Bethe Ansatz for Yangian Invariants – Towards Super Yang-Mills Amplitudes

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1 Motivation

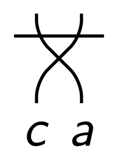
- Last decade: Overwhelming evidence that planar $\mathcal{N}=4$ super Yang-Mills theory is integrable [Minahan, Zarembo 2003] [Beisert, Kristjansen, Staudacher 2003] [Gromov, Kazakov, Vieira 2009]
  - Access non-perturbative regimes of a 4-dimensional quantum field theory
- Recently: Remarkable closed formula for scattering amplitudes of the theory
  - Amplitudes are invariant under action of ∞-dimensional Yangian algebra

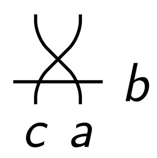
⇒ How to use standard methods of integrable systems to construct amplitudes?
  - First step: We show that certain Yangian invariants can be obtained from a Bethe ansatz
  - Fruitful exchange of ideas between amplitudes and integrable systems

2 Basic Methods and Structures in Integrable Systems

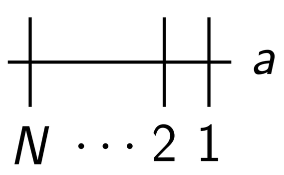
Quantum Inverse Scattering Method [Faddeev et al. 1980s]

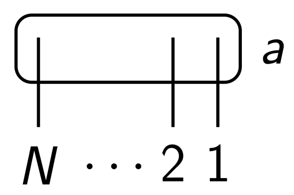
- Powerful algebraic approach to integrable systems
- Main problem: Diagonalize the Hamiltonian of a spin chain
- Central object: R-matrix satisfying a Yang-Baxter equation



$$= R_{ab}(z-y)R_{ac}(z)R_{bc}(y) = R_{bc}(y)R_{ac}(z)R_{ab}(z-y) =$$


- Introduce monodromy and transfer matrix for a spin chain with N sites

$$M_a(z) = R_{aN}(z-v_N)\cdots R_{a1}(z-v_1) =$$


$$T(z) = \operatorname{tr}_a M_a(z) = M_{11}(z) + \ldots + M_{mm}(z) =$$


where $R(z) = z1 + E_{\alpha\beta}\otimes J_{\beta\alpha}$ with $\mathfrak{gl}_m$-generators $J_{\alpha\beta}$ and $(E_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu}$

⇒ Recover Hamiltonian as logarithmic derivative of transfer matrix

Bethe Ansatz

- Method to diagonalize the transfer matrix
- Example: algebra $\mathfrak{gl}_2$ and representation with $\mathfrak{gl}_2$-weights $\Lambda = (\lambda, 0)$
  - Bethe roots $u_1, \ldots, u_S$ are solutions of Bethe equations [Bethe 1931]

$$\prod_{i=1}^{N}\frac{u_k - v_i}{u_k - v_i + \lambda} = -\prod_{i=1}^{S}\frac{u_k - u_i + 1}{u_k - u_i - 1}$$

  ⇒ Bethe vector Ψ is an eigenvector of the transfer matrix given in terms of matrix elements of the monodromy acting on a vacuum state

$$\Psi = M_{12}(u_1)\cdots M_{12}(u_S)\Omega$$

Yangian [Drinfeld 1985]

- Symmetry algebra of many integrable systems
- Yangian $\mathcal{Y}(\mathfrak{gl}_m)$ is defined by the Yang-Baxter-like relation

$$R_{ab}(z-y)M_a(z)M_b(y) = M_b(y)M_a(z)R_{ab}(z-y)$$

where the $\mathcal{Y}(\mathfrak{gl}_m)$-generators are obtained from the expansion

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}1 + M^{(1)}_{\alpha\beta}z^{-1} + M^{(2)}_{\alpha\beta}z^{-2} + \ldots$$

⇒ Monodromy of a spin chain provides a realization of the Yangian

3 Details on the Bethe Ansatz for Yangian Invariants

Characterization of Yangian Invariants

- Aim: Construct vectors Ψ annihilated by all generators of Yangian $\mathcal{Y}(\mathfrak{sl}_m)$
- Invariants are characterized by condition in terms of $\mathcal{Y}(\mathfrak{gl}_m)$-generators

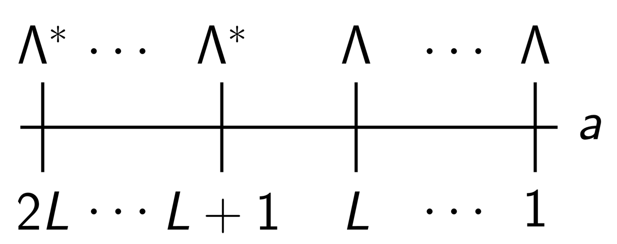
$$M_{\alpha\beta}(z)\Psi \propto \delta_{\alpha\beta}1\Psi$$

  - Characterization implies directly that invariants are special eigenvectors of the transfer matrix

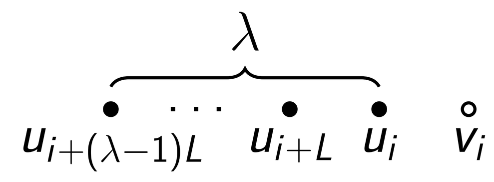
  ⇒ Construct invariants Ψ using Bethe ansatz

Bethe Ansatz

- Example: algebra $\mathfrak{gl}_2$ and a monodromy with even number of sites $N = 2L$
  - Need at least two different kinds of sites: L sites with representation $\Lambda = (\lambda, 0)$ and L sites with dual representation $\Lambda^* = (0, -\lambda)$
  - Need special choice of inhomogeneities: $v_{L+i} = v_i - \lambda - 1$

$$M_a(z) =$$


⇒ Bethe equations can be solved explicitly!
  - Bethe roots are $u_{i+(j-1)L} = v_i - j$ where $i = 1, \ldots, L$ and $j = 1, \ldots, \lambda$

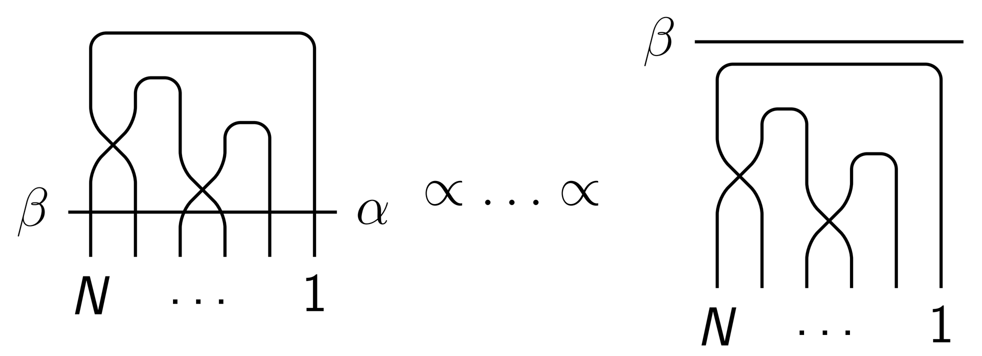


  - Bethe vector Ψ is a Yangian invariant
- Construction can be generalized
  - Arbitrary order of sites and odd number of sites
  - Nested Bethe ansatz for algebra $\mathfrak{gl}_m$

Graphical Interpretation

- Understand Yangian invariant Ψ as "network" of R-matrices
  - Pair of sites i, j with representations Λ, Λ^* and inhomogeneities v_i, $v_j = v_i - \lambda - 1$ is connected by a line
  - Intersection of two lines corresponds to an R-matrix satisfying a Yang-Baxter equation and "unitarity and crossing relations"

⇒ Properties of R-matrices imply immediatly Yangian invariance of Ψ

$$M_{\alpha\beta}(z)\Psi =$$


$$= \delta_{\alpha\beta}1\Psi$$

4 Main Results

- Invariants of Yangian $\mathcal{Y}(\mathfrak{sl}_m)$ with certain irreducible $\mathfrak{gl}_m$-representations at each site of the monodromy can be constructed using Bethe ansatz
- Generalization of perimeter Bethe ansatz [Baxter 1987]: Invariants can be understood as partition functions of vertex models on random lattices
- Progress towards more explicit "coordinate expressions" of Bethe vectors for $\mathfrak{gl}_3$ with mixed representations Λ and Λ^*

5 Towards Super Yang Mills Amplitudes

Scattering amplitudes

- Tree-level amplitudes for N particles in sector with K negative helicity gluons [Arkani-Hamed et al. 2009]

$$\int \mathrm{D}^{K(N-K)}c \prod_{a=1}^{K}\prod_{\alpha=1}^{8}\delta\Big(\sum_{i=1}^{N}c_{ai}(\mathcal{W}_\alpha)_i\Big)$$

  - $\mathcal{W}_\alpha$ are supertwistor variables transforming in the fundamental representation of the superconformal algebra $\mathfrak{psu}_{2,2|4}$
  - Integration along contour in Grassmannian $G(K,N) = \{K\text{-planes in } \mathbb{C}^N\}$
- Amplitudes are invariant under Yangian $\mathcal{Y}(\mathfrak{psu}_{2,2|4})$ [Drummond et al. 2009]

Outlook

- How to obtain amplitudes from a Bethe ansatz?

  ⇒ Are amplitudes "generating functions" for Yangian invariants with irreducible representations at each site?
- Excitingly Grassmannians play an important role in different areas of integrable systems
  - τ-functions of KP-hierarchy [Sato 1981], partition function of vertex models [Foda et al. 2009], master T-operator [Alexandrov et al. 2011]

  ⇒ Relation to the Grassmannian appearing in scattering amplitudes?

Benefits from Graduiertenkolleg

- Financial support to attend international conferences and schools
- Block courses with lectures on various topics in experimental and theoretical particle physics help to broaden the perspective

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