lagrangian would be expressed as a series of terms with increasing number of derivatives of the pion

$$\sum_{n=1} \frac{c_n\left(\pi\right)}{\Lambda^{2n-4}}\left[\partial_\mu \pi(x)\right]^{2n}.$$

  This will be our general prescription in constructing an *EFT* for the light mesons.

In general, non-linear relizations of symmetry group are the most effective way to represent a symmetry that has been spontaneously broken. The technology to do this was first worked out by Callan, Coleman, Wess and Zumino, and therefore is named after them as the *CCWZ* prescription.

3.2 The most general *GB*-Lagrangian

The *CCWZ* prescription is a generic way to parametrize the *GB* π^a arising from a G/H symmetry breaking pattern. Let $\phi(x)$ be a set of scalar fields which transforms linearly under the global symmetry G:

$$g : \phi \to g\,\phi.$$

If T^a are the generators of H, and X^a are the generators of the coset G/H, the *CCWZ* prescription is to parametrize $\phi(x)$ as

$$\phi(x) = \xi(x)\cdot\langle\phi\rangle = e^{i\pi^a(x)\cdot X^a/f}\,\langle\phi\rangle \qquad (4)$$

where π^a are the *GB* fields, and $\langle\phi\rangle$ is the vacuum expectation value which realizes the breaking $G \to H$. Notice that the definition (4) is independent from the particular representation of ϕ under G.

Naïvely one would say that even $\xi(x)$ transforms linearly as $\phi(x)$ under the action of $g \in G$, but this is not true in general: under a global symmetry transformation g, the matrix $\xi(x)$ is transformed to the new matrix $g\,\xi(x)$, but this new matrix is in general no longer in the standard form

$$g : \xi(x) \to g\,\xi(x) \neq e^{i\pi'(x)\cdot X^a/f}.$$

In order to have a well-defined linear transformation law for $\phi(x)$, one can use the fact that the vacuum $\langle\phi\rangle$ is invariant under H transformations,

$$h\,\langle\phi\rangle = \langle\phi\rangle \quad \forall\, h \in H$$

and find an h transformation U_H such that $g\,\xi(x)\,U_H^\dagger(g,\pi)$ is in the standard form:

$$g\,\phi(x) = g\,\xi(x)\cdot\langle\phi\rangle = g\,\xi(x)\,U_H^\dagger(g,\pi)\,U_H(g,\pi)\,\langle\phi\rangle = g\,\xi(x)\,U_H^\dagger(g,\pi)\,\langle\phi\rangle = \xi'(x)\cdot\langle\phi\rangle.$$

$U_H^\dagger(g,\pi)$ depends on g, but also on the *GB* fields: therefore under a transformation $g \in G$ the *GB* transform *non-linearly* as

$$g : \xi(x) \to g\xi(x)U_H^\dagger(g,\pi). \tag{5}$$

On the other hand, under a transformation of the unbroken group H, $\xi(x)$ transforms linearly

$$h : \xi(x) \to h\xi(x)h^{-1}.$$

Let's do an example. Consider a theory of a single scalar field[3] ϕ where the symmetry breaking pattern $SU(N) \to SU(N-1)$ is realized: following the *CCWZ* prescription, we parametrize ϕ as

$$\phi = \xi\cdot\langle\phi\rangle = e^{i\pi^a X^a/f}\langle\phi\rangle = \exp\left\{\frac{i}{f}\left(\begin{array}{ccc|c} \pi^0 & & & \pi_1 \\ & \ddots & & \vdots \\ & & \pi^0 & \pi_{N-1} \\ \hline \pi_1^* & \dots & \pi_{N-1}^* & -(N-1)\pi_0 \end{array}\right)\right\}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where the field π_0 is real whereas the fields $\bar\pi = (\pi_1,\ldots,\pi_{N-1})$ are complex, representing the $2N-1$ *GB* of the theory. We now show how the *GB* transform under the broken and unbroken symmetries, neglecting for simplicity the real *GB* π^0.

Let's consider first the unbroken $SU(N-1)$ transformations: ϕ transforms linearly under the whole group G, and thus we have

$$\phi \to U_{N-1}\,\phi = \left(U_{N-1}\,e^{i\pi^a X^a/f}\,U_{N-1}^\dagger\right)U_{N-1}\,\phi_0 = e^{i/f\left(U_{N-1}\,\pi^a X^a\,U_{N-1}^\dagger\right)}\langle\phi\rangle$$

where in the second equality we used the fact that the vacuum $\langle\phi\rangle$ is invariant under the unbroken U_{N-1} transformations. Thus the *GB* transform linearly under $SU(N-1)$

$$\pi^a X^a \to U_{N-1}\,\pi^a X^a\,U_{N-1}^\dagger.$$

Explicitly, a generic $SU(N-1) \subset SU(N)$ transformation can be written as

$$U_{N-1} = \left(\begin{array}{c|c} \hat U_{N-1} & 0 \\ \hline 0 & 1 \end{array}\right)$$

and we can see that the $N-1$ complex *GB* transform in the fundamental representation of $SU(N-1)$:

$$\left(\begin{array}{c|c} 0 & \bar\pi \\ \hline \bar\pi^\dagger & 0 \end{array}\right) \to U_{N-1}\left(\begin{array}{c|c} 0 & \bar\pi \\ \hline \bar\pi^\dagger & 0 \end{array}\right)U_{N-1}^\dagger = \left(\begin{array}{c|c} 0 & \hat U_{N-1}\bar\pi \\ \hline \bar\pi^\dagger\hat U_{N-1}^\dagger & 0 \end{array}\right).$$

[3]we drop the space-time dependence x for simplicity

Under a symmetry transformation of the coset G/H, remembering that ϕ transforms linearly under G, and that $\langle\phi\rangle$ is H-invariant, we have

$$\begin{aligned} U_{G/H}\, e^{i\pi}\langle\phi\rangle &= \exp\left\{ i \begin{pmatrix} 0 & \bar{\alpha} \\ \bar{\alpha}^\dagger & 0 \end{pmatrix} \right\} \exp\left\{ \frac{i}{f} \begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^\dagger & 0 \end{pmatrix} \right\} \langle\phi\rangle \\ &= \exp\left\{ i \begin{pmatrix} 0 & \bar{\alpha} \\ \bar{\alpha}^\dagger & 0 \end{pmatrix} \right\} \exp\left\{ \frac{i}{f} \begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^\dagger & 0 \end{pmatrix} \right\} U_H^\dagger(\alpha,\pi)\langle\phi\rangle = \\ &= \exp\left\{ \frac{i}{f} \begin{pmatrix} 0 & \bar{\pi}' \\ \bar{\pi}'^\dagger & 0 \end{pmatrix} \right\} \langle\phi\rangle \end{aligned} \tag{6}$$

defining a non-linear transformation law for the *GB*, as already discussed in the general case. One can notice that to linear order in α the transformation (6) reduces to a *shift* transformation:

$$\bar{\pi} \to \bar{\pi}' = \bar{\pi} + \bar{\alpha} \cdot f + \mathcal{O}(\alpha^2).$$

The prescription to construct the most general *EFT* for only *GB*s degrees of freedom (with all other heavy fields integrated out), is now to write down all Lorentz- and G-invariant terms with increasing number of derivatives of the *GB* matrix, just as for the $U(1)$ case. However for general G and H, this is not trivial. Let's consider first the two-derivatives term. Naïvely one would write a two-deriatives term using the field ξ in the parametrization (4), i.e.

$$f^2\, \mathrm{tr}\left|\partial_\mu \xi\right|^2,$$

but in general this is not invariant under G

$$f^2\, \mathrm{tr}\left|\partial_\mu \xi\right|^2 \to f^2\, \mathrm{tr}\left|\partial_\mu \left(\xi(x) U^\dagger(x)\right)\right|^2$$

because of the dependence of x in $U(g,\pi) \in H$. Using a little bit of algebra one obtains

$$\mathrm{tr}|\partial_\mu\xi|^2 = \mathrm{tr}\left[\left(\partial_\mu \xi^\dagger\right)\xi\xi^\dagger\left(\partial^\mu\xi\right)\right] = \mathrm{tr}\left[\left(\xi^\dagger\partial_\mu\xi\right)^\dagger\left(\xi^\dagger\partial^\mu\xi\right)\right]$$

and it can be shown that the object $\xi^\dagger\partial_\mu\xi$ decomposes as

$$\xi^\dagger\partial_\mu\xi = v_\mu^a T^a + p_\mu^a X^a \tag{7}$$

with the objects $v_\mu = v_\mu^a T^a$ and $p_\mu^a X^a$ transforming as

$$\begin{aligned} v_\mu &\to U(v_\mu + \partial_\mu)U^\dagger \\ p_\mu &\to U p_\mu U^\dagger. \end{aligned}$$

The field v_μ transforms like a connection, while p_μ is suitable to construct a G-invariant two-derivatives term: the only non-trivial term is given by

$$\mathcal{L}_2 = f^2 \mathrm{tr}\left[p^\mu p_\mu^\dagger\right]. \tag{8}$$

However the form of p_μ and v_μ depends heavily on the specific groups G and H.

Everything simplifies if the Lie algebra G/H is a *symmetric space*. By definition, a symmetric space has an involutive automorphism on the generators

$$T^a \to T^a, \qquad X^a \to -X^a,$$

and applying the automorphism to Eq. (7), we find that the object p_μ is simply given by

$$p_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger\right).$$

We can thus rewrite the two-derivative term of Eq. (8) as

$$\mathcal{L}_2 = \frac{f^2}{4} \mathrm{tr}\left|\partial_\mu \Sigma\right|^2 \tag{9}$$

which indeed contains the *GB* kinetic term canonically normalized, where we have defined

$$\Sigma = \xi\tilde{\xi}^\dagger = \xi^2 = e^{2i\pi^a X^a/f},$$

with $\tilde{\xi}$ the image of ξ under the automorphism. From Eq. (5), we see that Σ transforms as

$$\Sigma \to g\Sigma\tilde{g}^\dagger, \tag{10}$$

where $\tilde{g}$ is the image of g under the automorphism. Therefore, in symmetric spaces we can construct a Goldstone matrix Σ that is an element of G/H but transforms linearly under G.

We can then summarize the *CCWZ* formalism with a prescription for constructing the most general *EFT* of only *GB*s degrees of freedom:

- identify the groups G and H describing the spontaneous symmetry breaking pattern
- construct the *GB* matrix $\xi(x)$ and consequently the quantities p_μ, v_μ or $\Sigma(x)$ depending on whether the coset G/H is a symmetric group or not
- write all Lorentz and G-invariant terms with p_μ, v_μ (or $\partial_\mu\Sigma$) as building blocks, with increasing number of derivatives
- identify the finite *cut-off* up to which the theory is valid

4 Chiral Perturbation Theory

4.1 Approximate Chiral Symmetry of *QCD*

Now that we have developed the formalism to construct an *EFT* of only *GB*s, let us return to the original question for a low energy theory of light mesons. In the Introduction, we already discussed how the observed spectrum of the light mesons, in the limit in which their masses are degenerate and equal to zero, points toward considering these light degrees of freedom as *GB* of a spontaneous symmetry breaking pattern.

This is probably the only case in which we know the underlying theory behind the effective low-energy theory we are constructing: for this reason we will study carefully the low-energy *QCD* limit, in order to identify a possible global symmetry breaking which could give rise to the eight light mesons as *GB*.

Let us consider the *QCD* Lagrangian for the u, d and s quarks only. Defining the flavour vector $q_i = (u, d, s)$, we can write the three flavour $SU(3)_C$-invariant Lagrangian as

$$\mathcal{L}_{\text{QCD, 3 fl.}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{q}_i \left(i\not{D} - m_i\right) q_i$$

where each quark transforms as a triplet under $SU(3)_C$ and with $G^a_{\mu\nu}$ the field strength tensor for the gluon fields. Let us focus on the quark sector of the Lagrangian, expanding the flavour vector in the chiral basis:

$$\mathcal{L}_{\text{QCD, 3 fl.}} \supset \bar{q}^i_L \, i\not{D} q^i_L + \bar{q}^i_R \, i\not{D} q^i_R + m_i(\bar{q}^i_L q^i_R + \bar{q}^i_R q^i_L).$$

It is clear that the mass terms for the quarks mix left- and right-chiralities, but if we consider the so called *chiral-limit* where the three quark masses are set to zero, a large chiral $U(3)_L \otimes U(3)_R$ symmetry is restored. The independent $U(3)$ rotations of the left- and right-chiral flavour vectors leave indeed the Lagrangian invariant:

$$\begin{aligned} {q_L}^i &\rightarrow {U_L}^i{}_j\, {q_L}^j \\ {q_R}^i &\rightarrow {U_R}^i{}_j\, {q_R}^j \end{aligned} \tag{11}$$

At energies much smaller than $\Lambda_{QCD} \sim 1$ GeV, setting the masses of the three quarks to zero is indeed a good approximation

$$\begin{aligned} m_u &\sim \text{2-4 MeV} \\ m_d &\sim \text{4-8 MeV} \\ m_s &\sim \text{80-130 MeV} \end{aligned}$$

and therefore they are only a *small* breaking of the $U(3)_L \otimes U(3)_R$ symmetry, which is therefore an *approximate* symmetry of the three-flavours QCD Lagrangian. It's to be noted that the mass of the next-to-light c quark causes on the other hand a huge breaking of a possible $U(4)_L \otimes U(4)_R$ symmetry:

$$m_c \sim 1.3 \text{ GeV}.$$

Actually, a $U(3)$ group could be decomposed into the direct product of $SU(3) \otimes U(1)$, and therefore the (approximate) chiral symmetry of the three flavours Lagrangian reads

$$[SU(3)_L \otimes U(1)_L] \otimes [SU(3)_R \otimes U(1)_R]$$

and the transformation laws can be expressed as

$$\begin{aligned} {q_L}^i &\rightarrow \left[e^{i\alpha_L}\, e^{i\alpha^a_L \lambda^a}\right]^i_j {q_L}^j \\ {q_R}^i &\rightarrow \left[e^{i\alpha_R}\, e^{i\alpha^a_R \lambda^a}\right]^i_j {q_R}^j \end{aligned}$$

where λ^a are $SU(3)$ generators, and $\alpha_{L/R}$, $\alpha^a_{L/R}$ are parameters of the respective transformations. An equivalent formulation of the chiral symmetry is in terms of vector and axial transformations of the flavour vectors, described by the following transformation laws respectively

$$\begin{aligned} q^i &\rightarrow \left[e^{i\alpha}\, e^{i\alpha^a \lambda^a}\right]^i_j q^j \\ q^i &\rightarrow \left[e^{i\alpha\gamma_5}\, e^{i\alpha^a \lambda^a \gamma_5}\right]^i_j q^j, \end{aligned}$$

and the conserved currents associated to these symmetries are indeed related by the following relations

$$\begin{aligned} J^a_{\mu\,V} &= \bar{q}\gamma_\mu \lambda^a q = J^a_{\mu\,L} + J^a_{\mu\,R} && (SU(3)_V \text{ current}) \\ J^a_{\mu\,A} &= \bar{q}\gamma_\mu \gamma_5 \lambda^a q = J^a_{\mu\,L} - J^a_{\mu\,R} && (SU(3)_A \text{ current}) \\ J_{\mu\,V} &= \bar{q}\gamma_\mu q = J_{\mu\,L} + J_{\mu\,R} && (U(1)_V \text{ current}) \\ J_{\mu\,A} &= \bar{q}\gamma_\mu \gamma_5 q = J_{\mu\,L} - J_{\mu\,R} && (U(1)_A \text{ current}). \end{aligned}$$

The (approximate) chiral symmetry of the three flavours Lagrangian could then be expressed as

$$[SU(3)_V \otimes SU(3)_A] \otimes [U(1)_V \otimes U(1)_A]\,. \tag{12}$$

The global $U(1)_A$ symmetry is however not a good symmetry at quantum level (it is *anomalous*) and therefore in the following we will not consider it anymore. Neglecting also the $U(1)_V$ symmetry (Baryon number conservation) since it will have no influence on the physics that we are discussing, we will focus only on the $SU(3)_V \otimes SU(3)_A$ symmetry of the Lagrangian.

We have however a tremendous amount of phenomenological and theoretical evidence (e.g. from lattice QCD) that the $SU(3)_A$ axial symmetry is spontaneously broken. The origin of this spontaneous symmetry breaking pattern could be found in the dynamics of QCD: at low energies QCD is a strongly coupled theory, and one assumes the phenomenon of *confinement* which allows the creation of bound states of quarks. Heuristically, if $q, \bar{q}$ have small masses ($m_q \ll \Lambda_{\text{QCD}}$), then it doesn't cost too much energy to create $q\bar{q}$ pairs, usually called *quark condensate*: the vacuum expectation value of a quark condensate is therefore not vanishing

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle \neq 0.$$

The vacuum is thus clearly not invariant under simultaneous left- and right- transformations of the quark fields (11), but in the case in which the left- and right- components transform in the same way. In other words, the non-vanishing vacuum expectation value of the quark condensate forces the global chiral symmetry to be spontaneously broken down to the diagonal $SU(3)_D$ subgroup

$$SU(3)_L \otimes SU(3)_R \to SU(3)_D.$$

under which the left- and right- chiralities transform equally, i.e. with $\alpha_L = \alpha_R$. Equivalently, the quark condensate is invariant only under the vectorial $SU(3)_V$ global symmetry, spontaneously breaking the axial $SU(3)_A$ part:

$$SU(3)_V \otimes SU(3)_A \to SU(3)_V.$$

This spontaneously breaking of the axial symmetry is called *dynamical* symmetry breaking, because of the (assumed) dynamical origin of the quark condensate.

We expect thus eight new GB associated to the symmetry breaking, and we identify them with the eight light meson fields. Of course they will be *pseudo*-Goldstone bosons because the axial symmetry was only *approximately* realized. $SU(3)_L \otimes SU(3)_R \to SU(3)_D$ is therefore the symmetry breaking pattern that we want to describe using the $CCWZ$ formalism.

4.2 Constructing the Chiral Lagrangian

We want now to describe the dynamics of GB arising from the symmetry breaking pattern $G \to H = SU(3)_L \otimes SU(3)_R \to SU(3)_D$. Notice that the G/H coset is a symmetric space: the automorphism defining the symmetric space just interchanges the left- and right- generators. Following the $CCWZ$ prescriptions, we introduce a scalar field

$$\phi(x) = \xi(x) \cdot \langle\phi\rangle = e^{i\pi^a X^a/f} \cdot \mathbb{1}_{3\times 3}$$

where X^a are $SU(3)$ generators, f is a mass dimension 1 parameter, and $\mathbb{1}_{3\times 3}$ is the three-dimensional identity matrix. The explicit representation of the pion matrix is the following

$$\pi^a X^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^3 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^3 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \tag{13}$$

where we have already identified the different GB combinations with the eight light meson fields.

From the previous discussion we identify ϕ with the quark condensate, assuming ϕ to have the same quantum numbers under $SU(3)_L \otimes SU(3)_R$:

$$\langle q\bar{q}\rangle \sim (3,\bar{3}) \quad \Rightarrow \quad \phi \sim (3,\bar{3}).$$

Remembering that under the automorphism we have $\tilde{g}_L = g_R$, and using Eq. (10), we obtain that the field Σ has the same quantum numbers as ϕ:

$$\Sigma = \xi\tilde{\xi}^{\dagger} = e^{2i\pi^a X^a/f}, \quad \Sigma \to L\Sigma R^{\dagger}. \tag{14}$$

The first non-trivial derivative term is the one with two partial derivatives, see Eq. (9)

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}|\partial_\mu \Sigma|^2 \tag{15}$$

where the prefactor $f^2/4$ assures the correct normalization for the pion kinetic term

$$\frac{1}{2}\partial_\mu \pi^a\, \partial^\mu \pi^a.$$

By expanding until third order the Σ field, one obtains the following interaction terms for the GB fields:

$$\mathcal{L}_2 \supset \frac{1}{2}\partial_\mu \pi^a\, \partial^\mu \pi^a + \frac{1}{24 f^2}\left[(\bar{\pi}\cdot\partial_\mu\bar{\pi})^2 - \bar{\pi}^2\,(\partial_\mu\bar{\pi})^2\right] \tag{16}$$

where the first term is the GB kinetic term, and the second term describes a four-GB interaction at order p^2/f^2.

Now we want to understand the physical meaning of the dimensionfull parameter f. There are two ways to identify it: the first is to calculate the expectation value of the axial current $J^a_{\mu,A}$ between an initial pion state and the vacuum as final state. Indeed we know that it is proportional to the pion decay constant f_π

$$\langle 0|J^a_{\mu,A}|\pi^b\rangle = i\delta^{ab} p_\mu f_\pi$$

and by explicitly calculating the matrix element, we would be able to identify the f parameter with the pion decay constant f_π.

The second way to identify the value of f is by explicitly calculating the decay width of the leptonic pion decay $\pi^+ \to \mu^+\nu_\mu$ through the W^+ gauge boson. To do so we have to introduce a local gauge invariance under the SM-like $SU(2)_L \otimes U(1)_Y$ group: under the following identifications

$$\begin{aligned} SU(2)_L &\subset SU(3)_L \\ U(1)_Y &= T_{3R} + \left(\frac{B-L}{2}\right) \end{aligned}$$

we can promote the partial derivatives to covariant derivatives

$$\partial_\mu \Sigma \to D_\mu \Sigma = \left(\partial_\mu - igW^a_\mu T^a_{SU(2)_L} + ig' T_{3R} B_\mu\right)\Sigma$$

and rewrite the two-derivative term as

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}|D_\mu \Sigma|^2. \tag{17}$$

By expanding the Σ fields at linear order in the pion fields, one obtains the following terms:

$$\mathcal{L}_2 \supset \partial_\mu \pi^+\, \partial^\mu \pi^- - gf\left[\left(\partial_\mu \pi^-\right) W^+_\mu + \left(\partial_\mu \pi^+\right) W^-_\mu\right].$$

Adding to $\mathcal{L}_2$ the usual interaction term of the Ws with lepton fields, as well as a mass term for the W

$$m_W^2 W_\mu^+ W^{\mu -} - \frac{g}{\sqrt{2}} \left[W_\mu^+ J^{\mu -} + W_\mu^- J^{\mu +} \right]$$

and integrating out the gauge boson W, we finally obtain the low energy effective Lagrangian describing the $\pi^+ - \mu^+ \nu_\mu$ interaction:

$$\mathcal{L}_{\text{eff}} \supset -4 G_F f \left[\left(\partial_\mu \pi^- \right) J_\mu^+ + \left(\partial_\mu \pi^+ \right) J_\mu^- \right]$$

with $G_F/\sqrt{2} = g^2/8m_W^2$. The decay rate is found to be[4]

$$\Gamma_{\pi^+ \to \mu^+ \nu_\mu} = \frac{G_F^2}{4\pi} f^2 m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2} \right)^2$$

which allows us to identify again the f parameter with the pion decay constant f_π, which measured value is

$$f \sim 93 \text{ GeV}.$$

It's to be noted that even if the decay into the first generation of leptons $\pi^+ \to e^+ \nu_e$ has a larger accessible phase space, the decay rate is highly suppressed with respect to the decay into the second generation of leptons

$$\frac{\Gamma_{\pi^+ \to e^+ \nu_e}}{\Gamma_{\pi^+ \to \mu^+ \nu_\mu}} \sim 1.23 \cdot 10^{-4}.$$

This is the well-known *helicity-suppression* phenomenon.

Another important side remark arising from this discussion, is whether the chiral symmetry breaking of *QCD* $SU(3)_L \otimes SU(3)_R \to SU(3)_D$ could trigger also the electroweak symmetry breaking *EWSB*, since $SU(2)_L \subset SU(3)_L$. The answer is yes, but the contribution to the *EWSB* is too small to reproduce the observed phenomenology. In particular, from the kinetic term (17) one obtains the following mass terms for the gauge bosons

$$\mathcal{L}_2 \supset \frac{g^2 f^2}{4} W_\mu^+ W^{\mu -} + \frac{g^2 + g'^2}{8} f^2 Z_\mu Z^\mu$$

which are the same terms as in the *SM* but with $v \to f$, giving a mass to the gauge bosons which is $\sim$1000 times smaller than the observed one. The conclusion is that *QCD* dynamically breaks the electroweak symmetry, also preserving the tree-level custodial-symmetry relation $\rho = 1$, but the contribution is too small to assume that it is the only source of *EWSB*.

So far we have assumed a perfect $SU(3)_L \otimes SU(3)_R$ symmetry, with the 8 *GB* as massless. As discussed before, the quark mass term of *QCD* is an explicit breaking of the chiral symmetry,

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{18}$$

In order to incorporate the consequences of Eq. (18) into the effective Lagrangian framework, one makes use of a *spurion* analysis. Although M is in reality just a constant matrix and does not transform along with the quark fields, $\mathcal{L}_M$ *would be* invariant *if* M is transformed as

$$M \to R M L^\dagger. \tag{19}$$

[4]we restored the pion mass in the kinematic calculations

One then constructs the most general Lagrangian $\mathcal{L}(U, M)$ which is invariant under (14) and (19) and expand this function in powers of M. At lowest order in M one obtains

$$\mathcal{L}_{\text{s.b.}} = \frac{f^2\mu}{2}\text{tr}\left(M\Sigma^\dagger + \Sigma M^\dagger\right)$$

where the subscript *s.b.* indicates that this term is an explicit breaking of the original $SU(3)_L \otimes SU(3)_R$ symmetry, and μ is a mass-dimension 1 parameter. In order to determine the masses of the Goldstone bosons, we identify the terms of second order in the fields in $\mathcal{L}_{\text{s.b.}}$,

$$\mathcal{L}_{\text{s.b.}} \supset -\frac{\mu}{2}\text{tr}\left(\pi^2 M\right)$$

and using the explicit expression for the pion matrix (13), we find

$$\begin{aligned} \text{tr}\left(\pi^2 M\right) &= 2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K^0\bar{K}^0 + \\ &+ (m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{m_u + m_d + 4m_s}{3}\eta\eta. \end{aligned} \tag{20}$$

For simplicity we can take the limit $m_u = m_d = m$ so that there is no $\pi^0 - \eta$ mixing, and the different masses satisfy the so called *Gell-Mann Okubo relation*

$$4m_K^2 = 3m_\eta^2 + m_\pi^2$$

independent of the value of μ, and consistent with the experimental observation.

One has to say also that the $SU(2)_L \otimes U(1)_Y$ gauging is another source of chiral symmetry breaking: loop diagrams involving a photon and the charged meson propagtor are responsible to another (small) contribution to the charged meson masses, which is indeed consistent with the observed phenomenology.

4.3 Higher Order Effects and Naïve Dimensional Analysis

Consider now higher order effects in p/f that enter into a calculation: they can come from loop diagrams with multiple insertions of 2-derivative operators, or from 4-derivative (and higher) operators that we ignored so far. What are the coefficients of these higher dimensional operators? Just by dimensional analysis, an operator of dimension d must have a coefficient that goes like

$$c\frac{f^2}{\Lambda_\chi^{d-2}},$$

where c is a dimensionless number, and the f^2 factor is there to match the normalization we have used defining Σ. Λ_χ is the *chiral symmetry breaking scale* and represents the energy where the expansion breaks down.

We are thus effectively doing an expansion in E/Λ_χ: as long as we reduce ourself to energies much smaller than Λ_χ, higher derivative operators are irrelevant in the low-energy behaviour, and the main features of *EFT* could be captured by only the first terms of the derivative expansion. Naïvely we would thus expect Λ_χ to be at order f.

The dimensionless coefficient c can in principle be computed in *QCD*, but in practice is either measured or computed on a lattice. Since there are no more large or small numbers to play with, it must be that these coefficiens are $\mathcal{O}(1)$. If we were to measure them to be substantially different from unity, it would imply that we are missing some physics. This is called the *naturalness* argument.

It turns out that the most general $SU(3)$ chiral Lagrangian involving four derivatives is

$$\begin{aligned}\mathcal{L}_4 &= c_1 \cdot \left[\mathrm{tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger\right)\right]^2 + c_2 \cdot \mathrm{tr}\left(\partial_\mu \Sigma\, \partial_\nu \Sigma^\dagger\right) \cdot \mathrm{tr}\left(\partial^\mu \Sigma\, \partial^\nu \Sigma^\dagger\right) + \\ &+ c_3 \cdot \mathrm{tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger\, \partial_\nu \Sigma\, \partial^\nu \Sigma^\dagger\right). \end{aligned} \tag{21}$$

Let us now consider the $\pi\pi \to \pi\pi$ scattering for the Goldstone bosons coming from the interaction vertices in $\mathcal{L}_2$, Eq. (16), and in $\mathcal{L}_4$, Eq. (21):

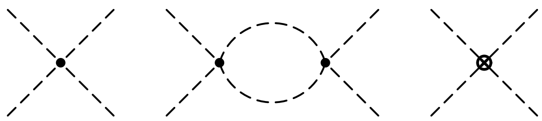

Figure 1: $\pi\pi \to \pi\pi$ scattering: the solid dot and circlecross vertices come from $\mathcal{L}_2$ and $\mathcal{L}_4$ terms, respectively. Not shown are the crossing diagrams of B.

Without considering all the details, we can compute

$$\begin{aligned}\mathcal{M}_A &\sim \frac{p^2}{f^2} \\ \mathcal{M}_B &\sim \int \frac{d^4 q}{(2\pi)^4} \frac{(p+q)^2/f^2 \cdot (p-q)^2/f^2}{(q^2)^2} \sim \frac{1}{16\pi^2}\left[\left(\frac{\Lambda}{f}\right)^4 + \frac{p^2}{f^2}\left(\frac{\Lambda}{f}\right)^2 + \frac{p^4}{f^4}\log\frac{\Lambda}{\mu}\right] \\ \mathcal{M}_C &\sim c\frac{p^4}{f^2 \Lambda_\chi^2}\end{aligned}$$

where Λ is a momentum cut-off not to be confused with the chiral symmetry breaking scale Λ_χ, and μ is a renormalization factor necessary for dimensional analysis. Summing up these contributions we find

$$\mathcal{M}(\pi\pi \to \pi\pi) \sim \frac{\Lambda^4}{16\pi^2 f^4} + \frac{p^2}{f^2}\left[1 + \frac{1}{16\pi^2}\left(\frac{\Lambda}{f}\right)^2\right] + \frac{p^4}{f^4}\left[\frac{1}{16\pi^2}\log\left(\frac{\Lambda}{\mu}\right) + c\frac{f^2}{\Lambda_\chi^2}\right].$$

The first term is a renormalization of the cosmological constant (there are no counterterms with zero powers of derivatives). The second term is a pure renormalization of f: the loop diagram using order $\mathcal{O}(p^2/f^2)$ vertices contributes therefore like an order $\mathcal{O}(p^4/f^4)$ tree-level diagram. This is a general behaviour, and therefore if one decides to consider corrections only to a certain order $\mathcal{O}(p^n/f^n)$, then loops with only a finite number of vertices are necessary.

Recalling that the final answer cannot depend on μ, we understand that the c coefficients of $\mathcal{L}_4$ should also have a μ dependence. Requiring that the one-loop contribution is at maximum at the same order of the tree-level contribution, we can finally have an estimation of the chiral symmetry breaking scale:

$$\Lambda_\chi \sim 4\pi f.$$

This reasoning is called *naïve dimensional analysis* (*NDA*). For our chiral-Lagrangian we have $f \sim 93$ MeV, so we predict $\Lambda_\chi \sim 1$ GeV: the *EFT* should then be trusted only for energies much below this scale. And this is indeed the case, as we have seen with the previous low-energy predictions, and recalling that it does not describe a particle, the ρ meson, with a mass at 770 MeV. At Λ_χ, it is thus no longer correct to identify the mesons as the physical degrees of freedom: chiral symmetry is restored and one has to work with a theory of quarks, which fortunately becomes (nearly) perturbative at that scale.

5 Conclusions

In this notes we have seen how to build a low energy theory for the light mesons. The main points that we have learned are the following:

- the GB are derivatively coupled, i.e. have no potential, and therefore are massless
- *non-linear realizations* of symmetries allow to construct EFTs which capture the geometrical properties of GBs
- these results are completely general once we know the SSB pattern
- the resulting EFT has a finite range of validity
- new categories of composite models incorporating this formalism are able to propose an appealing solution to the fine-tuning problem of the SM: if the Higgs is a (pseudo)-Goldstone Boson of an enlarged global symmetry, a radiatively unstable Higgs mass is prevented by the shift symmetry acting on the GB!

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