Planar Two-loop Master Integrals for the Production of Two Equal-mass Particles at the LHC

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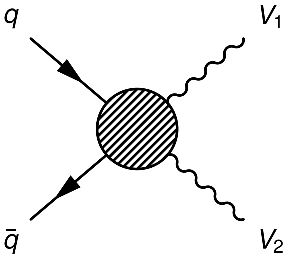
September 4, 2013

based on work together with *Thomas Gehrmann* and *Lorenzo Tancredi*:
JHEP 08 (2013) 070 (arXiv:1306.6344)

What is diboson production?

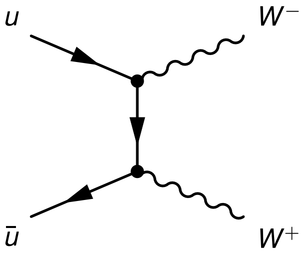
The production of two electroweak gauge bosons (γ,$W^{+,-}$,Z)

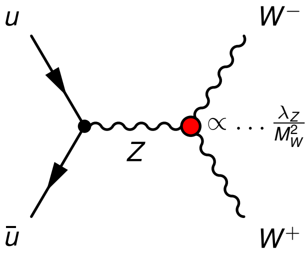


- Background for Higgs boson searches, Beyond the Standard Model (BSM) Physics searches
- Study of electroweak symmetry breaking, unitarization of $W_L W_L$ scattering
- Indirect probe for new physics

Diboson production at leading order

Example: Production of a W^+ W^- pair





Triple Gauge Coupling: modified by New Physics?

Triple Gauge Couplings

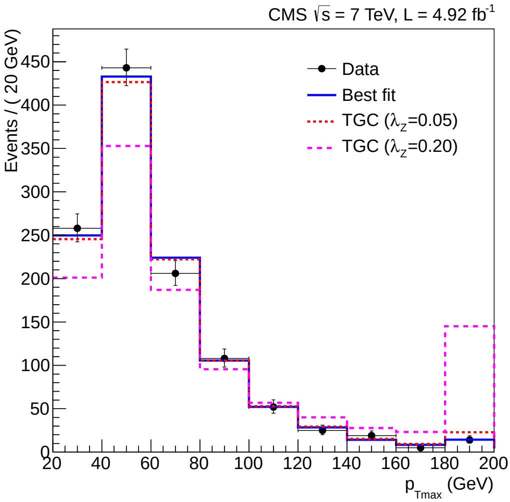
- Probe for physics above the LHC scale (i.e. a few TeV)
- present couplings: modified by BSM physics?
- new couplings: generated by BSM physics?
- in general: modifications very small

We need

- to study distributions
- precise measurements and predictions

Triple Gauge Couplings

Influence of a modified *ZWW* coupling



Leading lepton p_T distribution in *WW* events at the LHC (CMS) [1306.1126]

Status of higher order computations

- **Electroweak corrections**: done at NLO for all processes

  [Accomando et al. (2005), Bierweiler et al. (2013)]

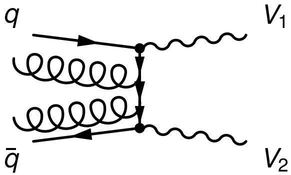
- **QCD NLO**: done for all processes

  [Ohnemus et al. (1993), Baur et al. (1993,1998), Dixon et al. (1998,1999)]

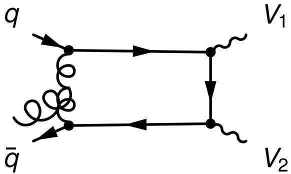
- **QCD NNLO**:
  - complete: only $\gamma\,\gamma$ [Catani et al. (2012)]
  - MINLO / VBFNLO approximation: $W\,Z$ [Campanario & Sapeta (2012)]

Ingredients of the NNLO computation

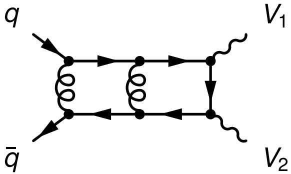
- double-real corrections: known for all processes



- real-virtual corrections: known for all processes [Dittmaier, Kallweit, Binoth, Campanario, ...]



- virtual corrections:
  - $\gamma\ \gamma$ [Bern et al. (2001)]
  - $W\ \gamma$ and $Z\ \gamma$ [Gehrmann, Tancredi, Weihs (2012/13)]
  - $W\ W$ (high energy limit) [Chachamis et al. (2008)]



NNLO virtual contributions for two equal-mass particles

- write down **Feynman diagrams** (143 for $q\,\bar{q} \to Z\,Z$)

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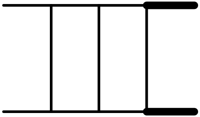
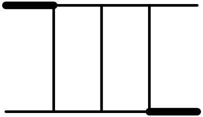
- computation of **tensor coefficients** using projectors

NNLO virtual contributions for two equal-mass particles

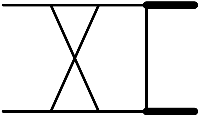
- write down **Feynman diagrams** (143 for $q\,\bar{q} \to Z\,Z$)
- computation of **tensor coefficients** using projectors
- Result: $\approx$ 4600 scalar integrals, classified into 3 topologies



Topo A



Topo B



Topo C

Solution of the Integrals

- **derive relations among them** exploiting analytic structure, Lorentz covariance and symmetries
- **reduction to Master Integrals (MIs)** using Laporta's algorithm implemented in Reduze [Studerus, v. Manteuffel (2012)]

  Topo A: 26 MIs   Topo B: 13   Topo C: 16
- **solution of MIs:** Method of differential equations

  [Kotikov (1991), Remiddi (1997), Remiddi, Gehrmann (2000)]

Idea:

derive differential equation for the integral with respect to external invariants

Solution of the Differential Equations

$$\frac{\partial}{\partial s_\alpha} M_j(D, \mathbf{s}) = \sum_k A_k(D, \mathbf{s}) M_k(D, \mathbf{s}) + \mathit{Inhom.}$$

Solution of the Differential Equations

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1. **expansion** in $\epsilon = \frac{1}{2}(4-D)$
2. **decoupling** of differential equations
3. **solve** by Euler's method of variation of constants
4. **fix boundary conditions** by imposing regularity at pseudo-thresholds

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Challenges

- Variable transformations
- Taking limits
- Analytical continuations

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Solution [C. Duhr]

Make use of algebraic structure of underlying functions!

Multiple Polylogarithms

- **Multiple Polylogarithms** (MPLs) are defined as iterated integrals:

$$G(a_1,\dots,a_n;x) \equiv \int_0^x \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t)$$

- Many curious properties, e.g. shuffle product

$$G(a_1,\dots,a_{n_1};x)\, G(a_{n_1+1},\dots,a_{n_1+n_2};x) = \sum_{\sigma\in\Sigma(n_1,n_2)} G(a_{\sigma(1)},\dots,a_{\sigma(n_1+n_2)};x)$$

- They form an algebra
- Can be extended to a **Hopf algebra**, i.e. an algebra with additional structure: coproduct [A. B. Goncharov (2002)]

Idea for manipulating MPLs [C. Duhr (2012)]

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Translate the problem to an algebraic tensor product space

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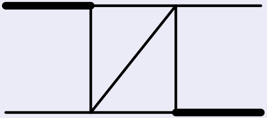
Translate the result back to the function space

Algorithm is described in

[Anastasiou, Duhr et al. (1302.4379)], [Gehrmann, Tancredi, Weihs (1306.6344)]

An example problem

Integration of



- Transform the differential equation from given in (z, y) to (z, x)
- obtain terms like

$$\int_0^z \mathrm{d}z' \frac{1}{z'-x} G\left(0, -1, -2-z'; y \to \frac{1+x^2}{x} - z'\right)$$

Easy to integrate in terms of MPLs if their z' dependence is only $G(\ldots; z')$!

Result

$$
\begin{aligned}
G\left(0,-1,-2-z';y\to\frac{1+x^2}{x}-z'\right) = {} & -G(0;x)^3/6+(G(0;x)^2G(J_x^{-1};z'))/2-G(J_x^{-1};z')(2G(0,-1;x) \\
& -2G(-c,-1;x)+G(-c,0;x)-2G(-\bar{c},-1;x)+G(-\bar{c},0;x)) \\
& +G(-2;z')(-G(0;x)^2/2+G(0,-c;x)+G(0,-\bar{c};x)+G(-I,0;x) \\
& -(G(-c;x)+G(-\bar{c};x))(G(-2,J_x^{-1};z')-G(-1,J_x^{-1};z') \\
& +G(J_x^{-1},-2;z')-G(J_x^{-1},-1;z'))+G(0;x)(G(-2,J_x^{-1};z') \\
& -G(-1,-2;z')+G(J_x^{-1},-2;z')-G(J_x^{-1},I_x^{-1};z')) \\
& \ldots \\
& -G(i,-\bar{c},0;x)-G(J_x^{-1},-2,I_x^{-1};z')+G(J_x^{-1},-1,-2;z') \\
& +G(J_x^{-1},-1,I_x^{-1};z')-G(J_x^{-1},I_x^{-1},-2;z')+\pi^2((-2G(-2;z') \\
& -G(0;x)+G(-I;x)+G(i;x)+G(J_x^{-1};z'))/12-\log(2)/6) \\
& +(-G(0;x)^2/2+G(-1;z')(-G(0;x)+G(-I;x)+G(i;x)) \\
& +(G(0;x)-G(-c;x)-G(-\bar{c};x))G(J_x^{-1};z')-G(-1,-2;z') \\
& +G(-1,J_x^{-1};z')+G(0,-c;x)+G(0,-\bar{c};x)+G(-I,0;x) \\
& -G(-I,-c;x)-G(-I,-\bar{c};x)+G(i,0;x)-G(i,-c;x) \\
& -G(i,-\bar{c};x)+G(J_x^{-1},-1;z')-G(J_x^{-1},I_x^{-1};z'))\log(2) \\
& -(G(-1;z')\log(2)^2)/2+(7\zeta(3))/8
\end{aligned}
$$

The "Coproduct enhanced symbol formalism"

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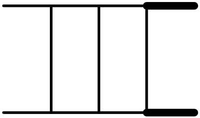
- straightforward procedure
- possible to automatise
- it works!



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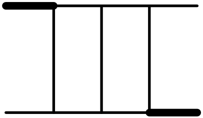
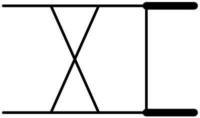


Topo B



Topo C

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- Results checked with
  - **FIESTA** [A. V. Smirnov, V. A. Smirnov, M. Tentyukov]
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- next steps:
  - use planar integrals to compute **leading color amplitudes**
  - solve **nonplanar master integrals** for the full result

Backup slides

Alternative formulation of an algebra using a tensor product

A "textbook" result from linear algebra

$\mathcal{A}$: a general algebra

$$\begin{tikzcd} \mathcal{A} \times \mathcal{A} \arrow[r, "\tau"] \arrow[dr, "\beta"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\ & \mathcal{A} \end{tikzcd}$$

$$G(a;x)G(b;x) = G(a,b;x) + G(b,a;x)$$

becomes

$$\mu(G(a;x) \otimes G(b;x)) = G(a,b;x) + G(b,a;x)$$

Alternative formulation of an algebra using a tensor product

Associativity

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad (a, b, c \in \mathcal{A})$$

becomes ($\beta \equiv$ bilinear algebra multiplication)

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{A} \times \mathcal{A} \arrow[r, "\mathrm{id} \times \beta"] \arrow[d, "\beta \times \mathrm{id}"'] & \mathcal{A} \times \mathcal{A} \arrow[d, "\beta"] \\
\mathcal{A} \times \mathcal{A} \arrow[r, "\beta"] & \mathcal{A}
\end{tikzcd}$$

Alternative formulation of an algebra using a tensor product

Associativity

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad (a, b, c \in \mathcal{A})$$

becomes ($\mu \equiv$ linear tensor multiplication)

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"] & \mathcal{A}
\end{tikzcd}$$

the Hopf-Algebra

A Hopf algebra $\mathcal{H}$ has one more structure:

$$\begin{tikzcd}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & & \mathcal{H} \otimes \mathcal{H} \arrow[ll, "\mathrm{id} \otimes \Delta"'] \\
\mathcal{H} \otimes \mathcal{H} \arrow[u, "\Delta \otimes \mathrm{id}"] & & \mathcal{H} \arrow[ll, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

Hopf algebra : $\mathcal{H}$ coproduct : Δ

Δ is coassiociative

$\Rightarrow$ unique way to iterate the coproduct:

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\mathrm{id}\otimes\Delta} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Delta} \dots$$

Hopf Algebra

Hopf algebra:

1. at the same time algebra (associative product)
2. and coalgebra (coassociative coproduct) (= bialgebra)
3. connected (the field over which the algebra is generated is included in the algebra)

- Product and coproduct are not dual any more!
- We require their compatibility

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b),$$

with $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) \equiv (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$.

the Multiple Polylogarithm Hopf Algebra

- This can be done for the MPLs [Alexander B. Goncharov, 2002]
- some examples for Δ:

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1 \,,$$

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!} \,.$$

Notation

- $\Delta_{2,1}$ = Apply coproduct; select tensors with weights $2 \otimes 1$.
- $\Delta_{1,1,1}$ = Apply coproduct two times; select tensors with weights $1 \otimes 1 \otimes 1$.

Ideas for manipulating MPLs [C. Duhr (2012)]

(obvious) observation

Equal expressions involving MPLs have an equal coproduct!
i.e. at weight 3:

$$F_3 = G_3$$

$$\Delta_{2,1}(F_3) = \Delta_{2,1}(G_3) \qquad \Delta_{1,2}(F_3) = \Delta_{1,2}(G_3)$$

$$\Delta_{1,1,1}(F_3) = \Delta_{1,1,1}(G_3)$$

Advantage: only identities of lower weight needed

An Algorithm to simplify/transform MPLs [C. Duhr (2012)] :

compute the "symbol" $\Delta_{1,\ldots,1}(F)$

$\Downarrow$

integrate it in terms of MPLs (yielding G)

$\Downarrow$

compute $\Delta_{2,1\ldots,1}(F - G)$, etc. to recover the parts lost by the symbol map ($\propto \pi^n$, $\zeta(n)$)

$\Downarrow$

determine missing constant term using PSLQ algorithm or guess