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Conformal Anomaly

Content:

0. Introduction

Last week we have seen that theories with only dimensionless coupling constants are scale invariant on the classical level. Today we will show that this invariance is broken on the quantum level which leads to new effects.

Consider for example QCD in the chiral limit. It contains no mass parameters and is therefore classically scale invariant. Without the anomaly the spectrum would consist of only massless states. It is due to the anomaly which introduces a mass parameter that QCD has a rich spectrum with massive states.

I Recap of last week

Scale transformations $\alpha : x \to e^{\alpha} x$ act on fields of dimension d according to

$$\alpha : \phi(x) \to e^{\alpha d} \phi(e^{\alpha} x)$$

and for infinitessimal transformations we obtain

$$\delta\phi = (d + x^{\mu}\partial_{\mu})\phi \tag{1}$$

Under these transformations the variation of the action is given by

$$\delta S = \int d^4x \; \Delta \tag{2}$$

with Δ being the divergence of the scale current s^μ and equal to the trace of the energy-momentum tensor

$$\Delta = \partial_\mu s^\mu = \Theta^\mu{}_\mu \, . \tag{3}$$

In theories with only dimensionless coupling constants this trace is equal to zero on the classical level.

From the Ward identities we found the relation

$$-i \int d^4y \; T^* \langle 0| \Delta(y)\, \phi(x_1) \dots \phi(x_n) |0\rangle = T^* \langle 0| \delta\phi(x_1)\, \phi(x_2) \dots \phi(x_n) |0\rangle + \dots + T^* \langle 0| \phi(x_1) \dots \delta\phi(x_n) |0\rangle \tag{4}$$

II Prerequisites

To illustrate the breaking of scale invariance on the quantum level let us consider ϕ^4-theory:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{\lambda_0}{4!} \phi^4 - \mu_0^2 \phi^2 \tag{5}$$

where the subscript 0 indicates the bare couplings. Although $d = 1$ in this example we will write d in favour of a more general formula.

By $\Gamma^{(n)}(p_1, \dots, p_n)$ we denote the renormalized Green's functions with n external legs and incoming momenta p_i such that $\sum_{i=1}^{n} p_i = 0$, i.e.

$$(2\pi)^4 \, \delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big) \, \Gamma^{(n)}(p_1, \dots, p_n) = \prod_{i=1}^{n} \int d^4x_i \; e^{ip_i x_i} \; T^* \langle 0| \phi(x_1) \dots \phi(x_n) |0\rangle \tag{6}$$

Analogously we consider by $\Gamma^{(n)}_\Delta(k; p_1 \dots p_n)$ a renormalized Green's function with an insertion of Δ, carrying momentum k. When going to momentum space, we see that $k = 0$:

$$\begin{aligned} \int d^4y \; T^* \langle 0| \Delta(y)\, \phi(x_1) \dots \phi(x_n) |0\rangle &= \int d^4q \, d^4y \; e^{-iqy} \; T^* \langle 0| \tilde{\Delta}(q)\, \phi(x_1) \dots \phi(x_n) |0\rangle \\ &= \int d^4q \; \delta^{(4)}(q) \; T^* \langle 0| \tilde{\Delta}(0)\, \phi(x_1) \dots \phi(x_n) |0\rangle \\ &= T^* \langle 0| \tilde{\Delta}(0)\, \phi(x_1) \dots \phi(x_n) |0\rangle \end{aligned} \tag{7}$$

From the physics point of view this is not surprising since Δ is nothing but a mass insertion (here: $\mu_0^2\phi^2$) which should not carry any momentum.

With these definitions we find the following relation for renormalized Green's functions:

$$-i\,\Gamma_\Delta^{(n)}(0;p_1,\dots,p_n) = -i\int d^4p_n\ \delta\Big(\sum_{j=1}^{n}p_j\Big)\ \Gamma_\Delta^{(n)}(0;p_1,\dots,p_n)$$

$$\overset{(6)}{=} -\frac{i}{(2\pi)^4}\int d^4p_n \int d^4x_1\dots d^4x_n\ e^{ip_1x_1}\dots e^{ip_nx_n}\int d^4y\ T^*\langle 0|\Delta(y)\,\phi(x_1)\dots\phi(x_n)|0\rangle$$

$$\overset{(4)}{=} \frac{1}{(2\pi)^4}\int d^4p_n\int d^4x_1\dots d^4x_n\ e^{ip_1x_1}\dots e^{ip_nx_n}\Big(T^*\langle 0|\delta\phi(x_1)\dots\phi(x_n)|0\rangle + \dots\Big)$$

$$\overset{(7)}{=} \frac{1}{(2\pi)^4}\int d^4p_n\int d^4x_1\dots d^4x_n\ e^{ip_1x_1}\dots e^{ip_nx_n}\Big(\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i} + nd\Big)\,T^*\langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle$$

$$\overset{(6)}{=} \int d^4p_n\Big(\sum_{i=1}^{n}p_i\frac{\partial}{\partial p_i} + nd\Big)\,\delta^{(4)}\Big(\sum_{j=1}^{n}p_j\Big)\ \Gamma^{(n)}(p_1,\dots,p_n)$$

$$= \Big(\sum_{i=1}^{n-1}p_i\frac{\partial}{\partial p_i} + nd\Big)\Gamma^{(n)}(p_1,\dots,p_n) + \int d^4p_n\ p_n\cdot\frac{\partial}{\partial p_n}\Big(\delta^{(4)}\Big(\sum_{j=1}^{n}p_j\Big)\cdot\Gamma^{(n)}(p_1,\dots,p_n)\Big)$$

$$\overset{\text{Int. by parts}}{=} \Big(\sum_{i=1}^{n-1}p_i\frac{\partial}{\partial p_i} + nd - 4\Big)\Gamma^{(n)}(p_1,\dots,p_n)$$

$$\Rightarrow\quad \boxed{-i\,\Gamma_\Delta^{(n)}(0;p_1,\dots,p_n) = \Big(\sum_{i=1}^{n-1}p_i\frac{\partial}{\partial p_i} + nd - 4\Big)\Gamma^{(n)}(p_1,\dots,p_n)} \qquad (8)$$

From dimensional analysis we can see that the Green's functions can be written as

$$\Gamma^{(n)}(p_1,\dots,p_n) = s^{\frac{4-n}{2}}\ F^{(n)}\Big(\frac{s}{\mu^2},\lambda,\frac{p_ip_j}{s}\Big) \qquad (9)$$

with $s=\sum_{i=1}^{n}p_i^2$ and $F^{(n)}$ some dimensionless function. Inserting this into equation (8) and writing

$$\frac{\partial}{\partial p_i} = \frac{\partial s}{\partial p_i}\frac{\partial}{\partial s} + \frac{\partial(s/\mu^2)}{\partial p_i}\frac{\partial}{\partial\mu} + \sum_{j,k=1}^{n}\frac{\partial(p_jp_k/s)}{\partial p_i}\ \frac{\partial s}{\partial\big(\frac{p_jp_k}{s}\big)}\ \frac{\partial}{\partial s} \qquad (10)$$

we find

$$\boxed{\left[\mu\frac{\partial}{\partial\mu} + n(1-d)\right]\Gamma^{(n)}(p_1,\dots,p_n) = i\,\Gamma_\Delta^{(n)}(0;p_1,\dots,p_n)} \qquad (11)$$

At zeroth order in perturbation theory we find in ϕ^4- theory $(d=1)$

$$\Gamma^{(2)} = \text{——} = -i(p^2-\mu^2)$$

$$\Gamma_\Delta^{(2)} = \text{—×—} = 2\mu^2$$

$$\Rightarrow \left(\mu\frac{\partial}{\partial\mu} + 0\right)(-i)\,(p^2-\mu^2) = 2i\mu^2 \quad \checkmark$$

and

$$\Gamma^{(4)} = \times = -i\lambda$$

$$\Gamma_\Delta^{(4)} = \qquad = 0$$

$$\Rightarrow \left(\mu\frac{\partial}{\partial\mu} + 0\right)(-i\lambda) = 0 \quad \checkmark$$

III Deep Euclidean region

In the Euclidean region of multi-particle ~~space~~ momentum space all four-momenta are Euclidean, i.e. have real space and imaginary time parts. By deep Euclidean region we mean this part of the Euclidean region where

- $s = \sum p_i^2$ is large
- $\frac{p_i p_j}{s}$ is fixed and
- no partial sum of the momenta is zero. ~~In this region all external~~

In this region all external lines are far off-shell as well as the momentum transferred between two halves of a diagram. S. Weinberg (Phys. Rev. **118**, 838 (1960)) was the first to discover that in this region $\Gamma^{(n)}$ scales at most like $s^{\frac{4-n}{2}}\mathcal{P}(\ln s/\mu^2)$ at all finite orders in perturbation theory, where $\mathcal{P}$ is some polynomial. Due to the additional propagator coming from the mass insertion, $\Gamma_\Delta^{(n)}$ grows no faster than $s^{\frac{n-2}{2}}\mathcal{P}(\ln s/\mu^2)$. In the deep Euclidean region we can therefore neglect the r.h.s. of (11) and obtain asymptotically

$$\left(\mu\frac{\partial}{\partial\mu} + n(1-d)\right)\Gamma^{(n)}_{as} = 0$$

here $d=1$ $\Longleftrightarrow$

$$\boxed{\mu\frac{\partial}{\partial\mu}\,\Gamma^{(n)}_{as} = 0} \qquad (12)$$

This equation tells us, that $\Gamma^{(n)}_{as}$ has to be free of logarithms in every order of renormalized perturbation theory! This is obviously wrong (loops lead to lns) and therefore (11) from which we derived (12) has to be false.

$\Rightarrow$ The Ward identities of broken scale invariance contain anomalies!

The reason for the occurrence of the anomalies is clear: In order to prove the Ward identities we have to introduce a cutoff to get rid of the divergencies. Obviously this cutoff has to be chosen such that the symmetry is preserved. However a cutoff always involves a large mass and masses spoil scale invariance.

IV Anomalous dimensions (S. Coleman & R. Jackiw: Ann. of Physics, 67, 552 (1971))

In the meson-nucleon theory

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi + \frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}\mu_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 + g_0\bar\psi\gamma_5\psi\phi$$

the validity of (11) can be restored at order g^2 by changing the ~~scaling~~ scale dimension of ϕ to $d_\phi = 1 + \frac{g^2}{8\pi^2}$ and of ψ to $d_\psi = 3/2 + \frac{g^2}{32\pi^2}$. One says that the fields acquire anomalous dimensions. This must not be confused with the dimension obtained from dimensional analysis which remains 1 and 3/2 respectively.

Unfortunately the introduction of anomalous dimensions is not sufficient. If it were so to all orders in perturbation theory we would get from (11) for any d

$$\mu\frac{\partial}{\partial\mu}\left[\frac{\Gamma^{(4)}_{as}}{(\Gamma^{(2)}_{as})^2}\right] = 0. \qquad (13)$$

Considering again ϕ^4-theory we get up to $\mathcal{O}(\lambda^2)$

$$\Gamma^{(2)}_{as} =$$  $$+ \cdots \sim s + \mathcal{O}(\lambda^2) \qquad (14)$$

$$\Gamma^{(4)}_{as} =$$  $$+ \cdots \sim \lambda + a\lambda^2 \ln \frac{s}{\mu^2} + b\lambda^2 +$$

where loop contributions to $\Gamma^{(2)}_{as}$ are absorbed in a mass renormalization. The ratio in (14) is then

$$\frac{\Gamma^{(4)}_{as}}{(\Gamma^{(2)}_{as})^2} \sim \frac{\lambda}{s^2} + \frac{a\lambda^2}{s^2} \ln \frac{s}{\mu^2} + \frac{b\lambda}{s^2} + \mathcal{O}(\lambda^3)$$

which clearly violates (13), pointing to more anomalies.

V Callan - Symanzik equations

(C. G. Callan: Phys. Rev., **D2**, 1514 (1970)
K. Symanzik: Comm. Math. Phys. **18**, 227 (1970))

To deal with these anomalies Callan and Symanzik derived independently the Callan-Symanzik equations. To obtain them we first note that the relation between bare and renormalized Green's functions is given by

$$\Gamma^{(n)}(p_1, \ldots, p_n) = (Z_3)^{n/2}\, \Gamma^{(n)}_u(p_1, \ldots, p_n) \qquad (15)$$

with the subscript u indicating the unrenormalized function and Z_3 being the wave-function renormalization constant. The unrenormalized version of (11) is then

$$i\,\Gamma^{(n)}_{u\Delta}(0; p_1, \ldots, p_n) = \mu_0 \frac{\partial}{\partial \mu_0} \Gamma^{(n)}_u(p_1, \ldots, p_n). \qquad (16)$$

Further we define

$$\Gamma^{(n)}_{\Delta}(k; p_1, \ldots, p_n) = Z\,(Z_3)^{n/2}\, \Gamma^{(n)}_{u\Delta}(k; p_1, \ldots, p_n) \qquad (17)$$

which makes the l.h.s. cutoff-independent in the limit of high cutoff, for an appropriate choice of Z (up to a constant factor).

From (15) - (17) we get

$$i\Gamma_{\Delta}^{(n)}(0;p_1,\dots,p_n) = \left(Z\mu_0 \frac{\partial}{\partial\mu_0} - \frac{n}{2} Z\mu_0 \frac{\partial \ln Z_3}{\partial\mu_0}\right)\Gamma^{(n)}(p_1,\dots,p_n)$$

$$= \left[\left(Z\mu_0 \frac{\partial\mu}{\partial\mu_0}\right)\frac{\partial}{\partial\mu} + \left(Z\mu_0 \frac{\partial\lambda}{\partial\mu_0}\right)\frac{\partial}{\partial\lambda} - \frac{n}{2}\left(Z\mu_0 \frac{\partial \ln Z_3}{\partial\mu_0}\right)\right]\Gamma^{(n)}(p_1,\dots,p_n). \tag{18}$$

Choosing

$$\mu = Z\mu_0 \frac{\partial\mu}{\partial\mu_0}\,; \qquad \beta = Z\mu_0 \frac{\partial\lambda}{\partial\mu_0}\,; \qquad \gamma = \frac{1}{2} Z\mu_0 \frac{\partial \ln Z_3}{\partial\mu_0}$$

we obtain the Callan-Symanzik equations

$$\boxed{\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right]\Gamma^{(n)}(p_1,\dots,p_n) = i\Gamma_{\Delta}^{(n)}(0;p_1,\dots,p_n)} \tag{19}$$

VI Pure Yang-Mills theory

For a simple example let us consider pure Yang-Mills theory

$$\mathcal{L}_{YM} = -\frac{1}{4g_0^2} G^a_{\mu\nu} G^{a\mu\nu} \tag{20}$$

which is clearly scale invariant at the classical level. To derive the anomaly in this theory at one-loop level we use dimensional regularization and calculate in $4-\varepsilon$ dimensions. This does not leave the action $\int d^{4-\varepsilon} G^2_{\mu\nu}$ scale invariant. Its change is proportional to ε but also going from the unrenormalized g_0 to the renormalized g introduces a further dependence on ε. So after all we obtain

$$\delta S = \int d^{4-\varepsilon}x \left\{-\frac{1}{4}\left(\frac{1}{g^2} + \frac{\beta_0}{8\pi^2}\frac{1}{\varepsilon}\right)(\alpha^{-\varepsilon}-1)\, G^2_{\mu\nu}\right\}$$

$$\xrightarrow{\varepsilon\to 0} \int d^4x \, \ln\alpha \left(+\frac{\beta_0}{32\pi^2} G^2_{\mu\nu}\right)$$

with $\beta_0 = N\cdot\frac{11}{3}$ being the first coeficient of the β-function and α the scaling parameter introduced at the beginning.

VII Conclusion

- Classical scale invariance is broken at the quantum level which we showed for ϕ^4-theory in the deep Euclidean region.
- The introduction of only anomalous dimensions is not sufficient to deal with all anomalies
- The Callan-Symanzik equations solve this problem and introduce a second dynamical parameter
- To generalize the Callan-Symanzik equations to more complicated theories we need to introduce a β-like (γ-like) term for each coupling constant (field).

VIII Literature

- S. Coleman: "Aspects of Symmetry"
- C. Itzykson & J.-B. Zuber: "Quantum Field Theory"
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- M. Shifman: "Advanced Topics in Quantum Field Theory: A lecture course"
- H.-G. Kohrs: "Die konforme Anomalie im Standardmodell", Diplomarbeit