XXV. Workshop Beyond The Standard Model

# Superstring Tree Amplitudes

# Recycling Field Theory Structures

Oliver Schlotterer (AEI Potsdam & DAMTP Cambridge)

1106.2645, 1106.2646: C. Mafra, OS, St. Stieberger

Work in progress: J. Brödel, OS, St. Stieberger

19.03.2013

# Superstring tree amplitudes recycling field theory structures

I. Basics of tree amplitudes in string and field theory

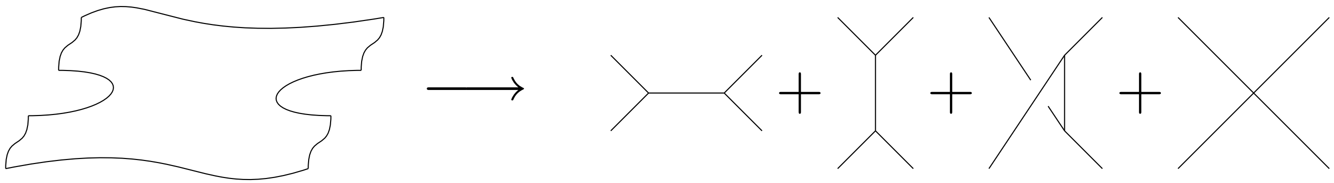
II. Open string tree amplitude

III. From field theory structures to string corrections

# I. Basics of tree amplitudes in string and field theory

- Open strings encompass gauge theory, closed strings encompass gravity.
- Regge slope $\alpha' \equiv \ell^2_{\text{string}}$ controls mass of additional vibration modes.
- String interactions enconded in worldsheet (generalizing worldlines)

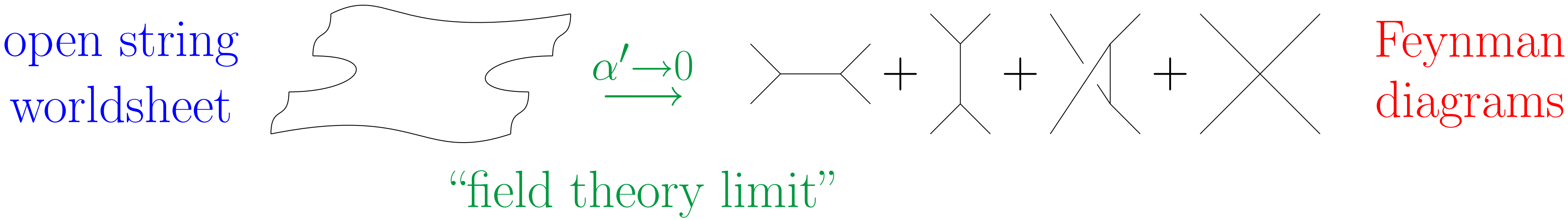
open string worldsheet



Feynman diagrams

# I. Basics of tree amplitudes in string and field theory

- Open strings encompass gauge theory, closed strings encompass gravity.
- Regge slope $\alpha' \equiv \ell^2_{\text{string}}$ controls mass of additional vibration modes.
- String interactions enconded in worldsheet (generalizing worldlines)



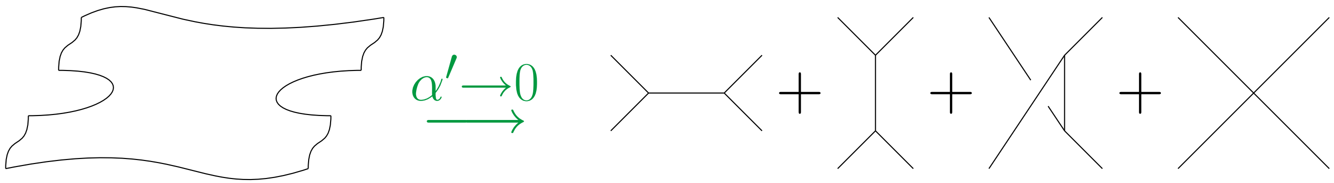
# I. Basics of tree amplitudes in string and field theory

- Open strings encompass gauge theory, closed strings encompass gravity.

- Regge slope $\alpha' \equiv \ell^2_{\text{string}}$ controls mass of additional vibration modes.

- String interactions enconded in worldsheet (generalizing worldlines)

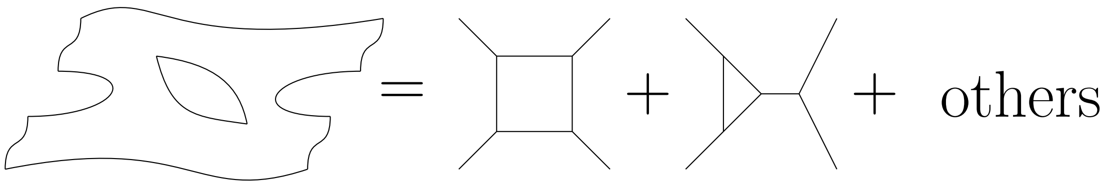
open string worldsheet




Feynman diagrams

"field theory limit"

- Perturbative loop expansion controlled by worldsheet topology




- In this talk: tree level (genus zero) interplay string- versus field theory

# I. 1 Color decomposition

Open string tree amplitudes from cyclically inequivalent disk orderings

$$\mathcal{M}_N^{\text{open}} = \sum_{\sigma\in S_{N-1}/\mathbb{Z}_2} \underbrace{\text{Tr}\{T^1\, T^{\sigma(2)} \ldots T^{\sigma(N)}\}}_{\text{gauge group generators}} \underbrace{\mathcal{A}^{\text{open}}\big(1,\sigma(2),\ldots,\sigma(N)\big)}_{\text{color stripped subamplitude}}$$

Cyclic & gauge invariant subamplitudes depend on kinematic d.o.f.

(polarizations $\xi^\mu, u^\alpha$; momenta $k^\mu$), NOT on color, e.g. $i \equiv (k_i^\mu, \xi_i^\mu)$.

# I. 1 Color decomposition

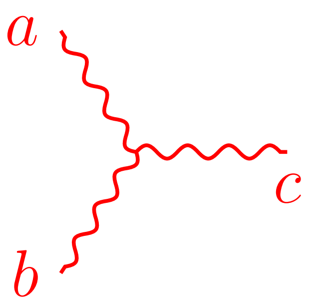
Open string & YM trees from cyclically inequivalent disk orderings

$$\mathcal{M}_N^{\left\{ \substack{\text{YM} \\ \text{open}} \right.} = \sum_{\sigma \in S_{N-1}/\mathbb{Z}_2} \underbrace{\text{Tr}\{T^1 T^{\sigma(2)} \ldots T^{\sigma(N)}\}}_{\text{gauge group generators}} \underbrace{\mathcal{A}^{\left\{ \substack{\text{YM} \\ \text{open}} \right.}\big(1, \sigma(2), \ldots, \sigma(N)\big)}_{\text{color stripped subamplitude}}$$

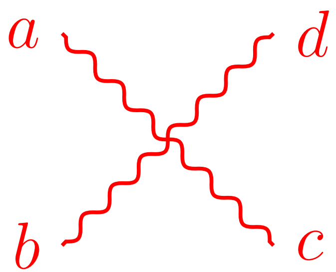
Cyclic & gauge invariant subamplitudes depend on kinematic d.o.f.

(polarizations $\xi^\mu, u^\alpha$; momenta $k^\mu$), NOT on color, e.g. $i \equiv (k_i^\mu, \xi_i^\mu)$.

Alternative color basis compared to YM Feynman rules $[T^a, T^b] = i f^{abc} T^c$.




$\sim \quad f^{abc}$ ,



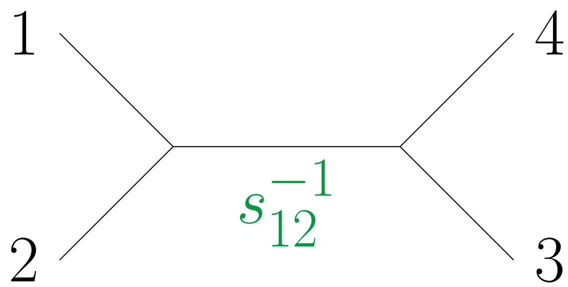

$\sim \quad \sum_e f^{abe} f^{ecd} \; + \;$ permutations

Throughout this talk, will use dimensionless Mandelstam variables:

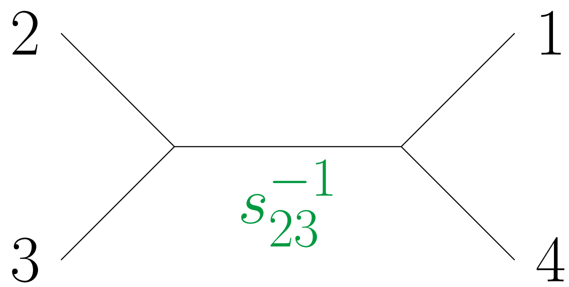
$$s_{ij} \;=\; \alpha' \,(k_i + k_j)^2 \,, \qquad s_{i_1 i_2 \ldots i_p} \;=\; \alpha' \,(k_{i_1} + k_{i_2} + \ldots + k_{i_p})^2$$

Propagators in $\mathcal{A}^{\left\{ {\rm YM} \atop {\rm open} \right.}\big(1, \sigma(2), \ldots, \sigma(N)\big)$ respect the cyclic ordering $\sigma$

$\mathcal{A}^{\left\{ {\rm YM} \atop {\rm open} \right.}(1,2,3,4) \;\sim$

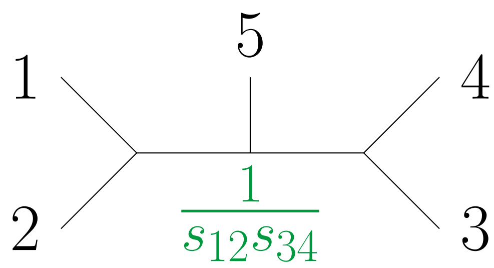



$+$




Note the absence of the $s_{13}^{-1}$ propagator in $\mathcal{A}^{\rm YM}(1,2,3,4)$!

$A^{\left\{ {\rm YM} \atop {\rm open} \right.}(1,2,3,4,5) \;\sim$




$+ \, {\rm cyc}(1,2,3,4,5)$

Can expand any YM subamplitude in a $(N-3)!$ element basis

$$\{\, \mathcal{A}^{\mathrm{YM}}\big(1, \pi(2), \ldots, \pi(N-2), N-1, N\big)\,,\ \pi \in S_{N-3}\,\}$$

using Kleiss Kuijf relations

$$\mathcal{A}^{\mathrm{YM}}(1,3,4,2,5) \;=\; -\mathcal{A}^{\mathrm{YM}}(1,2,3,4,5) \;-\; \mathcal{A}^{\mathrm{YM}}(1,3,2,4,5) \;-\; \underline{\mathcal{A}^{\mathrm{YM}}(1,3,4,5,2)}$$

Can expand any YM subamplitude in a $(N-3)!$ element basis

$$\{\, \mathcal{A}^{\mathrm{YM}}\big(1,\pi(2),\ldots,\pi(N-2),N-1,N\big)\,,\ \pi\in S_{N-3}\,\}$$

using Kleiss Kuijf relations and BCJ relations, e.g.

$$\mathcal{A}^{\mathrm{YM}}(1,3,4,2,5) \;=\; -\mathcal{A}^{\mathrm{YM}}(1,2,3,4,5) - \mathcal{A}^{\mathrm{YM}}(1,3,2,4,5) - \underline{\mathcal{A}^{\mathrm{YM}}(1,3,4,5,2)}$$

$$\underline{s_{25}\,\mathcal{A}^{\mathrm{YM}}(1,3,4,5,2)} \;=\; (s_{23}+s_{24})\,\mathcal{A}^{\mathrm{YM}}(1,2,3,4,5) + s_{24}\,\mathcal{A}^{\mathrm{YM}}(1,3,2,4,5)$$

**[Bern, Carrasco, Johansson 0805.3993]**

Can expand any YM subamplitude in a $(N-3)!$ element basis

$$\{\, \mathcal{A}^{\rm YM}\big(1,\pi(2),\ldots,\pi(N-2),N-1,N\big)\,,\ \pi\in S_{N-3}\,\}$$

using Kleiss Kuijf relations and BCJ relations, e.g.

$$\mathcal{A}^{\rm YM}(1,3,4,2,5) \ = \ -\mathcal{A}^{\rm YM}(1,2,3,4,5) \ - \ \mathcal{A}^{\rm YM}(1,3,2,4,5) \ - \ \underline{\mathcal{A}^{\rm YM}(1,3,4,5,2)}$$

$$\underline{s_{25}\,\mathcal{A}^{\rm YM}(1,3,4,5,2)} \ = \ (s_{23}+s_{24})\,\mathcal{A}^{\rm YM}(1,2,3,4,5) \ + \ s_{24}\,\mathcal{A}^{\rm YM}(1,3,2,4,5)$$

**[Bern, Carrasco, Johansson 0805.3993]**

They nicely follow from monodromy relations between string amplitudes

via $\alpha'\to 0$ limit $e^{i\pi s}=1+i\pi s+\ldots$ of the real- and imaginary part.

**[Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]**

$$0 \ = \ \mathcal{A}^{\rm open}(1,3,4,2,5) \ + \ e^{-i\pi s_{25}}\,\mathcal{A}^{\rm open}(1,3,4,5,2)$$

$$+ \ e^{i\pi s_{24}}\,\mathcal{A}^{\rm open}(1,3,2,4,5) \ + \ e^{i\pi(s_{23}+s_{24})}\,\mathcal{A}^{\rm open}(1,2,3,4,5)$$

# I. 2 KLT formula for gravity amplitudes

Closed string spectrum is double copy of (colorless) open string spectrum,

$$\text{e.g.} \quad |\text{graviton}\rangle_{\text{closed}} \quad = \quad |\text{gluon}\rangle_{\text{open}} \otimes |\text{gluon}\rangle_{\text{open}}$$

and this has consequences for scattering amplitudes.

# I. 2 KLT formula for gravity amplitudes

Closed string spectrum is double copy of (colorless) open string spectrum,

$$\text{e.g.} \quad |\text{graviton}\rangle_{\text{closed}} = |\text{gluon}\rangle_{\text{open}} \otimes |\text{gluon}\rangle_{\text{open}}$$

and this has consequences for scattering amplitudes.

KLT formula relates closed string tree amplitude to (open string tree)$^2$

$$\mathcal{M}_4^{\text{closed}} = \mathcal{A}^{\text{open}}(1,2,3,4)\sin(\pi s_{12})\,\tilde{\mathcal{A}}^{\text{open}}(1,2,4,3)$$

$$\mathcal{M}_5^{\text{closed}} = \mathcal{A}^{\text{open}}(1,2,3,4,5)\sin(\pi s_{12})\sin(\pi s_{34})\,\tilde{\mathcal{A}}^{\text{open}}(2,1,4,3,5) + (2 \leftrightarrow 3)$$

$$\mathcal{M}_N^{\text{closed}} \equiv \sum \mathcal{A}^{\text{open}}(\ldots)\left[\,\sin(\pi s_{ij})\,\right]^{N-3}\tilde{\mathcal{A}}^{\text{open}}(\ldots)$$

**[Kawai, Lewellen, Tye 1986]**

with $\sin(\pi s_{ij})$ determined by monodromy properties on the worldsheet.

Field theory limit $\mathcal{A}^{\text{open}}(\ldots) \to \mathcal{A}^{\text{YM}}(\ldots)$ and $\sin(\pi s_{ij}) \to \pi s_{ij}$

$\Rightarrow$ trees of perturbative gravity (coupled to dilaton and $B$-field)

$$\mathcal{M}_N^{\text{gravity}} = \sum_{\pi,\rho \in S_{N-3}} \mathcal{A}^{\text{YM}}(1,\pi,N-1,N)\, S[\pi,\rho]_1\, \tilde{\mathcal{A}}^{\text{YM}}(1,\rho,N,N-1)$$

**[Bern, Dixon, Perelstein, Rozowsky 9811140]**

$\exists$ many ways of rewriting, here we use momentum kernel $S[\pi,\rho]_1 \sim s_{ij}^{N-3}$

$$\begin{pmatrix} S[23|23]_1 & S[32|23]_1 \\ S[23|32]_1 & S[32|32]_1 \end{pmatrix} = \begin{pmatrix} s_{12}(s_{13}+s_{23}) & s_{12}s_{13} \\ s_{12}s_{13} & s_{13}(s_{12}+s_{23}) \end{pmatrix}$$

**[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove 1010.3933]**

Field theory limit $\mathcal{A}^{\text{open}}(\ldots) \to \mathcal{A}^{\text{YM}}(\ldots)$ and $\sin(\pi s_{ij}) \to \pi s_{ij}$

$\Rightarrow$ trees of perturbative gravity (coupled to dilaton and $B$-field)

$$\mathcal{M}_N^{\text{gravity}} = \sum_{\pi,\rho \in S_{N-3}} \mathcal{A}^{\text{YM}}(1,\pi,N-1,N)\, S[\pi,\rho]_1\, \tilde{\mathcal{A}}^{\text{YM}}(1,\rho,N,N-1)$$

**[Bern, Dixon, Perelstein, Rozowsky 9811140]**

$\exists$ many ways of rewriting, here we use momentum kernel $S[\pi,\rho]_1 \sim s_{ij}^{N-3}$

$$\begin{pmatrix} S[23|23]_1 & S[32|23]_1 \\ S[23|32]_1 & S[32|32]_1 \end{pmatrix} = \begin{pmatrix} s_{12}(s_{13}+s_{23}) & s_{12}s_{13} \\ s_{12}s_{13} & s_{13}(s_{12}+s_{23}) \end{pmatrix}$$

**[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove 1010.3933]**

At loop level, squaring relations $(\text{YM})^2 \to$ gravity act on cubic diagrams

**[Bern, Carrasco, Johansson 1004.0476]**

**[Bern, Carrasco, Dixon, Johansson, Roiban 1201.5366]**

# II. Open string tree amplitude

Color stripped tree amplitude for scattering $N$ massless open string states

$$\mathcal{A}^{\text{open}}(1,2,\ldots,N) \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\text{YM}}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\; F^\pi$$

**[Mafra, OS, Stieberger 1106.2645, 1106.2646]**

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\text{YM}}_{\pi\in S_{N-3}}$
- string effects ($\alpha'$ dependence) from generalized Euler integrals $F^\pi(\alpha')$

# II. Open string tree amplitude

Color stripped tree amplitude for scattering $N$ massless open string states

$$\mathcal{A}^{\text{open}}(1,2,\ldots,N) \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\text{YM}}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\, F^\pi$$

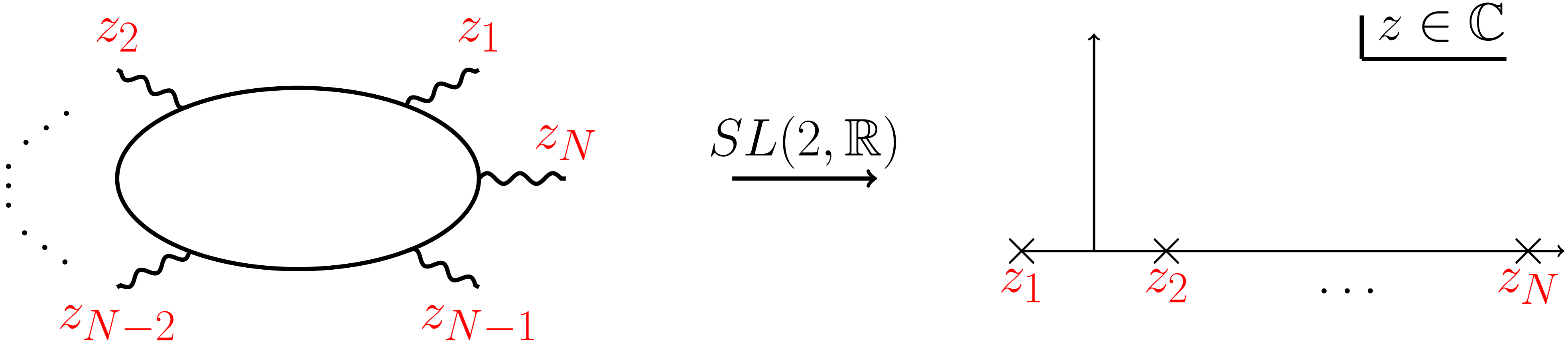
**[Mafra, OS, Stieberger 1106.2645, 1106.2646]**

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\text{YM}}_{\pi\in S_{N-3}}$
- string effects ($\alpha'$ dependence) from generalized Euler integrals $F^\pi(\alpha')$
- consistent with field theory limit: $F^\pi(\alpha'\to 0) \;=\; \delta^\pi_{(2,3,\ldots,N-2)}$
- valid for states of $\mathcal{N}=1$ SYM in $D=10$ (or $\mathcal{N}=4$ SYM in $D=4$)
- remain valid for the gluon's SUSY multiplet for $\mathcal{N}<4$ compactification

## II. 1 Disk integrals

$\mathcal{A}^{\text{open}}(1,2,\ldots,N)$ built from ordered integrals over worldsheet boundary

$$Z(1,\rho(2,3,\ldots,N)) \sim \int_{z_i<z_{i+1}} \frac{\prod_{i<j}^N |z_{ij}|^{s_{ij}}}{z_{1,\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-1),\rho(N)} z_{\rho(N),1}}$$




- totally symmetric Koba-Nielsen factor $\prod_{i<j}^N |z_{ij}|^{s_{ij}} = \langle \prod_{j=1}^N e^{ik_j\cdot X(z_j)} \rangle$
- $\rho \in S_{N-1}/\mathbb{Z}_2$ cycle of $N$ factors $(z_{ij})^{-1}$ with $z_{ij} := z_i - z_j$
- no subcycles within the $z_{ij}^{-1}$: They would cause tachyon propagation

In practice, fix three vertex positions $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$

$\Rightarrow$ mod out by conformal redundancy

$$Z(1, \rho(2, 3, \ldots, N)) = \int_{z_i < z_{i+1}} \prod_{k=2}^{N-2} \mathrm{d}z_k \, \frac{z_{1,N-1}\, z_{1,N}\, z_{N-1,N} \prod_{i<j}^{N} |z_{ij}|^{s_{ij}}}{z_{1,\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-1),\rho(N)} z_{\rho(N),1}}$$

In the subset $Z(1, \ldots, N-1)$, the Jacobian $z_{1,N-1} z_{1,N} z_{N-1,N}$ cancels

$$Z(1, \rho(2, \ldots, N-2, N), N-1) = \int_{z_i < z_{i+1}} \prod_{k=2}^{N-2} \mathrm{d}z_k \, \frac{\prod_{i<j}^{N} |z_{ij}|^{s_{ij}}}{\rho[\,(z_{pq})^{N-3}\,]}$$

# II. 2 KLT formula $\leftrightarrow$ open string disk amplitude

Recall KLT formula for gravity trees (momentum kernel $S[\pi,\rho]_1 \sim s_{ij}^{N-3}$)

$$\mathcal{M}_N^{\text{gravity}} = \sum_{\pi,\rho\in S_{N-3}} \mathcal{A}^{\text{YM}}(1,\pi,N-1,N)\, S[\pi,\rho]_1\, \tilde{\mathcal{A}}^{\text{YM}}(1,\rho,N,N-1)$$

# II. 2 KLT formula ↔ open string disk amplitude

Recall KLT formula for gravity trees (momentum kernel $S[\pi,\rho]_1 \sim s_{ij}^{N-3}$)

$$\mathcal{M}_N^{\text{gravity}} = \sum_{\pi,\rho \in S_{N-3}} \mathcal{A}^{\text{YM}}(1,\pi,N-1,N)\, S[\pi,\rho]_1\, \tilde{\mathcal{A}}^{\text{YM}}(1,\rho,N,N-1)$$

Translates to open string disk amplitude by replacing $\tilde{\mathcal{A}}^{\text{YM}} \mapsto Z$

$$\mathcal{A}^{\text{open}}(1,2,\ldots,N) = \sum_{\pi,\rho \in S_{N-3}} \mathcal{A}^{\text{YM}}(1,\pi,N-1,N)\, S[\pi,\rho]_1\, Z(1,\rho,N,N-1)$$

Manifests duality of kinematic factors $\tilde{\mathcal{A}}^{\text{YM}}$ to worldsheet integral $Z$.

**[Brödel, OS, Stieberger: work in progress]**

# II. 3 KK $\leftrightarrow$ partial fraction, BCJ $\leftrightarrow$ partial integration

The $\mathcal{A}^{\rm YM}$ and $Z$ obey dual systems of equations

| reduction to | $\mathcal{A}^{\rm YM}(\pi)$ | $Z(\pi)$ |
|---|---|---|
| $\frac{1}{2}(N-1)!$ | cyclicity & parity | conformal invariance & no tachyons |
| $(N-2)!$ | Kleiss Kuijf | partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ |
| $(N-3)!$ | BCJ | integration by parts: $\sum_{i\neq j} \frac{s_{ij}}{z_{ij}} \equiv 0$ |

# II. 3 KK $\leftrightarrow$ partial fraction, BCJ $\leftrightarrow$ partial integration

The $\mathcal{A}^{\rm YM}$ and $Z$ obey dual systems of equations

| reduction to | $\mathcal{A}^{\rm YM}(\pi)$ | $Z(\pi)$ |
|---|---|---|
| $\frac{1}{2}(N-1)!$ | cyclicity & parity | conformal invariance & no tachyons |
| $(N-2)!$ | Kleiss Kuijf | partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ |
| $(N-3)!$ | BCJ | integration by parts: $\sum_{i\neq j} \frac{s_{ij}}{z_{ij}} \equiv 0$ |

Worldsheet duals of KK relations $\equiv$ partial fraction in the integrand

$$0 \ = \ \mathcal{A}^{\rm YM}(1,2,3,4) \ + \ \mathcal{A}^{\rm YM}(1,3,2,4) \ + \ \mathcal{A}^{\rm YM}(1,2,4,3)$$

$$0 \ = \ Z(1,2,3,4) \ + \ Z(1,3,2,4) \ + \ Z(1,2,4,3)$$

$$\sim \int_{z_i<z_{i+1}} \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \left( \frac{1}{z_{12}z_{23}z_{34}z_{41}} + \frac{1}{z_{12}z_{24}z_{43}z_{31}} + \frac{1}{z_{13}z_{32}z_{24}z_{41}} \right)$$

Worldsheet duals the BCJ relations

$$0 \;=\; s_{13}\,\mathcal{A}^{\rm YM}(1,3,2,5,4) \;+\; (s_{13}+s_{23})\,\mathcal{A}^{\rm YM}(1,2,3,5,4) \;-\; s_{34}\,\mathcal{A}^{\rm YM}(1,2,5,3,4)$$

is the vanishing of

$$0 \stackrel{!}{=} \; s_{13}\,Z(1,3,2,5,4) \;+\; (s_{13}+s_{23})\,Z(1,2,3,5,4) \;-\; s_{34}\,Z(1,2,5,3,4)$$

# Worldsheet duals the BCJ relations

$$0 \;=\; s_{13}\,\mathcal{A}^{\rm YM}(1,3,2,5,4) \;+\; (s_{13}+s_{23})\,\mathcal{A}^{\rm YM}(1,2,3,5,4) \;-\; s_{34}\,\mathcal{A}^{\rm YM}(1,2,5,3,4)$$

is the vanishing of

$$\begin{aligned} 0 =&\; s_{13}\,Z(1,3,2,5,4) \;+\; (s_{13}+s_{23})\,Z(1,2,3,5,4) \;-\; s_{34}\,Z(1,2,5,3,4) \\ =& \int\limits_{z_i<z_{i+1}} \mathrm{d}z_3\,\mathrm{d}z_2 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}}\,\frac{1}{z_{12}}\left(\frac{s_{13}}{z_{13}}+\frac{s_{23}}{z_{23}}+\frac{s_{34}}{z_{43}}\right) \\ =& \int\limits_{z_i<z_{i+1}} \mathrm{d}z_3\,\mathrm{d}z_2\,\frac{\mathrm{d}}{\mathrm{d}z_3}\left[\frac{1}{z_{21}}\prod_{i<j}^{4}|z_{ij}|^{s_{ij}}\right] \end{aligned}$$

Total derivatives integrate to zero, regardless of the integration range!

# III. From field theory structures to string corrections

Complete disk amplitude encoded in $(N-3)!$ monodromy independent

$$\mathcal{A}_{\Sigma}^{\text{open}} \equiv \mathcal{A}^{\text{open}}(1, \Sigma(2, \ldots, N-2), N-1, N)$$

# III. From field theory structures to string corrections

Complete disk amplitude encoded in $(N-3)!$ monodromy independent

$$\mathcal{A}^{\rm open}_{\Sigma} \equiv \mathcal{A}^{\rm open}(1,\Sigma(2,\ldots,N-2),N-1,N)$$

Their string corrections can be aligned into $(N-3)!\times(N-3)!$ matrix

$$F_{\Sigma}{}^{\pi} = \sum_{\rho\in S_{N-3}} S[\,\pi\,|\,\rho\,]\,\underbrace{Z_{\Sigma}(1,\rho(2,\ldots,N-2),N,N-1)}_{\text{integrate over } z_{\Sigma(i)}<z_{\Sigma(i+1)}}$$

acting on $\mathcal{A}^{\rm YM}_{\pi} \equiv \mathcal{A}^{\rm YM}(1,\pi(2,\ldots,N-2),N-1,N)$:

$$\boxed{\mathcal{A}^{\rm open}_{\Sigma} = F_{\Sigma}{}^{\pi}\,\mathcal{A}^{\rm YM}_{\pi}}$$

Beautiful pattern in $\alpha'$ expansion of $F_{\Sigma}{}^{\pi}$ and multiple zeta values therein.

**[OS, Stieberger 1205.1516; Brödel, OS, Stieberger: work in progress]**

**[see talk by Johannes Brödel]**

Compute $\alpha'$ expansion of $F^\pi = S[\pi|\rho]_1 Z(1,\rho,N,N-1)$ at the level of

$$Z(1,\rho,N,N-1) \;=\; \int_{z_i<z_{i+1}} \prod_{k=2}^{N-2} \mathrm{d}z_k \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

Main information from Taylor expanding the Koba Nielsen factor

$$|z_{ij}|^{s_{ij}} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} (s_{ij} \, \ln |z_{ij}|)^n$$

which challenges to compute polylogarithmic integrals

$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} (\ln |z_{ij}|)^{n_{ij}} \; , \qquad \left\{ \begin{array}{c} n_{ij} \in \mathbb{N} \\ a_i \in \{0,1,z_{i+1},\ldots,z_{N-2}\} \end{array} \right.$$

**[see talk by Johannes Brödel]**

Compute $\alpha'$ expansion of $F^\pi = S[\pi|\rho]_1 Z(1,\rho,N,N-1)$ at the level of

$$Z(1,\rho,N,N-1) = \int_{z_i<z_{i+1}} \prod_{k=2}^{N-2} \mathrm{d}z_k \, \frac{\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-3),\rho(N-2)}}$$

Main information from Taylor expanding the Koba Nielsen factor

$$|z_{ij}|^{s_{ij}} = \sum_{n=0}^{\infty} \frac{1}{n!} (s_{ij} \ln |z_{ij}|)^n$$

which challenges to compute polylogarithmic integrals

$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} (\ln |z_{ij}|)^{n_{ij}} \ , \qquad \left\{ {n_{ij} \in \mathbb{N} \atop a_i \in \{0,1,z_{i+1},\ldots,z_{N-2}\}} \right.$$

**[see talk by Johannes Brödel]**

**But:** Kinematic poles in $Z$ spoil analyticity in the $s_{ij}$!

$\Rightarrow$ first subtract poles before Taylor expanding $\prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}$

# III. 1 Pole structure of the $Z$

Integration ranges $z_i \longrightarrow z_{i+1}$ give rise to kinematic poles in $Z(\ldots)$.

Let $f, g$ be regular at $x = 0, 1$,

$$\int_0^a \mathrm{d}x \, x^{s-1} f(x) \ \sim \ \frac{a^s}{s} f(0) \, , \qquad \int_{1-b}^1 \mathrm{d}x \, (1-x)^{s-1} g(x) \ \sim \ \frac{b^s}{s} g(1)$$

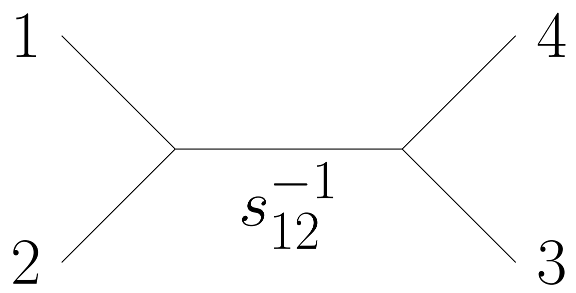
At $N = 4$ points, can immediately read off the poles in

$$Z(1,2,4,3) \ = \ \int_0^1 \mathrm{d}z_2 \, z_2^{s_{12}-1} (1-z_2)^{s_{23}} \ = \ \frac{1}{s_{12}} \ + \ \text{regular } \mathcal{O}(\alpha'^2)$$

$$Z(1,4,2,3) \ = \ \int_0^1 \mathrm{d}z_2 \, z_2^{s_{12}} (1-z_2)^{s_{23}-1} \ = \ \frac{1}{s_{23}} \ + \ \text{regular } \mathcal{O}(\alpha'^2)$$

To leading $\alpha'$ order, the $Z(\ldots)$ yield

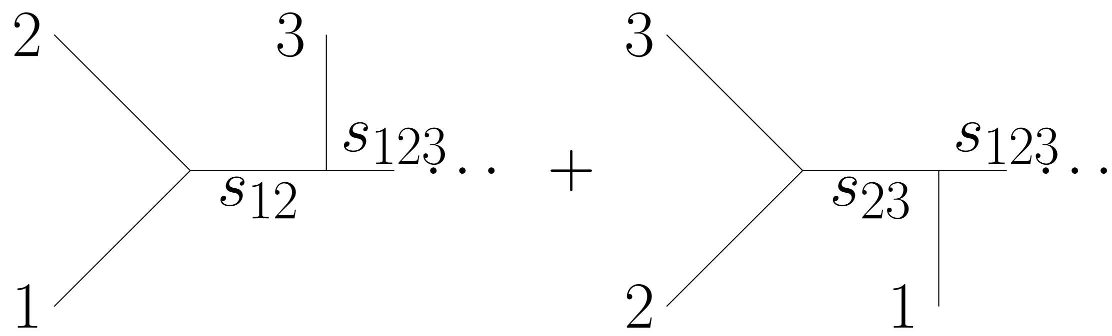
$N-3$ propagators in cubic YM diagrams



At higher $N$, have to take care of multiparticle channels, e.g.

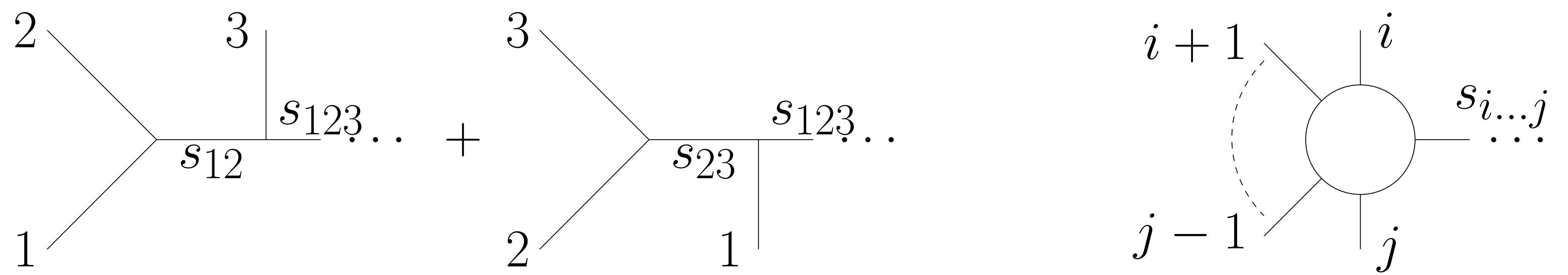
$$Z(1,2,3,5,4) = \int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\, |z_{12}|^{s_{12}-1}\, |z_{13}|^{s_{13}}\, |z_{23}|^{s_{23}-1}\, |z_{24}|^{s_{24}}\, |z_{34}|^{s_{34}}$$

$$= \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}} + \text{less singular } \mathcal{O}(\alpha'^2)$$



At higher $N$, have to take care of multiparticle channels, e.g.

$$Z(1,2,3,5,4) = \int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2\, |z_{12}|^{s_{12}-1}\, |z_{13}|^{s_{13}}\, |z_{23}|^{s_{23}-1}\, |z_{24}|^{s_{24}}\, |z_{34}|^{s_{34}}$$
$$= \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}} + \text{less singular } \mathcal{O}(\alpha'^2)$$




Keep track of poles occurring through the following "rule of thumb"

> $(j-i+1)$ particle pole $s_{i,i+1\dots j-1,j}$ triggered by
>
> $j-i$ factors of $z_{pq}$ in the range $i \le p < q \le j$.

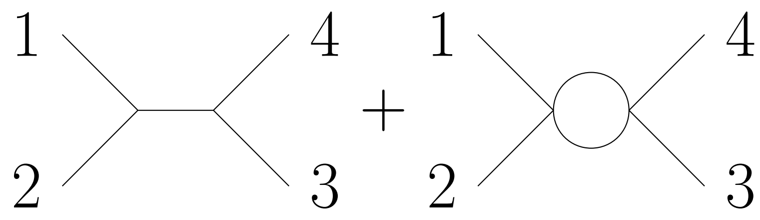
Can read off pole content from the permutation $Z(1,\rho,N-1)$.

# III. 2 Recursive structure in the residues

Reduce any massless pole residue to regular $< N$ point integrals $\mathcal{O}(\alpha'^{\geq 2})$

$$I^{\text{reg}}_{\{a_i\}} := \underbrace{\prod_{i=2}^{N-2}\int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i-a_i}}_{\text{reflects } \rho \text{ of } Z(\rho)} \underbrace{\prod_{i<j}^{N-1}\sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!}\left(s_{ij}\ \ln|z_{ij}|\right)^{n_{ij}}}_{\text{Taylor expanded Koba Nielsen}}$$

$$Z(1,2,4,3) = \frac{1}{s_{12}} + I^{\text{reg}}_0(k_1,k_2,k_3) =$$

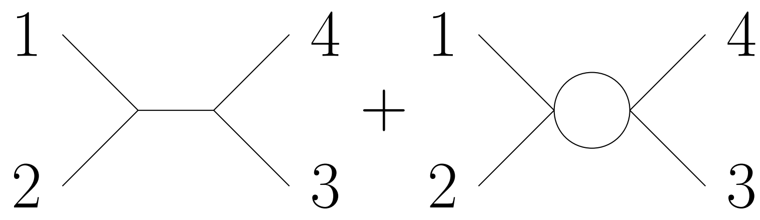


# III. 2 Recursive structure in the residues

Reduce any massless pole residue to regular $< N$ point integrals $\mathcal{O}(\alpha'^{\geq 2})$

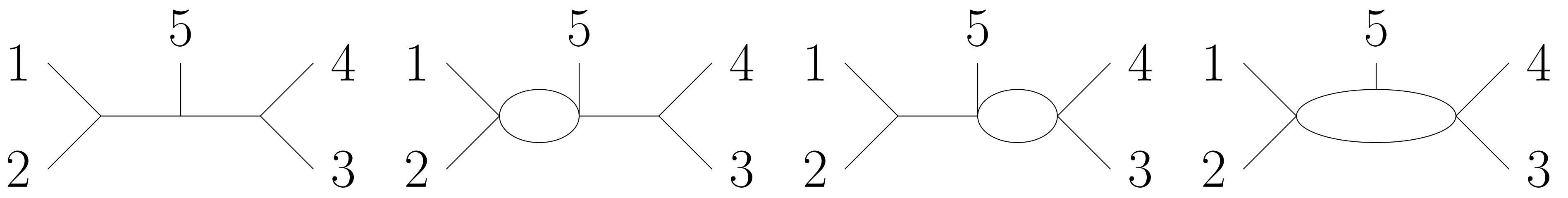
$$I^{\rm reg}_{\{a_i\}} \ := \ \underbrace{\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{{\rm d}z_i}{z_i - a_i}}_{\text{reflects } \rho \text{ of } Z(\rho)} \ \underbrace{\prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!} (s_{ij} \ \ln|z_{ij}|)^{n_{ij}}}_{\text{Taylor expanded Koba Nielsen}}$$

$$Z(1,2,4,3) \ = \ \frac{1}{s_{12}} \ + \ I^{\rm reg}_0(k_1,k_2,k_3) \ =$$




... possibly at shifted momenta $k_{ab} = k_a + k_b$ ...

$$Z(1,2,4,3,5) \ = \ \frac{1}{s_{12}s_{34}} \ + \ \frac{I^{\rm reg}_0(k_1,k_2,k_{34})}{s_{34}} \ + \ \frac{I^{\rm reg}_1(k_{12},k_3,k_4)}{s_{12}} \ + \ I^{\rm reg}_{0,1}$$



## Concluding remarks

- Open superstring tree amplitude strongly resembles KLT formula:

  Trade $\mathcal{A}^{\rm YM}(\pi) \mapsto$ disk integral $Z(\pi)$

- Disk integrals $Z(\pi)$ literally satisfy KK- and BCJ relations of the $\mathcal{A}^{\rm YM}(\pi)$

  $\Rightarrow$ $(N-3)!$ bases in both sectors

- Pole structure of $Z(\pi)$ can be understood recursively,

  $\alpha'$ expansion from polylogarithmic integrals over $\prod_{i<j} \ln |z_{ij}|^{n_{ij}}$

## Concluding remarks

- Open superstring tree amplitude strongly resembles KLT formula:

  Trade $\mathcal{A}^{\rm YM}(\pi) \mapsto$ disk integral $Z(\pi)$

- Disk integrals $Z(\pi)$ literally satisfy KK- and BCJ relations of the $\mathcal{A}^{\rm YM}(\pi)$

  $\Rightarrow$ $(N-3)!$ bases in both sectors

- Pole structure of $Z(\pi)$ can be understood recursively,

  $\alpha'$ expansion from polylogarithmic integrals over $\prod_{i<j} \ln |z_{ij}|^{n_{ij}}$

**Thank you for your attention !**