A first step toward a quantum duality map between F- and heterotic theories

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(work in progress with H. Jockers and T. Wotschke)

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Motivation

Duality: heterotic theory compactified on a K3 surface which is elliptically fibered over $\mathbb{P}^1$ with a section is dual to F-theory compactified on a 3 dim'l Calabi-Yau which is K3 fibered over $\mathbb{P}^1$ and also elliptically fibered with a section.

F-theory	heterotic theory
$\text{c.s. moduli of } K3$	$T^2 + \text{gauge bundle}$

moduli space

$$\mathcal{O}(\Gamma_{2,2+n})\backslash\mathcal{O}(2,2+n)/(\mathcal{O}(2)\times\mathcal{O}(2+n))$$

n: number of Wilson lines

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stable deg. limit		classical limit $\mathrm{vol}(T^2)\to\infty$

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Goal: understand the duality map beyond the boundary of the moduli space.

Confine ourselves to $E8\times E8$ heterotic theory with one Wilson loop.

Observation

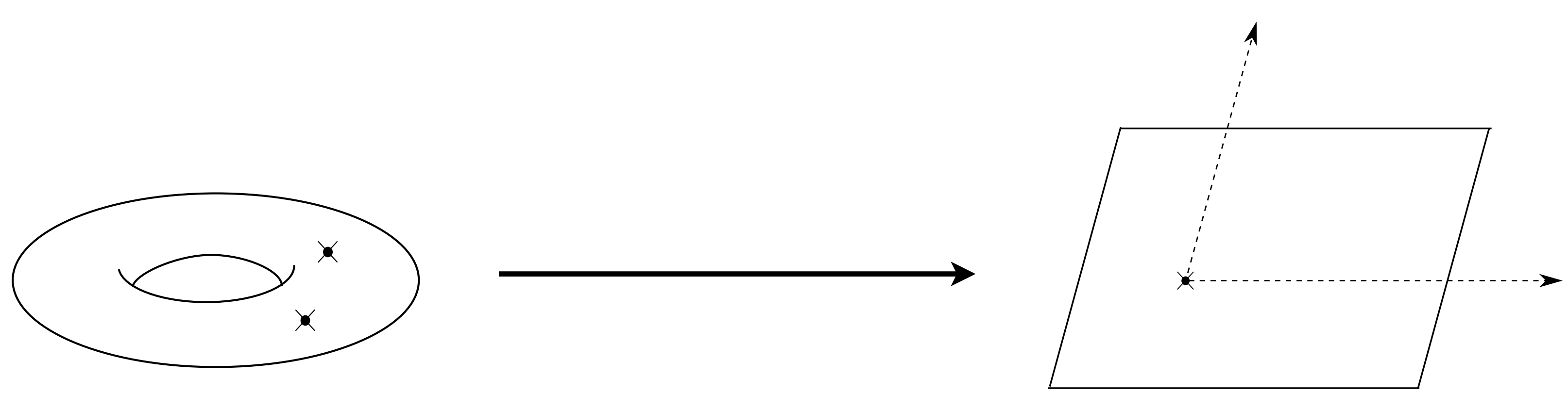
- No Wilson line $E8 \times E8\ (T^2)$

c.s.	τ	$SL(2,\mathbb{Z})$	modularity of torus
comp. Kähler	ρ	$SL(2,\mathbb{Z})$	T-duality
		$\tau \longleftrightarrow \rho$	T-duality on one 1-cycle

- Large volume with one Wilson line $E7 \times E8\ (T^2)$

Wilson line κ shift symmetry $\kappa \to \kappa + 1 \quad \kappa \to \kappa + \tau$

gauge $SU(2)$ bundle over an elliptic curve is represented by 2 points on the curve adding up to zero. κ is the Abel-Jacobi map of 1 of the 2 points.



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τ, κ, ρ $Sp(4,\mathbb{Z})$ [Mayr, Stieberger '95]

$$\begin{array}{cccc} T & S & \text{shift of } \kappa & \mathbb{Z}_2 \\ \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\tau+1\\ \kappa\\ \rho\end{pmatrix} & \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}-1/\tau\\ \kappa/\tau\\ \rho-\kappa^2/\tau\end{pmatrix} & \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\tau\\ \kappa+\tau\\ \rho+\tau+2\kappa\end{pmatrix} & \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\rho\\ \kappa\\ \tau\end{pmatrix} \end{array}$$

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τ, κ, ρ $Sp(4,\mathbb{Z}) \longrightarrow$ genus 2 curve [Mayr, Stieberger '95]

$$T: \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\tau+1\\ \kappa\\ \rho\end{pmatrix} \qquad S: \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}-1/\tau\\ \kappa/\tau\\ \rho-\kappa^2/\tau\end{pmatrix} \qquad \text{shift of } \kappa: \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\tau\\ \kappa+\tau\\ \rho+\tau+2\kappa\end{pmatrix} \qquad \mathbb{Z}_2: \begin{pmatrix}\tau\\ \kappa\\ \rho\end{pmatrix} \to \begin{pmatrix}\rho\\ \kappa\\ \tau\end{pmatrix}$$

Shioda-Inose correspondence

Shioda-Inose structure on $K3$: an involution ι preserving the holomorphic 2-form s.t. the quotient is a Kummer surface Y (after blowing up the fixed points).

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X \arrow[dr, dashed, "\iota"'] & \\
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Kummer surface: a $K3$ surface obtained by the quotient of an abelian surface $\mathcal{A}$ by -1 involution followed by blowing up the 16 fixed points. In particular $\mathcal{A}$ can be regarded as the Jacobian of a genus 2 curve $\mathcal{C}$.

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Necessary condition: $\mathrm{Pic}(X) \geq 17$ [Morrison '84]

✓An elliptically fibered $K3$ surface with singular fibers of type $E7$ and $E8$.

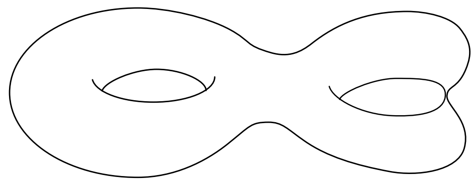
resolution has 17

F-theory & Genus 2 Invariants

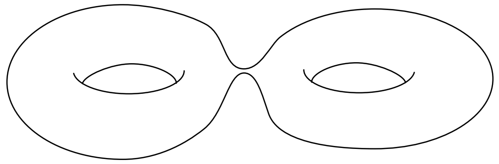
A genus 2 curve is always hyperelliptic $\mathcal{C} : y^2 = g_6(x) \quad \xrightarrow{\pi} x \in \mathbb{P}^1$

invariants: I_2, I_4, I_6, I_{10} moduli space: $\mathbb{P}^3_{(1,2,3,5)} \simeq \mathbb{P}(\langle I_2, I_4, I_6, I_{10} \rangle)$

- deg. case 1: pinch of a non-trivial cycle



- deg. case 2: split to two elliptic curves



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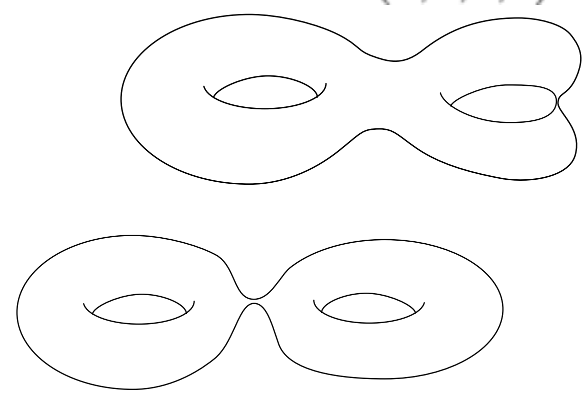

Kumar '08: Concrete construction of the Shioda-Inose structure on an elliptic $K3$ with singular fibers of type $E7$ and $E8$ in terms of invariants I_2, I_4, I_6, I_{10} of genus two curves.

$K3$ surface

$$y^2 = x^3 - \left(t^3 + \frac{1}{2} I_4 t^4\right) x + \left(\frac{1}{24} I_2 t^5 + \frac{1}{108}(I_2 I_4 - 3I_6) t^6 + \frac{1}{4} I_{10} t^7\right)$$

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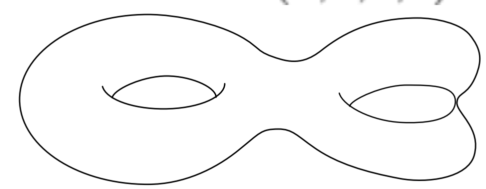

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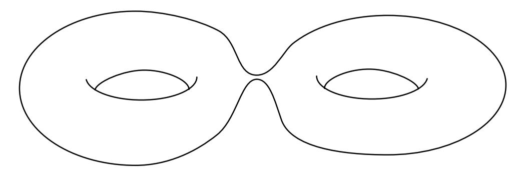

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Claim: this provides precisely the duality map between the moduli spaces of F-theory compactified on a $K3$ surface with $E7$ and $E8$ singular fibers and $E8 \times E8$ heterotic theory compactified on a torus with one Wilson loop turned on.

[Malmendier '11]

Case 1: Classical Limit

Het. theory: classical limit $\rho \to i\infty$

τ of elliptic curve, flat conn. κ of $SU(2)$ vector bundle on the elliptic curve

$SU(2)$ vector bundle: 2 pts (spectral cover) on the elliptic curve adding up to zero

F-theory: stable degeneration limit of $K3$

[Friedman, Morgan, Witten '97]

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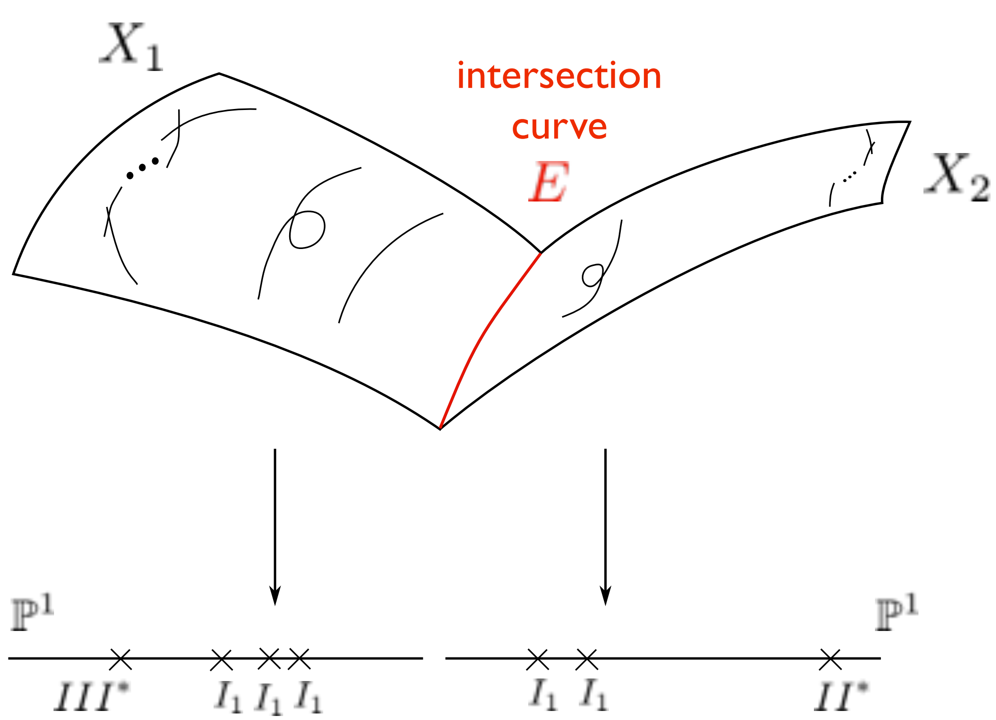

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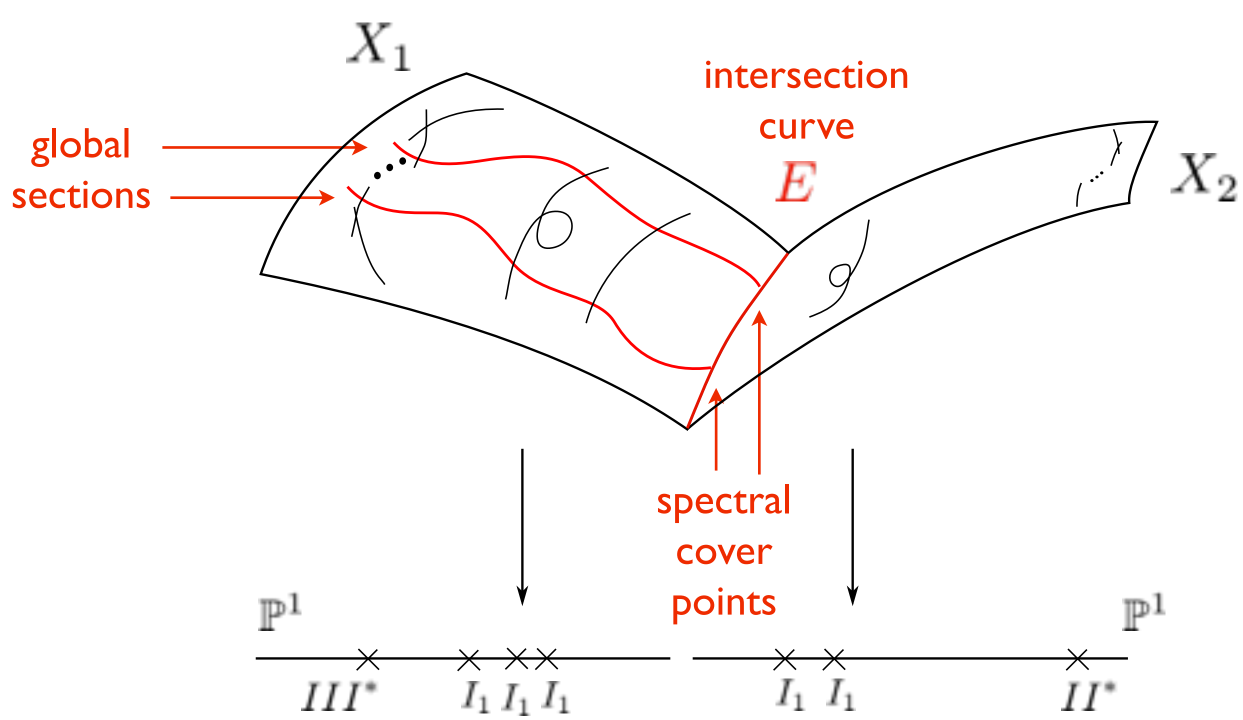

- Two rational elliptic surfaces X_1, X_2 intersecting at an elliptic curve E.

- Each rational surface responsible for one E8 factor.

- X_1, responsible for the $SU(2)$ gauge bundle, has two additional global sections, intersecting with E at two points.

Case 1: Classical Limit

✓ Degeneration of the Kumar formalism: $\rho \to i\infty$.

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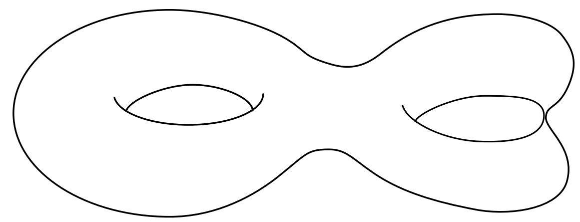

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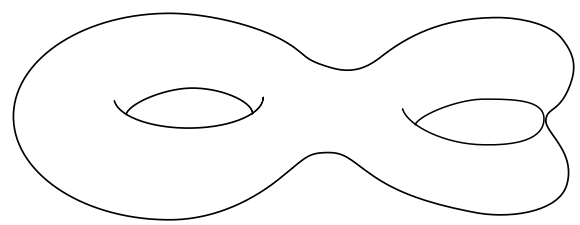

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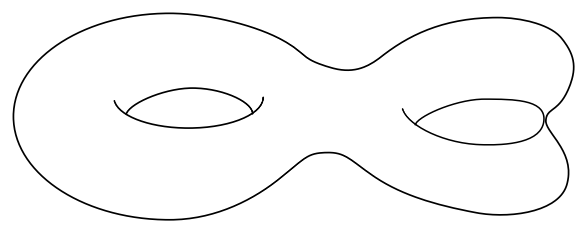

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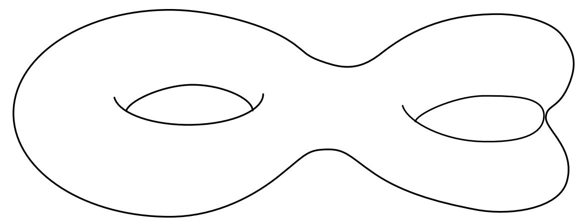

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intersection elliptic curve

$a_2 x + a_0 = 0$

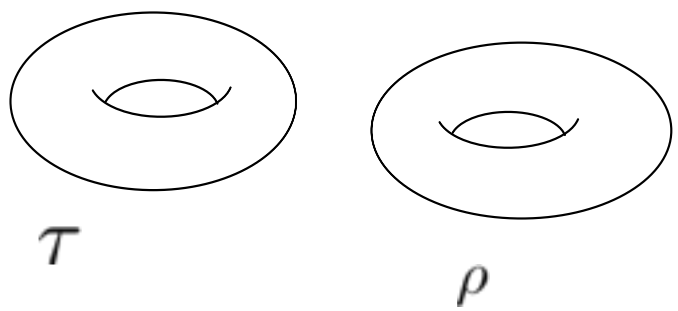
the point that represents the $SU(2)$ bundle

Case 2: E8xE8 Limit

✓ No Wilson loop: $\kappa \to 0$. $E8 \times E8$

Het. theory

[Cardoso, Curio, Lüst, Mohaupt '96]



Two tori given in the Weierstrass form

↓

$K3$ surface equation on F-theory side

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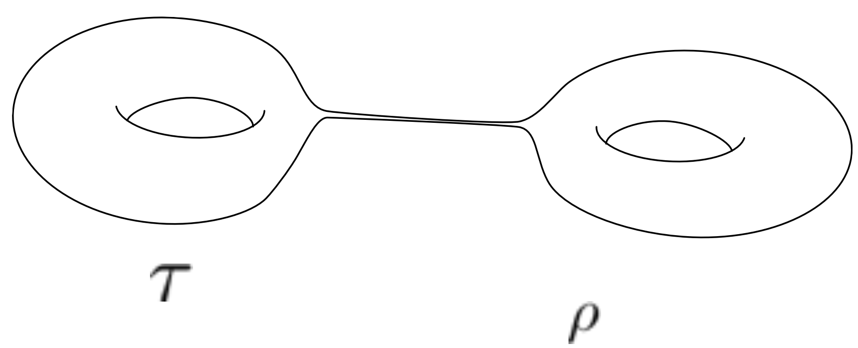

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$\downarrow$

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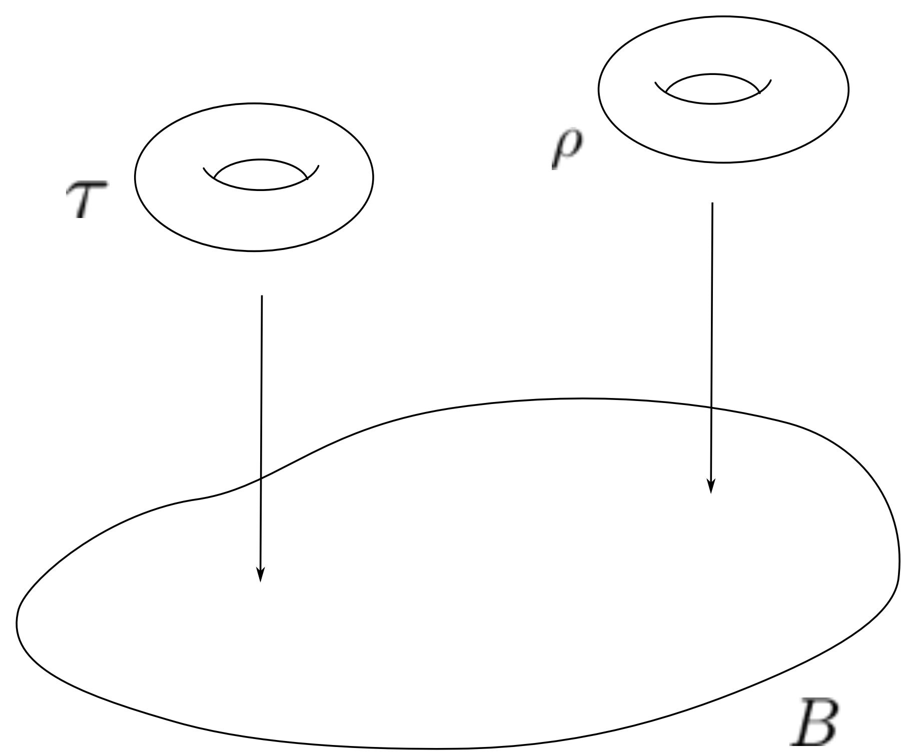
The Kumar formalism reproduces this result exactly in the $\kappa \to 0$ limit.

Non-geometric Fibration

[McOrist, Morrison, Sethi '10]

Non-geometric compactification

monodromy on ρ



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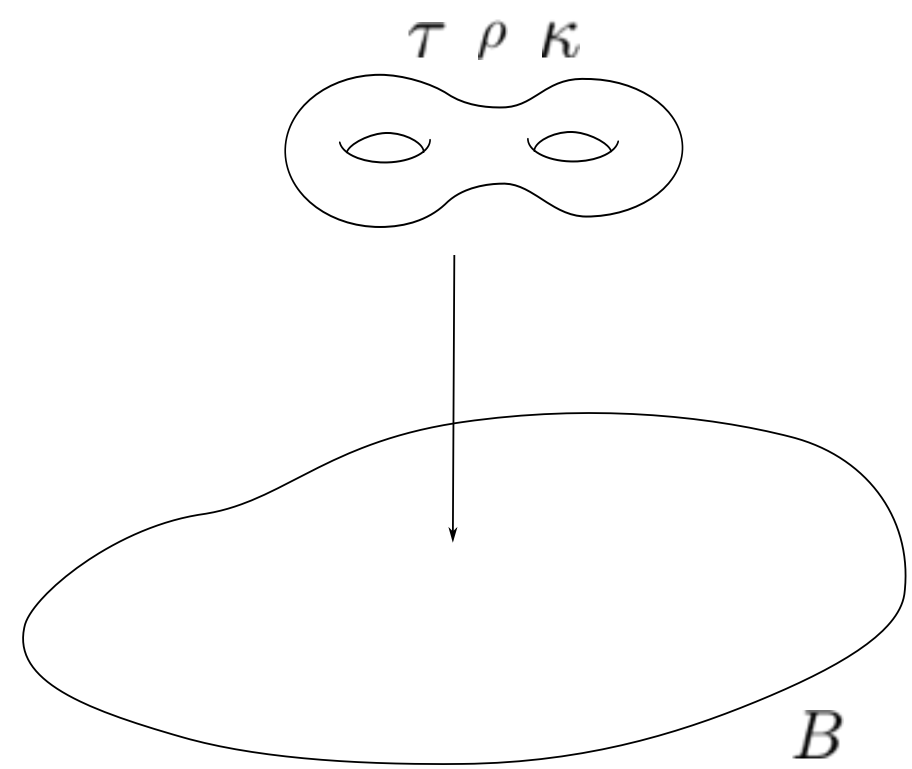

Non-geometric compactification

$Sp(4,\mathbb{Z})$ monodromy on $\tau \ \rho \ \kappa$

Mestre's Approach toward Fibration

F-theory: $y^2 = x^3 - (\lambda t^3 + \frac{1}{2} I_4 t^4)x + (\frac{1}{24}\lambda I_2 t^5 + \frac{1}{108}(I_2 I_4 - 3I_6)t^6 + \frac{1}{4}\frac{I_{10}}{\lambda}t^7)$

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Question: given I_2, I_4, I_6, I_{10} , can a_4, a_3, a_2, a_1, a_0 be constructed as analytic, or at least rational functions of I_2, I_4, I_6, I_{10} ?

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Mestre's theorem '90

If $\mathrm{Aut}(\mathcal{C}) \simeq C_2$, there exists an obstruction for the genus two curve $\mathcal{C}$ to live in the field of moduli: algebra $\mathbb{C}(I_2, I_4, I_6, I_{10})$.

Fibration via Mestre's Algorithm

Local fibration: yes; global patching up: difficult.

Summary

- The moduli space of E7xE8 heterotic theory compactified on a torus can be represented by a genus two algebraic curve.

- The (quantum) duality map between E7xE8 heterotic theory compactified on a torus and F-theory compactified on an elliptic K3 surface with E7 and E8 singular fibers is postulated to be given by the Shioda-Inose structure in the Kumar formalism.

- The formalism above is tested at two boundary points on the moduli space: the classical limit and the full gauge group limit.

- Fibration of genus two curves can achieve non-geometric compactification. But more clever ways than naive fibration are needed.

Prospects

- Analysis of genus two fibration.

- More Wilson lines.

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Thanks for your attention!

and, safe journey!

Backup

- Moduli Space of genus two curves

$$\mathrm{Proj}\,\mathbb{C}[I_2, I_4, I_6, I_{10}] = \mathbb{P}^3_{(1,2,3,5)}$$

- Kumar formalism

$K3$ surface

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scaling: $y \to \lambda_1^3\, y, \quad x \to \lambda_1^2\, x, \quad t \to \lambda_1\, t.$

- Fibration of K3 surfaces

$$I_2 \in \Gamma(\mathcal{O}(-2K_B)), I_4 \in \Gamma(\mathcal{O}(-4K_B)), I_6 \in \Gamma(-6K_B)$$

$$I_{10} = \lambda_1 \cdot \lambda_2, \lambda_1 \in \Gamma(\mathcal{O}(-4K_B + \Lambda)), \lambda_2 \in \Gamma(\mathcal{O}(-10K_B))$$

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\Lambda)) \to B$$