

Novel methods for $N=2$ BPS-saturated amplitudes in String Theory

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Based on work with
Carlo Angelantonj & *Boris Pioline*

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Max-Planck-Institut für Physik
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Physikzentrum Bad Honnef

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MAX-PLANCK-GESELLSCHAFT

Closed String perturbation theory

Topological expansion over closed Riemann surfaces



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$$\sum_{g=0}^{\infty} g_s^{2(g-1)} \int_{\text{moduli}} \int_{\text{insertions-}z_i} \int \mathcal{D}X \mathcal{D}\psi \dots \mathcal{V}_i(z_i) \dots e^{-S[X,\psi,g_{ab},\dots]}$$



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- Gauge modular group of large diffeomorphisms $PSL(2; \mathbb{Z})$
- Integration restricted over fundamental domain $\mathcal{F} = \{\tau \in \mathcal{H} : |\tau| \geq 1, |\tau_1| \leq 1/2\}$
- Invariant measure $d\mu := \frac{d^2\tau}{\tau_2^2}$
- $\mathcal{A}(\tau, \bar{\tau})$ modular invariant function



Closed String perturbation theory

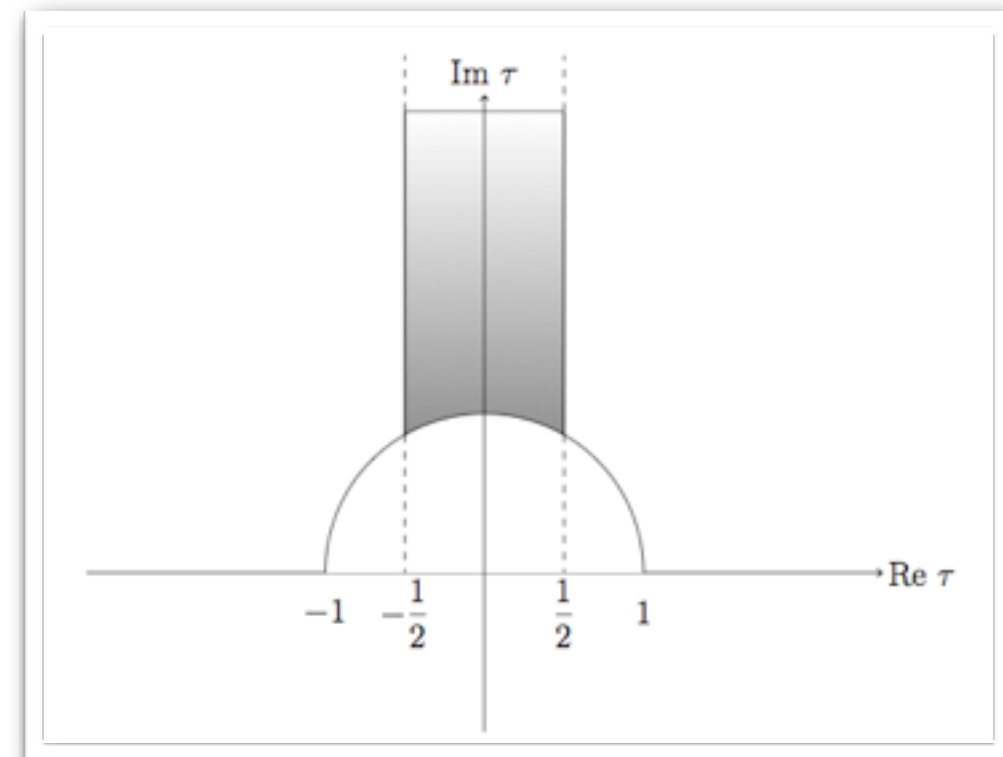
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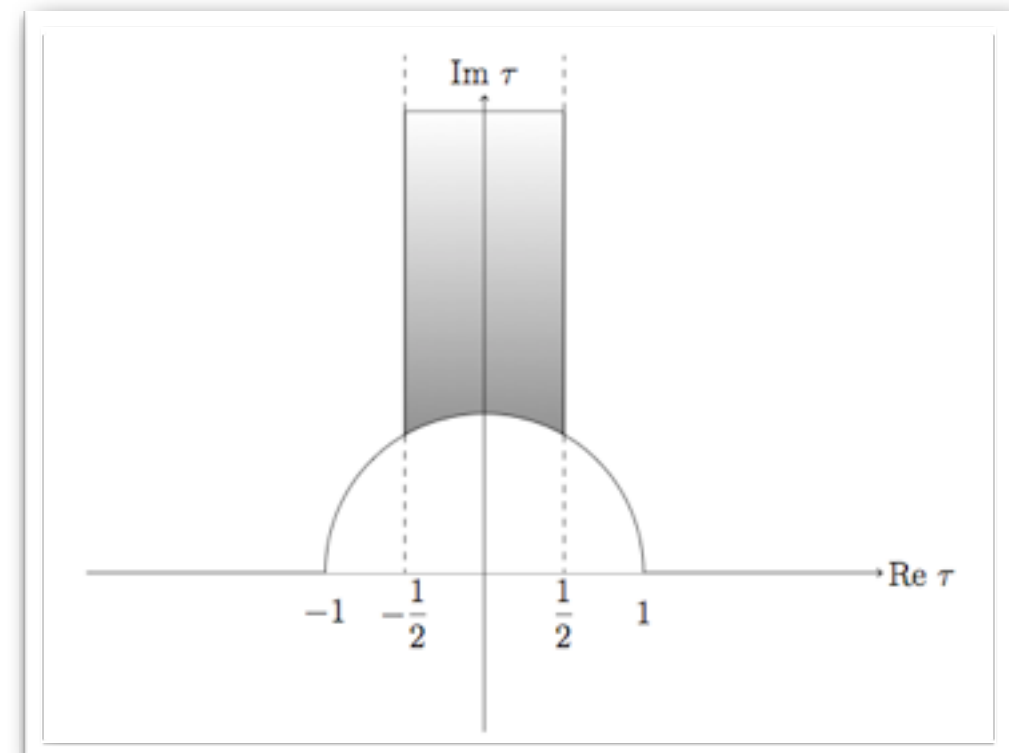
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$$\alpha' = 1$$



The problem at hand



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The known way out is a procedure that goes by
the name “**orbit method**” or simply “**unfolding**”

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- τ_1 : imposes **level matching**
- τ_2 : **Schwinger representation**

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NOT invariant under T-duality !



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Absolute convergence: **interchange integration with summation**



Traditional unfolding against the lattice

Traditionally : use **orbit decomposition of the lattice** (in **Lagrangian** rep.)

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Unfolding against the lattice is useful for extracting the **large volume behaviour** of the amplitude



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- Useful for extracting **asymptotic** behaviour in **large volume** limit



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Idea : Let's unfold against something else !

Goal: find some other way to unfold that **does not spoil** the manifest **T-duality** symmetries of the lattice

Look for representation that captures the behaviour around T-self-dual points

1 $\int_{\mathcal{F}} d\mu \Gamma_{d,d}(G, B; \tau, \bar{\tau})$ Rankin-Selberg-Zagier method

↓ deform

$\int_{\mathcal{F}_T} d\mu \Gamma_{d,d}(G, B; \tau, \bar{\tau}) E^*(s; \tau) \xrightarrow{\text{unfold}} \int_0^\infty d\tau_2 \tau_2^{s-2} \int_{-1/2}^{1/2} d\tau_1 \tau_2^{d/2} \sum_{m,n} e^{-2\pi\tau_2 \mathcal{M}^2} e^{2\pi i \tau_1 m^T n}$

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- No need for delicate regularization of degenerate orbit

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(unphysical tachyon)



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2 $\int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y; \tau, \bar{\tau}) \Phi(\tau)$ unfold the elliptic genus !

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$$\begin{aligned}\mathcal{F}(s, \kappa, w) &= \frac{1}{2} \sum_{\gamma \in SL(2; \mathbb{Z}) / \Gamma_{\infty}} (c\tau + d)^{-w} \mathcal{M}_{s,w}(-\kappa \operatorname{Im} \gamma \cdot \tau) e^{-2\pi i \kappa \operatorname{Re}(\gamma \cdot \tau_1)} \\ &= \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-w} \mathcal{M}_{s,w} \left(-\frac{\kappa \tau_2}{|c\tau + d|^2} \right) \exp \left\{ -2\pi i \kappa \left(\frac{a}{c} - \frac{c\tau_1 + d}{c|c\tau + d|^2} \right) \right\}\end{aligned}$$



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- **Converges absolutely** for $\operatorname{Re}(s) > 1$



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Spectrum is obtained by studying Fourier expansion & using **raising** and **lowering operators**



The spectrum of modular forms as limits of the N-P series



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Theorem



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$w = 0$

$$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2, 1, 0) - 5 \mathcal{F}(1, 1, 0) - 144$$

$$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \frac{1}{5} \mathcal{F}(3, 1, 0) - 4 \mathcal{F}(2, 1, 0) + 13 \mathcal{F}(1, 1, 0) + 144$$

$$\frac{\hat{E}_2^3 E_6}{\Delta} = \frac{3}{175} \mathcal{F}(4, 1, 0) - \frac{3}{5} \mathcal{F}(3, 1, 0) + \frac{33}{5} \mathcal{F}(2, 1, 0) - 17 \mathcal{F}(1, 1, 0) - 144$$

$$\frac{\hat{E}_2^4 E_4}{\Delta} = \frac{1}{1225} \mathcal{F}(5, 1, 0) - \frac{6}{175} \mathcal{F}(4, 1, 0) + \frac{18}{35} \mathcal{F}(3, 1, 0) - \frac{16}{5} \mathcal{F}(2, 1, 0) + \frac{29}{5} \mathcal{F}(1, 1, 0) + \frac{144}{5}$$

$$\frac{\hat{E}_2^6}{\Delta} = \frac{1}{1926925} \mathcal{F}(7, 1, 0) - \frac{3}{2695} \mathcal{F}(5, 1, 0) + \frac{6}{175} \mathcal{F}(4, 1, 0) - \frac{3}{7} \mathcal{F}(3, 1, 0) + \frac{12}{5} \mathcal{F}(2, 1, 0) - \frac{29}{7} \mathcal{F}(1, 1, 0) - \frac{144}{7}$$

$w = -2$

$$\frac{\hat{E}_2 E_4^2}{\Delta} = \frac{1}{40} \mathcal{F}(3, 1, -2) - \frac{1}{3} \mathcal{F}(2, 1, -2)$$

$$\frac{\hat{E}_2^2 E_6}{\Delta} = \frac{1}{525} \mathcal{F}(4, 1, -2) - \frac{1}{20} \mathcal{F}(3, 1, -2) + \frac{11}{30} \mathcal{F}(2, 1, -2)$$

$$\frac{\hat{E}_2^3 E_4}{\Delta} = \frac{1}{11760} \mathcal{F}(5, 1, -2) - \frac{1}{350} \mathcal{F}(4, 1, -2) + \frac{9}{280} \mathcal{F}(3, 1, -2) - \frac{2}{15} \mathcal{F}(2, 1, -2)$$

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$w = -4$

$$\frac{\hat{E}_2 E_6}{\Delta} = \frac{1}{2520} \mathcal{F}(4, 1, -4) - \frac{1}{120} \mathcal{F}(3, 1, -4)$$

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$w = -6$

$$\frac{\hat{E}_2 E_4}{\Delta} = \frac{1}{241920} \mathcal{F}(5, 1, -6) - \frac{1}{10080} \mathcal{F}(4, 1, -6)$$

$$\frac{\hat{E}_2^3}{\Delta} = \frac{1}{792792000} \mathcal{F}(7, 1, -6) - \frac{1}{887040} \mathcal{F}(5, 1, -6) + \frac{1}{50400} \mathcal{F}(4, 1, -6)$$

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$$\frac{\hat{E}_2^2}{\Delta} = \frac{1}{2854051200} \mathcal{F}(7, 1, -8) - \frac{1}{3991680} \mathcal{F}(5, 1, -8)$$

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Unfolding against the N-P series gives a BPS sum



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$$R.N. \int_F d\mu \Gamma_{(d+k,d)} \mathcal{F}(s, \kappa, -\frac{k}{2}) = \lim_{T \rightarrow \infty} \left[\int_{\mathcal{F}_T} d\mu \Gamma_{(d+k,d)} \mathcal{F}(s, \kappa, -\frac{k}{2}) + f_0(s) \frac{T^{\frac{d}{2} + \frac{k}{4} - s}}{s - \frac{d}{2} - \frac{k}{4}} \right]$$

$$= \int_0^\infty d\tau_2 \tau_2^{d/2-2} \mathcal{M}_{s, -\frac{k}{2}}(-\kappa\tau_2) \sum_{\text{BPS}} e^{-\pi\tau_2 (P_L^2 + P_R^2)/2}$$

for generic values of $s \neq \frac{d}{2} + \frac{k}{4}$

$$I = (4\pi\kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{d}{2} + \frac{k}{4} - 1)$$

$$\times \sum_{\text{BPS}} {}_2F_1 \left(s - \frac{k}{4}, s + \frac{d}{2} + \frac{k}{4} - 1; 2s; \frac{4\kappa}{P_L^2} \right) \left(\frac{P_L^2}{4\kappa} \right)^{1-s-\frac{d}{2}-\frac{k}{4}}$$

- For $\text{Re}(s) > d/2 + k/4$, sum converges absolutely, with a **simple pole** at $s = d/2 + k/4$
- Manifestly T-duality invariant
- Chamber independent



BPS state sums & Singularity Structure



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This is the appropriate representation to read-off the **singularity structure** of the **integral** around **extended symmetry points**



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Amplitudes involving linear
combinations of modular
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One-dimensional lattice



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$$\int_{\mathcal{F}} d\mu \, \Gamma_{(1,1)}(R) \, \mathcal{F}(1+n, 1, 0) = 2^{2+2n} \sqrt{\pi} \, \Gamma(n + \tfrac{1}{2}) \left(R^{1+2n} + \frac{1}{R^{1+2n}} - \left| R^{1+2n} - \frac{1}{R^{1+2n}} \right| \right)$$



Example of Gauge Threshold calculations



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$\mathcal{N} = 2$ heterotic vacuum at the **orbifold** point $T^2 \times T^4 / \mathbb{Z}_2$



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$\mathcal{N} = 2$ heterotic vacuum at the orbifold point $T^2 \times T^4/\mathbb{Z}_2$

In the absence of Wilson lines

$$E_8 \times E_8 \rightarrow E_8 \times E_7 \times SU(2)$$



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Without Wilson lines:



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Results **regular** at any point in moduli space **and in any chamber** !



One-loop BPS amplitudes with momentum insertions



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Consider modular integrals with **insertions** of left/right- moving **lattice momenta**:



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$$\int_{\mathcal{F}} d\mu \left[\tau_2^{-\lambda/2} \sum_{P_L, P_R} \rho(P_L \sqrt{\tau_2}, P_R \sqrt{\tau_2}) q^{\frac{1}{4} P_L^2} \bar{q}^{\frac{1}{4} P_R^2} \right] \Phi(\tau)$$



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Further simplification, when $\rho(x,y)$ is polynomial



An example from **non-compact** heterotic vacua



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Non-trivial integrals without moduli dependence



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L. Carlevaro, D. Israël 2012



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Unfold à la Niebur:

$$\frac{\hat{E}_2^2 E_4^2}{\Delta} - 2 \frac{\hat{E}_2^2 E_4 E_6}{\Delta} = \frac{1}{5} \mathcal{F}(3, 1, 0) - 6 \mathcal{F}(2, 1, 0) + 23j + 984$$



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Fourier Expansions & Borchers Products

$$\int_{\mathcal{F}} d\mu \, \Gamma_{(d,d)} \mathcal{F}(s, \kappa, 0) = (4\pi\kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{d}{2} - 1) \sum_{\text{BPS}} {}_2F_1 \left(s, s + \frac{d}{2} - 1; 2s; \frac{4\kappa}{P_L^2} \right) \left(\frac{P_L^2}{4\kappa} \right)^{1-s-\frac{d}{2}}$$



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 &= 2^{2s-1} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} \kappa^{s-\frac{1}{2}} \sigma_{1-2s}(\kappa) E(s; U) \\
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 &= -24 \log[\cancel{T_2 U_2} |\eta(T)\eta(U)|^4] - \log |j(T) - j(U)|^4
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Fourier Expansions & Borcherds Products

$$\begin{aligned}
 \int_{\mathcal{F}} d\mu \Gamma_{(d,d)} \mathcal{F}(s, \kappa, 0) &= (4\pi\kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{d}{2} - 1) \sum_{\text{BPS}} {}_2F_1 \left(s, s + \frac{d}{2} - 1; 2s; \frac{4\kappa}{P_L^2} \right) \left(\frac{P_L^2}{4\kappa} \right)^{1-s-\frac{d}{2}} \\
 &= 2^{2s-1} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} \kappa^{s-\frac{1}{2}} \sigma_{1-2s}(\kappa) E(s; U) \\
 &\quad + 2\sqrt{T_2} \sum_{N>0} \sum_{d|\kappa} \sqrt{\frac{d}{\kappa|N|}} e^{2\pi i N d T_1} K_{s-\frac{1}{2}}(2\pi|N|dT_2) \mathcal{F}_U(s, \frac{|N|\kappa}{d}, 0)
 \end{aligned}$$

Example: $s=\kappa=1$

$$\begin{aligned}
 \int_{\mathcal{F}} d\mu \Gamma_{(2,2)} (j + \cancel{24}) \\
 &= -12 \log [4\pi T_2 U_2 |\eta(U)|^4] + \sum_{N>0} \frac{1}{|N|} T_{|N|} \cdot (j(U) + 24) e^{2\pi(|N|T_2 + iNT_1)} + \text{c.c.} \\
 &= -24 \log [\cancel{T_2 U_2} |\eta(T)\eta(U)|^4] - \log |j(T) - j(U)|^4
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Example: $s=2$



Fourier Expansions & Borcherds Products

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$$\int_{\mathcal{F}} d\mu \Gamma_{(2,2)} \left(\frac{\hat{E}_2 E_4 E_6}{\Delta} + 5(j + 24) + 144 \right)$$



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Example: $s=2$

$$\begin{aligned}
 \int_{\mathcal{F}} d\mu \Gamma_{(2,2)} &\left(\frac{\hat{E}_2 E_4 E_6}{\Delta} + 5(j + 24) + 144 \right) \\
 &= \sum_{M,N} \tilde{c}(MN) \left[\text{Li}_1(q_T^N q_U^M) + \frac{1}{2\pi N T_2} \text{Li}_2(q_T^N q_U^M) \right] \\
 &\quad - c_2(MN) \left[\frac{3N}{\pi U_2} \text{Li}_2(q_T^N q_U^M) + \frac{3}{2\pi^2 T_2 U_2} \text{Li}_3(q_T^N q_U^M) \right] + \text{c.c.} \\
 &\quad + \frac{8\pi}{T_2} \left[\frac{\pi^2}{90} + \frac{\zeta(3)}{2\pi U_2} + \sum_{m>0} \frac{\sigma_3(m)}{m^2} \left(1 + \frac{1}{2\pi m U_2} \right) (q_U^m + \bar{q}_U^m) \right]
 \end{aligned}$$





Modular Integrals: Current Status



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- 1 $\int_{\mathcal{F}} d\mu \Phi(\tau)$
- 2 $\int_{\mathcal{F}} d\mu \Gamma_{d,d}(G, B; \tau, \bar{\tau})$
- 3 $\int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y; \tau, \bar{\tau}) \Phi(\tau)$
- 4 $\int_{\mathcal{F}} d\mu \mathcal{Z}(\tau, \bar{\tau})$



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Stokes theorem

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Unfold the elliptic genus
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- 3 $\int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y; \tau, \bar{\tau}) \Phi(\tau)$ Unfold the elliptic genus
(Niebur-Poincaré)
- 4 $\int_{\mathcal{F}} d\mu \mathcal{Z}(\tau, \bar{\tau})$ No general approach... yet !





Conclusions & Outlook



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- ☑ **Even in the absence of the lattice itself !**



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- ☑ Higher genus amplitudes ($g=2,3$)
- ☑ **Effective potential** of strings at finite temperature (String Cosmology)





Max-Planck-Institut für Physik
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Thank You !



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