Supercurrent for chiral and scale transformations

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Contents

This talk closely follows sections 49, 57-59 of [1] and section 26 of [2]. Section 3 follows [3].

1 Motivation

As we know from the Noether theorem, global continuous symmetries of a theory give rise to conserved currents. Therefore, if we study a Poincaré invariant field theory with global supersymmetry, we should be able to identify currents that correspond to these symmetries. These are called the energy-momentum tensor and the supercurrent of the theory, respectively. From the SUSY algebra, we know that

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu. \tag{1}$$

The Noether theorem furthermore tells us that the charges that generate a symmetry can be obtained by integrating the time-component of the corresponding current over a fixed time-slice, such that

$$Q_\alpha = \int_V d^3x S_{0\alpha}, \quad P_\mu = \int_V d^3x T_{0\mu}. \tag{2}$$

Plugging these expressions into Eq.(1) we find that

$$\left\{S_{0\alpha}, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} T_{0\mu}. \tag{3}$$

Eq.(3) shows that in supersymmetric theories the energy-momentum tensor and the supercurrent are not independent of each other, but rather seem to belong to one SUSY-multiplet. In this talk, we therefore want to study the relations supersymmetry imposes among symmetry currents and make precise the idea sketched above. In section 2 we study the problem for simple Wess-Zumino models, which will also allow us to set up notations and conventions. In section 3 we discuss recent results concerning the existence of supercurrent multiplets, before we study currents in SQCD in section 4. Given the topics of the preceding talks, we will be especially interested in how anomalies arise and how they arrange themselves in a multiplet.

2 Wess-Zumino models

2.1 Reminder: SUSY field theory

In this talk, we will work in $\mathcal{N} = 1$ superspace, which is the extension of Minkowski space by 4 Grassmann coordinates

$$\{x^\mu\} \to \left\{x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}\right\}, \tag{4}$$

where $\alpha, \dot\alpha = 1, 2$. On superspace, a SUSY transformation can be parametrised by[1]

$$\delta\theta_\alpha = \epsilon_\alpha, \quad \delta\bar\theta_{\dot\alpha} = \bar\epsilon_{\dot\alpha}, \quad \delta x_{\alpha\dot\alpha} = \left(\sigma^\mu\right)_{\alpha\dot\alpha} \delta x_\mu = -2i\theta_\alpha\bar\epsilon_{\dot\alpha} - 2i\bar\theta_{\dot\alpha}\epsilon_\alpha. \tag{5}$$

Two subspaces that are left invariant by such transformations are the chiral subspaces

$$\left\{x_L^\mu = x^\mu - i\theta^\alpha\left(\sigma^\mu\right)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \theta^\alpha\right\}, \quad \left\{x_R^\mu = x^\mu + i\theta^\alpha\left(\sigma^\mu\right)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \bar\theta^{\dot\alpha}\right\}. \tag{6}$$

Representations of the SUSY algebra are then formed by superfields $f\left(x^\mu, \theta, \bar\theta\right)$, fields which depend on all superspace coordinates. Due to the Grassmann nature of the θ, $\bar\theta$-variables, superfields can be expanded as

$$f\left(x^\mu, \theta, \bar\theta\right) = f_1\left(x^\mu\right) + \cdots + \theta^2\bar\theta^2 f_n\left(x^\mu\right), \tag{7}$$

[1] We will frequently switch between vectors and bi-spinors using the Pauli matrices: $v_{\alpha\dot\alpha} = \left(\sigma^\mu\right)_{\alpha\dot\alpha} v_\mu \leftrightarrow v^\mu = \frac{1}{2} v_{\alpha\dot\alpha}\left(\bar\sigma^\mu\right)^{\dot\alpha\alpha}$

where the $f_i(x^\mu)$ are regular fields, which only depend on the space-time coordinates. The action of the supercharges and superderivatives on superfields can then be represented by differential operators as

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} + \bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \theta^\alpha\partial_{\alpha\dot{\alpha}}, \tag{8}$$

as well as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}}. \tag{9}$$

To find the transformation behaviour of a given superfield, we just have to plug Eq.(5) in

$$\Phi + \delta\Phi = \Phi\left(x + \delta x, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}\right) \tag{10}$$

and compare the corresponding components. A generic feature is that the highest component of a superfield transforms as a full derivative, which we will use later to build SUSY-invariant actions.

A general superfield will form a reducible representation of the SUSY algebra. Therefore, we need to impose constraints on the superfields which are consistent with SUSY to find irreducible representations. Two constraints we will use later on are the reality condition $V^\dagger = V$, which leads to the vector superfield, and the chirality constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$. The latter constraint is equivalent to saying that $\Phi = \Phi(x_L, \theta)$ and can be expanded as

$$\Phi(x_L, \theta) = \phi(x_L) + \sqrt{2}\theta^\alpha\psi_\alpha(x_L) + \theta^2 F(x_L). \tag{11}$$

This represents the minimal supermultiplet. Analogously, we can define antichiral superfields by the constraint $D_\alpha\bar{\Phi} = 0$. The chirality constraint is consistent with SUSY due to $\left\{\bar{D}_{\dot{\alpha}}, Q_\alpha\right\} = 0$. It should be noted that sums and products of superfields are again superfields, as are space-time- and superderivatives of superfields.

To find supersymmetric Lagrangians we use integrals over Grassmann variables, which are defined by the properties

$$\int d\theta_i = 0, \quad \int d\theta_i\theta_j = \delta_{ij}. \tag{12}$$

We normalise integrals over the chiral subspaces such that

$$\int d^2\theta\theta^2 = \int d^2\bar{\theta}\bar{\theta}^2 = 1. \tag{13}$$

Note that Eq.(12) implies

$$d(c\theta) = c^{-1}d\theta. \tag{14}$$

Furthermore, it follows from Eq.(12) that

$$\int d^2\theta d^2\bar{\theta}V \tag{15}$$

over an arbitrary superfield V projects out the highest component of V. As mentioned before, this generically transforms as a total derivative. Therefore,

$$S = \int d^4x d^2\theta d^2\bar{\theta} V =: \int d^4x \mathcal{L} \tag{16}$$

is invariant under SUSY-transformations and can serve as an action for a SUSY field theory. Note, however, that it is only the action that is invariant, not the Lagrangian itself.

2.2 WZ Models

We now want to construct a simple field theory using a chiral and an antichiral superfield. Using Eq.(11) and the corresponding expression for an antichiral superfield $\bar{\Phi}$, we see that the highest component of $\bar{\Phi}\Phi$ gives rise to kinetic terms for the fields ϕ, ψ:

$$\int d^4x \left(\partial_\mu \bar{\phi} \partial^\mu \phi + i\bar{\psi}_{\dot{\alpha}} \partial^{\dot{\alpha}\alpha} \psi_\alpha + \bar{F}F\right) \tag{17}$$

Non-derivative terms such as mass terms and interactions are then introduced by a polynomial function of the superfields called superpotential $\mathcal{W}(\Phi, \bar{\Phi})$. We will focus on the case where the superpotential $\mathcal{W}(\Phi)$ only depends on Φ. In four dimensions, the highest power of Φ that can appear in $\mathcal{W}$ if the theory is to be renormalisable is three. We can then write the superpotential as

$$\mathcal{W}(\Phi) = \frac{m}{2}\Phi^2 - \frac{\lambda}{3}\Phi^3, \tag{18}$$

where the linear term was omitted, since it can always be removed by a shift in Φ. This defines the Wess-Zumino model,

$$S = \int d^4x \left(\int d^2\theta d^2\bar{\theta} \bar{\Phi}\Phi + \int d^2\theta \mathcal{W}(\theta) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\bar{\Phi}) \right), \tag{19}$$

whose Lagrangian in space-time reads

$$\mathcal{L} = \partial_\mu \bar{\phi} \partial^\mu \phi + i\bar{\psi}_{\dot{\alpha}} \partial^{\dot{\alpha}\alpha} \psi_\alpha + \bar{F}F + \left[F\mathcal{W}'(\phi) - \frac{1}{2}\mathcal{W}''(\phi)\psi^2 + \text{h.c.} \right]. \tag{20}$$

2.3 Currents in the Wess-Zumino models

What are the symmetries of Eq.(19)? By construction, the theory is Poincaré and SUSY invariant. Therefore, we can find the corresponding conserved currents $T_{\mu\nu}$ and $J_{\mu\alpha}$. Furthermore, we see that λ is dimensionless, so that the theory is scale invariant (and therefore conformal) for $m = 0$. In that case, we should have $T^\mu_\mu = 0$. Starting from the simplest case of vanishing superpotential, one finds the corresponding energy-momentum tensor

$$
\begin{aligned}
T_{\mu\nu} = &\partial_\mu \bar{\phi} \partial_\nu \phi + \partial_\nu \bar{\phi} \partial_\mu \phi - g_{\mu\nu} \left(\partial^\chi \bar{\phi} \partial_\chi \phi - F \bar{F} \right) + \text{fermions} \\
&+ \frac{1}{3} \left(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) \phi \bar{\phi}.
\end{aligned} \tag{21}
$$

The second line of Eq.(21) seems to be unnecessary. It is conserved by itself and gives no contribution to the charges P^μ of the energy-momentum tensor. However, as mentioned above, we need T^μ_μ to be proportional to the equations of motion, which is only fulfilled after including this term. Such a term is called an improvement. For the supercurrent, we can again use the Noether procedure and obtain

$$
\begin{aligned}
J_{\alpha\beta\dot{\beta}} = &2\sqrt{2} \Big(\left[\left(\partial_{\alpha\dot{\beta}} \bar{\phi} \right) \psi_\beta - i \epsilon_{\beta\alpha} F \bar{\psi}_{\dot{\beta}} \right] \\
&- \frac{1}{6} \left[\partial_{\alpha\dot{\beta}} \left(\psi_\beta \bar{\phi} \right) + \partial_{\beta\dot{\beta}} \left(\psi_\alpha \bar{\phi} \right) - 3 \epsilon_{\beta\alpha} \partial^\gamma_{\dot{\beta}} \left(\psi_\gamma \bar{\phi} \right) \right] \Big),
\end{aligned} \tag{22}
$$

where the second line again is an improvement term. Equipped with Eqs.(21) and (22) we can then study the real superfield

$$
\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{1}{3} \left(\bar{D}_{\dot{\alpha}} \bar{\Phi} \right) \left(D_\alpha \Phi \right) + \frac{2}{3} i \bar{\Phi} \overleftrightarrow{\partial_{\dot{\alpha}\alpha}} \Phi. \tag{23}
$$

Using the equations of motion, now for a general superpotential Eq.(18), one can show that

$$
D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} = 2 \bar{D}_{\dot{\alpha}} \left(\bar{\mathcal{W}} - \frac{1}{3} \bar{\Phi} \bar{\mathcal{W}}' \right) =: \bar{D}_{\dot{\alpha}} \bar{X}, \tag{24}
$$

which implies (using $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}}$)

$$
\partial^{\dot{\alpha}\alpha} \mathcal{J}_{\alpha\dot{\alpha}} = \frac{1}{2i} \left(D^2 X - \bar{D}^2 \bar{X} \right). \tag{25}
$$

Expanding the right-hand side of Eq.(23) in terms of components, we find the lowest components

$$
\mathcal{J}_{\alpha\dot{\alpha}} = R_{\alpha\dot{\alpha}} - \left[i\theta^\beta \left(J_{\beta\alpha\dot{\alpha}} - \frac{2}{3} \epsilon_{\beta\alpha} \epsilon^{\gamma\delta} J_{\delta\gamma\dot{\alpha}} \right) + \text{h.c.} \right] + ..., \tag{26}
$$

as well as the $\theta\bar{\theta}$-component (for convenience written in vectorial notation)

$$
\mathcal{J}_\mu |_{\theta\bar{\theta}} = \left[\bar{\theta}_{\dot{\alpha}} \left(\bar{\sigma}^\nu \right)^{\dot{\alpha}\alpha} \theta_\alpha \right] \left(2T_{\mu\nu} - \frac{2}{3} g_{\mu\nu} T^\chi_\chi - \frac{1}{2} \epsilon_{\nu\mu\rho\sigma} \partial^\rho R^\sigma \right). \tag{27}
$$

As hinted at in the motivation, we indeed see that the energy-momentum tensor Eq.(21) and the supercurrent Eq.(22) appear in one multiplet. It should be noted that $\bar{X}$ in Eq.(24) vanishes for vanishing or purely cubic superpotential, i.e. for massless theories. Let us now look at the lowest component of the supercurrent multiplet, which is given by

$$
R^\mu = -\frac{1}{3} \bar{\psi}_{\dot{\alpha}} \left(\bar{\sigma}^\mu \right)^{\dot{\alpha}\alpha} \psi_\alpha + \frac{2}{3} i \bar{\phi} \overleftrightarrow{\partial}^\mu \phi. \tag{28}
$$

This looks like a U(1) current for the transformation $\phi \to e^{\frac{2}{3}i\alpha}\phi$, $\psi \to e^{-\frac{1}{3}i\alpha}\psi$, which is an R-symmetry transformation. Remember that R-symmetry corresponds to symmetry under automorphisms of the supercharges, which for $\mathcal{N}=1$ SUSY just corresponds to a U(1)-rotation,

$$[R, Q_\alpha] = -Q_\alpha, \quad \left[R, \bar{Q}_{\dot{\alpha}}\right] = \bar{Q}_{\dot{\alpha}}, \tag{29}$$

which can be implemented on superspace as a transformation of the Grassmann directions $\theta \to e^{i\alpha}\theta$, $\bar{\theta} \to e^{-i\alpha}\bar{\theta}$. Therefore, R-symmetry is, like translations and SUSY, a geometric symmetry and the three symmetries are naturally grouped together in Eq.(23). The R-charge we assign to a superfield $\Phi(x,\theta,\bar{\theta}) \to e^{i\alpha r}\Phi\left(x, e^{i\alpha}\theta, e^{-i\alpha}\bar{\theta}\right)$ corresponds to the R-charge of its lowest component. The R-charge of all higher components is then fixed. For a general superpotential

$$\int d^2\theta \mathcal{W}(\Phi) \to \int \left(e^{-2i\alpha} d^2\theta\right) e^{i\alpha r_{\mathcal{W}}} \mathcal{W}\left(e^{i\alpha r}\Phi\right) \tag{30}$$

this means that the theory can only be invariant under R-symmetry if $\mathcal{W}$ has R-charge 2. For the above case Eq.(28), we have $\Phi \to e^{\frac{2}{3}i\alpha}\Phi$, which leads to an invariant Lagrangian for a purely cubic superpotential. If the Lagrangian is R-symmetric the lowest component of $D^2X - \bar{D}^2\bar{X} = 0$ and the R-current is conserved, but generically this is not the case.
As a last comment, note that the higher components of the supercurrent multiplet are trivially conserved.

3 Supercurrent multiplets

In the last section, we saw that the geometric currents of Wess-Zumino models arrange themselves in a supermultiplet, the Ferrara-Zumino multiplet [4]. Since WZ-models constitute a special class of SUSY field theories, it is a natural question whether this multiplet structure is a generic feature of SUSY theories. This turns out to be true, as is shown in [3] and we review the central results here.
SUSY field theories can be divided in two classes: those with an exact R-symmetry and those without. For the latter, it is (almost) always possible to find a multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$, called FZ-multiplet, with the properties

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X, \text{ with } \bar{D}_{\dot{\alpha}}X = 0. \tag{31}$$

The multiplet we studied in section 2 is an example of such a multiplet. The choice of $\mathcal{J}_{\alpha\dot{\alpha}}$, however, is not unique, since the defining properties Eq.(31) are conserved under the shifts

$$\mathcal{J}'_{\alpha\dot{\alpha}} = \mathcal{J}_{\alpha\dot{\alpha}} + \left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]\left(\Xi + \bar{\Xi}\right), \tag{32}$$

$$X' = X + \frac{1}{2}\bar{D}^2\bar{\Xi}, \tag{33}$$

for any chiral Ξ. Under these shifts, the currents in the multiplet change by improvement terms. In [3] it is shown that if there is a solution to the equation

$$X = -\frac{1}{2}\bar{D}^2\bar{\Xi}, \tag{34}$$

so that X is 'pure gauge', the theory is superconformal and the bottom component of $\mathcal{J}_{\alpha\dot{\alpha}}$ is the conserved R-symmetry current. In fact, it is true that the theory is superconformal if and only if the theory has a supercurrent multiplet which is conserved, $D^\alpha \mathcal{J}_{\alpha\dot{\alpha}} = 0$. The FZ-multiplet exists for most SUSY theories, but can be ill-defined for more general Kähler potentials or when FI-terms are present.
For theories with a continuous R-symmetry, another multiplet exists and is given by a real superfield $\mathcal{R}_{\alpha\dot{\alpha}}$. It is defined by the properties

$$\begin{aligned} &\bar{D}^{\dot{\alpha}}\mathcal{R}_{\alpha\dot{\alpha}} = \chi_\alpha, \quad \text{with} \\ &\bar{D}_{\dot{\alpha}}\chi_\alpha = 0 \quad \text{and} \quad \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - D^\alpha\chi_\alpha = 0. \end{aligned} \tag{35}$$

It is easy to show that Eqs.(35) immediately imply that the bottom component of $\mathcal{R}_{\alpha\dot{\alpha}}$ is conserved. Again, the choice of $\mathcal{R}_{\alpha\dot{\alpha}}$ is not unique and we have a shift symmetry under

$$\mathcal{R}'_{\alpha\dot{\alpha}} = \mathcal{R}_{\alpha\dot{\alpha}} + \left[D_\alpha, \bar{D}_{\dot{\alpha}}\right] J, \tag{36}$$

$$\chi'_\alpha = \chi_\alpha + \frac{3}{2}\bar{D}^2 D_\alpha J, \tag{37}$$

where J is a real linear superfield. As in the FZ-case, the currents change by improvement terms under the shift. It should be noted that the existence of the FZ- and the $\mathcal{R}$-multiplets is not exclusive. If a theory has a FZ-multiplet and a conserved R-current, the two multiplets are the same and differ only by a shift transformation.
The main result of [3] is the construction of a supercurrent multiplet $\mathcal{S}_{\alpha\dot{\alpha}}$ that exists for every SUSY theory. It is defined by the properties

$$\bar{D}^{\dot{\alpha}} S_{\alpha\dot{\alpha}} = D_\alpha X + \chi_\alpha, \tag{38}$$

$$\bar{D}_{\dot{\alpha}} X = 0, \tag{39}$$

$$\bar{D}_{\dot{\alpha}}\chi_\alpha = \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - D^\alpha\chi_\alpha = 0, \tag{40}$$

and therefore interpolates between the FZ- and the $\mathcal{R}$-multiplet. It reduces to the these cases if the solutions of $\bar{D}^2 U = -2X$ or $\bar{D}^2 D_\alpha U = -\frac{2}{3}\chi_\alpha$ for real U exist globally. It is interesting to note that while the FZ- and the $\mathcal{R}$-multiplet are known for more than 20 years, it was only two years ago that the most general multiplet was found.

4 SQCD

In the last part of the talk, we study the supercurrent multiplet for SUSY gauge theories, focusing on SQCD. While our discussion of the WZ-models was purely classical, we will be especially interested in quantum anomalies that show up in the currents.

SQCD is a $\mathcal{N}=1$, SU(N) gauge theory. Matter fields are represented by a pair of a chiral and an antichiral superfield Q_i, $\bar{Q}_i$ for each flavour. The chiral superfields are in the fundamental representation of SU(N), therefore the colour index above runs from $i=1,...,N$. The gauge fields live in a vector superfield V with field strength tensor

$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{-V}\left(D_\alpha e^V\right). \tag{41}$$

The Lagrangian of SQCD then reads

$$\mathcal{L} = \left(\frac{1}{4g^2}\int d^2\theta W^{\alpha a}W^a_\alpha + \text{h.c.}\right) + \sum_f \int d^2\theta d^2\bar{\theta}\bar{Q}_f e^V Q_f + \left(\int d^2\theta W\left(Q_f\right) + \text{h.c.}\right). \tag{42}$$

4.1 Gluodynamics

As a first step, let us only consider the gluonic part of Eq.(42), which explicitly reads

$$\mathcal{L}_{\text{glue}} = -\frac{1}{4g^2}G^a_{\mu\nu}G^{\mu\nu a} + \frac{i}{g^2}\lambda^{\alpha a}\mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{a\dot{\beta}} + \frac{\theta}{32\pi^2}G^a_{\mu\nu}\bar{G}^{\mu\nu a}. \tag{43}$$

This theory is similar to one-flavour QCD with the difference that the fermion here is in the adjoint and not in the fundamental representation. Since the gluinos are massless, the Lagrangian is invariant under $\lambda \to e^{i\alpha}\lambda$. This is just R-symmetry for the R-charges $R(V)=0$, $R(W)=1$. Therefore, we have a conserved R-current and, as we know from section 3, the $\mathcal{R}$-multiplet exists and is given by

$$\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{4}{g^2}\text{Tr}\left(e^V W_\alpha e^{-V}\bar{W}_{\dot{\alpha}}\right), \quad \text{with} \quad \bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = 0. \tag{44}$$

At quantum level, however, the R-symmetry is broken. This comes as no surprise, as we have seen the breaking of chiral symmetry for massless fermions before (cf. Tigran's talk). The explicit form of the non-conservation of the R-current is given by

$$\partial_\mu R^\mu = \frac{N}{16\pi^2}G^a_{\mu\nu}\bar{G}^{\mu\nu a}. \tag{45}$$

By anticommuting this expression with the supercharges, one finds the general expression for the supercurrent multiplet,

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{N}{8\pi^2}D_\alpha\left(\text{Tr}W^2\right) \neq 0, \tag{46}$$

as was pointed out in [5]. Since the left-hand side of Eq.(46) contain the term T^μ_μ, we can read off a possible scale anomaly and, indeed, we find

$$T^\mu_\mu = -\frac{3N}{16\pi^2}\text{Tr}\left(G^a_{\mu\nu}G^{\mu\nu a}\right). \tag{47}$$

This fits nicely in the picture of section 3: The R-current is no longer conserved and therefore the theory is no longer superconformal, as we can clearly see from a violation of scale invariance.

It is very interesting to see that not only the symmetry currents, but also the anomalies arrange themselves in one superfield. However, this gives rise to a profound problem: the R-current violation should be 1-loop exact by the SUSY version of the Adler-Bardeen theorem [6], [7]. On the other hand, we know from the preceding talks that the scale anomaly is proportional to the β-function, $T^\mu_\mu \sim \beta(g)$. Therefore, either the β-function is 1-loop exact or R-symmetry and T^μ_μ cannot reside in the same multiplet. The solution of this problem, according to [8], is that the anomalies, understood as operator equations, are indeed 1-loop exact in the Wilsonian picture. However, once matrix elements are computed, infrared effects enter and lead to higher-order corrections in the β-function (which is known exactly in gluodynamics)[2]. We will meet this problem again in the next section.

4.2 Once again with matter...

We now want to repeat the analysis of section 4.1 and study the changes that arise when matter fields are included. However, we will still work with vanishing superpotential. Therefore, we find a new U(1) symmetry $Q_f \to e^{i\alpha} Q_f$ in each flavour subsector, in addition to the geometric symmetries. The matter fields then give additional contributions to the supercurrent multiplet, which now reads

$$\mathcal{J}_{\alpha\dot\alpha} = \frac{4}{g^2} \mathrm{Tr} \left(\bar W_{\dot\alpha} e^V W_\alpha e^{-V} \right) - \frac{1}{3} \sum_f \bar Q_f \left(\overleftarrow{\bar\nabla}_{\dot\alpha} e^V \nabla_\alpha - e^V \bar D_{\dot\alpha} \nabla_\alpha + \overleftarrow{\bar\nabla}_{\dot\alpha} \overleftarrow{D}_\alpha e^V \right) Q_f, \quad (48)$$

and we find additional symmetry currents

$$R_{f\mu} = -\psi_f \sigma_\mu \bar\psi_f - \phi_f i \overleftrightarrow{\mathcal{D}}_\mu \bar\phi_f. \quad (49)$$

This turns out to be the $\theta\bar\theta$-component of the Konishi operator $\mathcal{J}_f = \bar Q_f e^V Q_f$. To make contact with the geometric supercurrent, we transform this into a bi-spinor via

$$\mathcal{J}_{f\alpha\dot\alpha} = -\frac{1}{2} \left[D_\alpha, \bar D_{\dot\alpha} \right] \mathcal{J}_f. \quad (50)$$

In contrast to the geometric supercurrent, all higher components of the Konishi current are conserved trivially. Including matter fields also modifies the anomaly on the right-hand side of Eq.(46). Explicitly,

$$X = -\frac{N}{8\pi^2} \mathrm{Tr} W^2 \to -\frac{2}{3} \left[\frac{3N - \frac{N_f}{2}}{16\pi^2} \mathrm{Tr} W^2 + \frac{1}{8} \sum_f \gamma_f \bar D^2 \left(\bar Q_f e^V Q_f \right) \right], \quad (51)$$

[2]It should be mentioned that there is some criticism concerning this solution. References that provide more information are [9] and [10].

where

$$\gamma_f = -\frac{d(\log Z_f)}{d(\log M_{\text{UV}})} \tag{52}$$

are the anomalous dimensions of the flavour fields. The first term of Eq.(51) is easy to understand: the violation of the R-current comes from fermion triangles. We now get additional contributions from the matter fermions, which contribute with a relative factor compared to the gauge fermions. The second term is a little bit more involved. Classically, this term should vanish because of $\bar{D}^2\mathcal{J}_f = 0$, but since we will find anomalies in the Konishi operator, as well, we keep the term. Due to the anomalous dimensions, this term includes all-loop order information. It can be understood when interpreting the original Lagrangian Eq.(42) to be formulated in the UV. Letting the theory flow to a renormalisation scale μ in the IR, [8] obtain the effective action

$$\mathcal{L}_{\text{eff}} = \left(\frac{1}{2g_0^2} - \frac{3N - \frac{N_f}{2}}{16\pi^2}\log\frac{M_{\text{UV}}}{\mu}\right)\int d^2\theta \text{Tr}W^2 + \sum_f \frac{1}{8}Z_f(\mu)\int d^2\theta D^2\bar{Q}_f e^V Q_f + \text{h.c.}, \tag{53}$$

where $Z_f(\mu)$ are the field renormalisation constants and M_{UV} is the UV cutoff. The first term in Eq.(53) shows explicitly what we mentioned at the end of the last section: Only the first term of the β-function appears in the operator equation. The appearance of terms like $\log M_{\text{UV}}$ signals the breakdown of scale invariance, as is to be expected due to UV divergences. Looking at the effective action, we see that the matter fields contribute terms proportional to $\frac{d\log Z_f}{d\log M_{\text{UV}}}$ when changing the scale, which explains the structure of the second term in Eq.(51).

As mentioned above, there are also anomalies in the flavour currents. Again, this is no surprise - the broken symmetry is a chiral symmetry for massless fermions. The anomaly reads

$$\bar{D}^2\mathcal{J}_f = \frac{1}{4\pi^2}\text{Tr}W^2 \tag{54}$$

and is exact to all-loop order. Since this is an exact result, we can plug Eq.(54) in Eq.(51) to find

$$X = -\frac{2}{3}\left[\frac{1}{16\pi^2}\left\{3N - \frac{1}{2}\sum_f(1-\gamma_f)\right\}\text{Tr}W^2\right], \tag{55}$$

which leads to the non-conservation equation

$$\partial^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = \frac{i}{48\pi^2}D^2\left[3N - \frac{1}{2}\sum_f(1-\gamma_f)\right]\text{Tr}W^2 + \text{h.c.} \tag{56}$$

This, however, allows us to define a new current

$$\tilde{\mathcal{J}}_{\alpha\dot{\alpha}} = \mathcal{J}_{\alpha\dot{\alpha}} - \frac{3N - \frac{1}{2}\sum_f(1-\gamma_f)}{\left(\frac{3N_f}{2}\right)}\sum_f \mathcal{J}_{f\alpha\dot{\alpha}}, \tag{57}$$

which is conserved $\partial^{\alpha\dot{\alpha}}\tilde{\mathcal{J}}_{\alpha\dot{\alpha}} = 0$. Furthermore, we can define $N_f - 1$ anomaly-free flavour currents by

$$\mathcal{J}^{fg}_{\alpha\dot{\alpha}} = \frac{1}{2}\left(\mathcal{J}_{f\alpha\dot{\alpha}} - \mathcal{J}_{g\alpha\dot{\alpha}}\right). \tag{58}$$

Note that from Eq.(55) we find

$$D^{\alpha}\tilde{\mathcal{J}}_{\alpha\dot{\alpha}} \sim 3N - \frac{1}{2}\sum_{f}\left(1-\gamma_f\right), \tag{59}$$

which means that, depending on γ_f, the theory can flow to the conformal limit in the IR.

5 Conclusions

Let us summarise our discussion. We have seen that the currents of the geometric symmetries, i.e. R-symmetry, supersymmetry and translation symmetry, arrange themselves in a supermultiplet. This is true for all SUSY field theories, whether it has a conserved R-current or not. In the explicit example of SQCD we studied the effects of quantum anomalies. We saw that the anomalies in the geometric currents are accompanied by matter anomalies, allowing the definition of anomaly-free currents, which are a combination of the geometrical and flavour currents.

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