Geometric Engineering of BPS States

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Goal:

Quantum spectrum of magnetic monopoles of $\mathcal{N} = 2$, $SU(N)$ Yang-Mills theory

Motivation:

- charge quantization
- electric-magnetic duality
- confinement
- gravity
- ...

Based on:

1301.3065 with W.-Y. Chuang, D.-E. Diaconescu, G. W. Moore and Y. Soibelman

Classical monopoles

Action:

$$S[A] = \frac{1}{g^2}\int \mathrm{Tr}\, F \wedge *F + \frac{i\theta}{8\pi^2}\mathrm{Tr}\, F \wedge F \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

$U(1)$:

$$Q_{\mathrm{e}} = \frac{1}{4\pi}\int_{S^2} *F \qquad Q_{\mathrm{m}} = \frac{1}{4\pi}\int_{S^2} F$$

Dirac monopole string

$SU(N \geq 2) + \text{adjoint scalar } \phi$:

- Smooth, solitonic monopoles
- Lower bound on energy: $E \geq |a||Q_{\mathrm{e}} + \tau Q_{\mathrm{m}}|$

't Hooft (1974), Polyakov (1976), Julia, Zee (1975), Prasad, Sommerfield (1975), Bogomolny (1975), Coleman, Parke, Neveu, Sommerfield (1977),...

Classical monopoles: Moduli space

Bogomolny equation:

$*_3 F + D\phi = 0 \quad \Rightarrow \quad$ saturates lower bound for E

Bogomolny (1978), Coleman *et al.* (1977)

BPS monopole moduli space:

$$\mathcal{M}_k = \mathbb{R}^3 \times (S^1 \times \tilde{\mathcal{M}}_k)/\mathbb{Z}_k$$

$\tilde{\mathcal{M}}_k$ $(4k-4)$-dimensional hyper-Kähler manifold

Atiyah, Hitchin (1985)

Quantum spectra $\Rightarrow$ $\mathcal{N} = 2$ supersymmetry

- facilitates explicit computations
- captures lots of interesting physics

BPS states: Supersymmetry

$\mathcal{N} = 2$ supersymmetry:

- anti-commuting extension of the Poincaré algebra:

$$\begin{aligned} \{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ}, \qquad I, J = 1, 2 \\ \{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} &= 2\varepsilon_{\alpha\beta}\, \varepsilon^{IJ}\, Z(\gamma, t) \end{aligned}$$

- vector multiplets: (F^A, X^A, ψ^A_i), $A = 0, \dots, b_2$
- electric-magnetic charge: $\gamma \in \Gamma \cong \mathbb{Z}^{2b_2+2}$
- moduli $t^a = X^a/X^0 \in \mathcal{M}$
- central charge $Z(\gamma, t) : (\Gamma, \mathcal{M}) \to \mathbb{C}$

BPS states: Mass

Properties of BPS states:

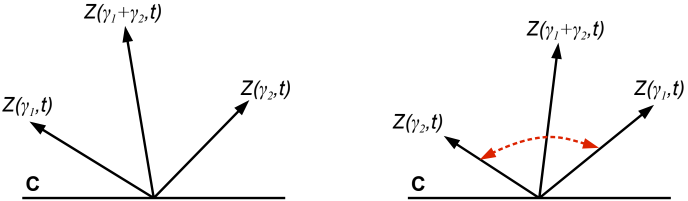
- Invariant under part of supersymmetry
- Carry minimal energy: $M(\gamma; t) = |Z(\gamma; t)|$

Stable for generic values of t:

Triangle inequality: $M(\gamma_1, t) + M(\gamma_2, t) \geq M(\gamma_1 + \gamma_2, t)$

BPS states: Wall-crossing

BPS states might decay or recombine across walls of marginal stability:



If $Z(\gamma_1, t) || Z(\gamma_2, t) \Rightarrow M(\gamma_1, t) + M(\gamma_2, t) = M(\gamma_1 + \gamma_2, t)$

Cecotti, Vafa (1992), Seiberg, Witten (1994), Denef (2000)...

BPS states: Index

$$\Omega(\gamma; t) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(\gamma,t)}\,(2J_3)^2\,(-1)^{2J_3}$$

- J_3 is a generator of the $SU(2)_{\mathrm{spin}}$ group
- BPS Hilbert space: $\mathcal{H}(\gamma, t)$

Generically a protected quantity $\Rightarrow$ independent of:

- coupling constant
- hypermultiplet scalars

$\Rightarrow$ Useful tool to study non-perturbative objects:

- charged black holes
- instantons
- monopoles

BPS states: Refined index

If $\exists$ $SU(2)_R$ symmetry $\Rightarrow$ Refined index:

$$\Omega(\gamma, y; t) = \frac{\mathrm{Tr}_{\mathcal{H}(\gamma,t)}\,(2J_R)\,y^{2J_3}(-y)^{2J_R}}{y - y^{-1}}$$
$$= \sum_{n\in\mathbb{Z}} \Omega_{\mathrm{ref},n}(\gamma; t)\,y^n$$

- Only an index with $SU(2)_R$ symmetry $\Rightarrow$ not in sugra
- No exotics conjecture: all BPS states on Coulomb branch are $SU(2)_R$ singlets

  Gaiotto, Moore, Neitzke (2010)
- $SU(2)_{\mathrm{spin}}$ character

BPS monopoles in $\mathcal{N} = 2$ gauge theory

$\Omega(\gamma; t)$ = Dirac index of dyon moduli space $\mathbb{R}^3 \times (S^1 \times \tilde{\mathcal{M}}_k)/\mathbb{Z}_k$

Beautiful results for $k = 2$

Sethi, Stern, Zaslow (1995), Gauntlett, Harvey (1996), Gauntlett, Kim, Park, Yi (2000),...

Hard for general $SU(N)$ and k

$\Rightarrow$ Geometric engineering in string theory:
Specific choice and limit of Calabi-Yau such that the physics in $\mathbb{R}^4$ is $\mathcal{N} = 2$ gauge theory

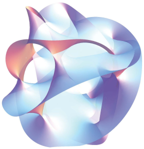
Kachru, Vafa (1995); Katz, Klemm, Vafa (1996); Katz, Mayr, Vafa (1997),...

Geometric Engineering: String theory

Compactification:

10d IIA string theory on $X = \mathrm{CY}_3$:

Kähler moduli: $\tilde{s}_a$



3-dim. projection of 6-dim. CY.
en.wikipedia.org

Prepotential: $\mathcal{F}(\tilde{s}) = \mathcal{F}^{\mathrm{pert}}(\tilde{s}) + \frac{\zeta(3)\chi(X)}{(2\pi\sqrt{-1})^3} + \mathcal{F}^{\mathrm{inst}}(\tilde{s})$

See your notes of Hans Jockers' talk

Central charges:

D0: M_s D2 on Q_a: $M_s \tilde{s}_a$

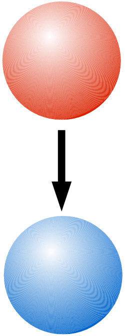
D4 on P^a: $M_s \frac{\partial \mathcal{F}}{\partial \tilde{s}_a}$ D6: $M_s \mathcal{F}$

Large volume limit:

$$Z(F,t) = -M_s \int_{X_N} e^{-\tilde{s}} \mathrm{ch}(F) \sqrt{\mathrm{Td}(X)}$$

Calabi-Yau toric resolution:

$$\begin{tikzcd} (\mathbb{C}^2/\mathbb{Z}_{N+1})_{\mathrm{res}} \arrow[r] & X_N \arrow[d] \\ & \Sigma_0 \end{tikzcd}$$



Compact homology:

4-cycles: $S_i \cong \mathbb{F}_{2i}, i = 1, \ldots, N-1$

2-cycles: $\Sigma_0, C_i \cong \mathbb{P}^1$

0−cycle

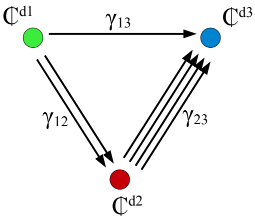
Quiver

D-branes $\Leftrightarrow$ Quivers with potential

Beilinson (1978), Bondal (1990), Douglas, Moore (1996), Brunner, Douglas, Lawrence, Romelsberger (2000), Aspinwall, Melnikov (2004), Herzog, Karp (2006), ...

Quiver:

1. Vertices $i \in V$
2. Arrows $i \to j \in A$



Quiver representation:

A set of vector spaces $V_i \cong \mathbb{C}^{d_i}$ and $\forall i \to j$ a linear map $V_i \to V_j$

Stable representation:

Given $\vec{\theta} \in \mathbb{R}^{|V|}$, a rep. F is stable if $\forall F' \subset F$, $\frac{\vec{\theta}\cdot\vec{d}'}{|d'|} < \frac{\vec{\theta}\cdot\vec{d}}{|d|}$

Index: $\Omega(\gamma; t) = \chi(\mathcal{M}_{\vec{\theta}}(\vec{d}))$

with $\mathcal{M}_{\vec{\theta}}(\vec{d})$ moduli space of stable representations

Geometric engineering: Quiver for X_N

X_N:

- Exceptional collection of $\mathbb{C}^2/\mathbb{Z}_{N+1}$

  Kapranov, Vasserot (2000)

- Exceptional collection of toric fibration

  Costa, Di Rocco, Miró-Roig (2009)

$\Rightarrow$ Fractional branes (P_i, Q_i), $i = 1 \dots N$

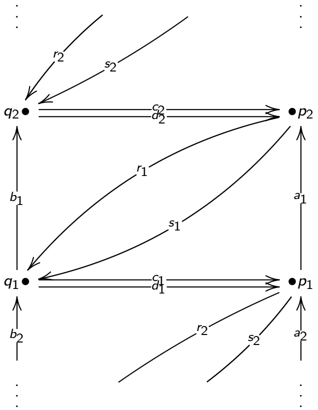
$\Rightarrow$ Quiver

Chuang *et al.*

Orbifold methods:

Douglas, Moore (1996), Fiol (2000),...

E.g. $N = 2$:

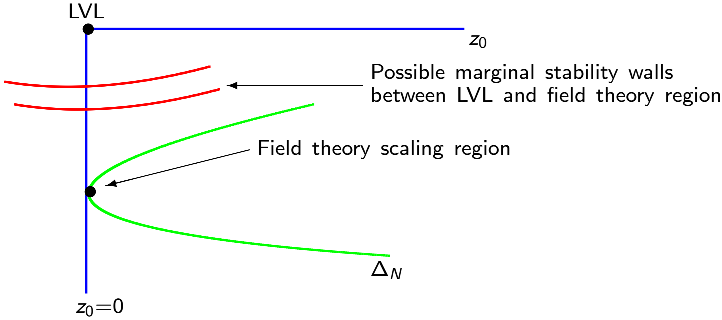


Superpotential:

$$W = r_1(a_1 c_1 - c_2 b_1) + s_1(a_1 d_1 - d_2 b_2) + 1 \leftrightarrow 2$$

Geometric Engineering: Limit

Kähler moduli space:



Geometric Engineering: Limit

Field theory limit: $\lim \epsilon \to 0$

$\tilde{s}_0 = -\frac{N\sqrt{-1}}{\pi}(c_0 + \ln \epsilon) \qquad \tilde{s}_i = \frac{a_i}{M_0}\epsilon \qquad M_s = \frac{M_0}{\epsilon}$

$M_0, M_{\Sigma_0} \to \infty$

$\Rightarrow$ charges disappear from spectrum

Electric-magnetic charges:

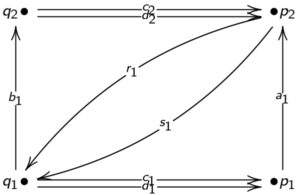
Magnetic: D4-brane/vector bundle on S_i

Electric: D2-brane on C_i

Geometric Engineering: Truncation of quiver

Field theory limit: nodes P_N and Q_N decouple from X_N quiver

E.g. $X_{N=3}$:



Fiol (2000); Cecotti, Neitzke, Vafa (2010); Alim *et al.* (2011); Chuang *et al.* (2011)

Geometric engineering: Dyon spectra

Weak coupling spectra:

$SU(2)$:

Kronecker quiver $\Rightarrow$ $(n_{\mathrm{m}}, n_{\mathrm{e}}) = (1, 2n)$ $n \in \mathbb{Z}$

Seiberg, Witten (1994), Bilal, Ferrari (1996), Denef (2000)...

$SU(3)$:

Determined subset of spectrum:

$$\sum_{r_2 \geq 0} \Omega((1, r_2, n_1, r_2 n_2); y)\, q^{r_2} = \prod_{s=0}^{n_1 - n_2 - 1} (1 + y^{n_2 - n_1 + 2s + 1} q)$$

For $r_2 = 1$: Gauntlett, Kim, Park, Yi (1999), Stern, Yi (2000); For $r_2 \geq 2$: Chuang *et al.* (2013)

- magnetic charges ≥ 1
- arbitrary high spin

Geometric engineering: "No walls" conjecture

Large volume limit (LVL):

$$\mathrm{Im}(\tilde{s}_i) \to \infty$$

D-branes are semi-stable coherent sheaves

Limit weak coupling (LWC):

$$\lim_{\lambda \to \infty} \left\{ \begin{array}{l} a_i(\lambda) = \mathrm{Re}(a_i) + \lambda\sqrt{-1}\,\mathrm{Im}(a_i) \\ \tau_0(\lambda) = \mathrm{Re}(\tau_0) + \lambda\sqrt{-1}\,\mathrm{Im}(\tau_0) \end{array} \right.$$

Conjecture:

There are no walls of marginal stability between LVL and LWC.

Geometric engineering: “No walls” conjecture

Proved for $SU(2)$, $N_f = 0$
Chuang *et al.*

Also confirmed for flavors $N_f \leq 3$ using sheaf counting on B_{N_f}
Including dyon with $n_m = 2$ for $N_f = 3$
Haghighat, Vandoren (2012)

Sheaf counting also captures “heavy” D0/D2 charges

$$\Updownarrow$$

$(0, 4)$ Sigma model with target space $\mathcal{M}_k$

Conclusions

Geometric engineering + BPS quivers are very powerful tools to determine the BPS spectrum!

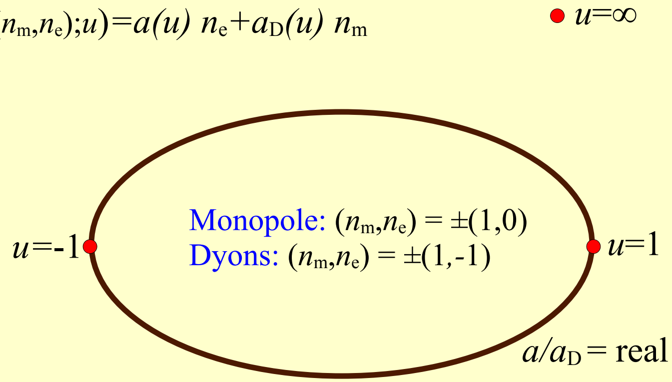
- We have shown the presence of BPS monopoles with charge ≥ 1 and higher spins
- We have found evidence for the absence of walls between LVL and LWC

Thank you for your attention!

$\mathcal{N} = 2$ $SU(2)$ Seiberg-Witten theory

Coulomb branch parametrized by $u = \mathrm{Tr}\,\phi^2$

$Z((n_\mathrm{m},n_\mathrm{e});u)=a(u)\, n_\mathrm{e}+a_\mathrm{D}(u)\, n_\mathrm{m}$



W-bosons: $(n_\mathrm{m},n_\mathrm{e}) = \pm(0,1)$

Dyons: $(n_\mathrm{m},n_\mathrm{e}) = \pm(1,n)$