Refined BPS invariants of del Pezzo and half K3 manifolds

Based on: M. X. Huang, A. Klemm, M. P.: arXiv:1303.????[hep-th]

Maximilian Poretschkin

Bethe Center for Theoretical Physics
Bonn University

March 18 2013

Aim and contents of this talk

Aim: How to compute the refined free energy

$$F = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}(t)$$

on local Calabi Yau manifolds whose base is a del Pezzo/$\frac{1}{2}K3$ surface and how to extract BPS numbers .

Contents

- Explain terms and definitions
- Present the necessary techniques

1. Generalized holomorphic anomaly equation
2. Generalized modular anomaly equation

Motivation

Mathematics

- Enumerative Geometry

Physics

- Seiberg-Witten theory with matter
- Stable degeneration limit of F-theory compactifications
- E-string

Topological string theory

Consider field content of $\mathcal{N} = (2,2)$ supersymmetric field theory

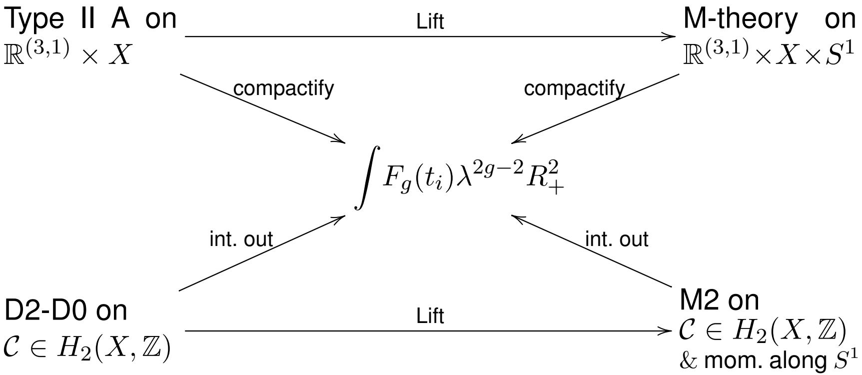
$$\begin{array}{rc} J, \bar{J} & \text{U(1) currents} \\ G^{\pm}, \bar{G}^{\pm} & \text{SUSY currents} \\ T, \bar{T} & \text{EM currents} \end{array}$$

Twisting

$$M_{E'} = M_E + F_V$$
$$M_{E'} = M_E + F_A$$

allows to define the free energies

$$F_g = \int_{\mathcal{M}_g} \langle | \prod_{i=1}^{3g-3} G^-(\mu_i) | \rangle$$



$$\lambda = \left\langle g_s dx_1 \wedge dx_2 + g_s dx_3 \wedge dx_4 \right\rangle + \delta\lambda \quad \text{self-dual}$$

Gopakumar, Vafa '98

Refined Topological string theory

Consider instead general field strength

$$\lambda = \left\langle \epsilon_1 dx_1 \wedge dx_2 - \epsilon_2 dx_3 \wedge dx_4 \right\rangle + \delta\lambda$$

Gopakumar, Vafa '98; Nekrasov '02

Little group for massive states in 5d

$$SO(4) = SU(2)_L \times SU(2)_R \supset U(1)_L \times U(1)_R$$

$$Z = \prod_{\beta} \prod_{\substack{j_{L/R}\\=0}}^{\infty} \prod_{\substack{m_{L/R}=\\-j_{L/R}}}^{j_{L/R}} \prod_{\substack{m_1=1,\\m_2=1}}^{\infty} \left(1 - q_L^{m_L} q_R^{m_R} e^{\epsilon_1(m_1-\frac{1}{2})} e^{\epsilon_2(m_2-\frac{1}{2})} Q^\beta\right)^{(-1)^{2(j_L+j_R)} N^\beta_{j_L j_R}}$$

$$\sum_{g_R, g_L} n^\beta_{g_R,g_L} I_R^{g_R} \otimes I_L^{g_L} = \sum_{j_R,j_L} N^\beta_{j_R,j_L} \left[\frac{j_R}{2}\right]_R \otimes \left[\frac{j_L}{2}\right]_L$$

$$I_*^n = \left(2[0]_* + \left[\frac{1}{2}\right]_*\right)^{\otimes n}$$

Mathematical interlude

D2-D0 bound state defines stable pair, i.e. a sheaf $\mathcal{F}$ together with a section $s \in H^0(\mathcal{F})$, s.t.

- $\mathcal{F}$ is pure of dimension 1 with $ch_2(\mathcal{F}) = \beta$, $\chi(\mathcal{F}) = n$
- s generates $\mathcal{F}$ outside a finite set of points

$SU(2)_L \times SU(2)_R$ operates on the moduli space of stable pairs $P_n(X, \beta)$ via Lefshetz decomposition.

$N^{\beta}_{j_L j_R}$ are the dimensions of the corresponding eigenspaces.

Choi, Katz, Klemm '12

Del Pezzo surfaces and half K3

Consider smooth divisor $P \subset X$ Calabi Yau mfd If $-K_P \cdot \mathcal{C} > 0$, $\mathcal{C}$ curve in X, P is called del Pezzo surface.

1. $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$
2. $B_n, \quad 0 \leq n \leq 8$
3. $n = 9 \; \frac{1}{2}K3$

$$\int_P c_1^2 = 9 - n$$

Homology
$\{1, A_1, A_1 \times A_2, A_4, D_5, E_6, E_7, E_8, \Gamma^{(1,1)} \oplus E_8\}$

↙ Modular symmetry

↘ Toric geometry

Getting hands on computations

B-model geometry of local Calabi-Yau manifold given by elliptic curve. Weierstrass normal form

$$y^2 = 4x^3 - g_2(u, m_i) - g_3(u, m_i)$$

$$j = 1728\frac{g_2^3(t, m_i)}{\Delta(t, m_i)} = \frac{1}{q} + 744 + \dots$$

$$\Delta = 27g_2^3(t, m_i) - g_3^2(t, m_i), \quad q = e^{2\pi i\tau}$$

$\Longrightarrow$ can solve for τ

$$\frac{dt}{du} = \sqrt{\frac{E_4(\tau)g_3(u, m_i)}{E_6(\tau)g_2(u, m_i)}}$$

The generalized holomorphic anomaly equation

The higher genus amplitudes can be calculated iteratively by

$$\bar{\partial}_{\bar{i}} F^{(n,g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \big(D_i D_k F^{(n,g-1)} + \sum_{m,h}{}' D_j F^{(m,h)} D_k F^{n-m,g-h} \big)$$

Bershadsky, Cecotti, Ooguri, Vafa '93
Huang, Klemm '10
Huang, Klemm, M.P. '13

1. Calculate $F^{(0,0)}, F^{(1,0)}, F^{(0,1)}$
2. Fix the holomorphic ambiguity

Fixing initial values and boundary condition

$$\frac{\partial^2}{\partial t^2} F^{(0,0)}(t, m_i) = -\frac{1}{2\pi i}\tau(t, m_i)$$

$$F^{(0,1)} = -\frac{1}{2}\log\left(G_{u,\bar{u}}(t, m_i)\left|\Delta u^{a_0} m_i^{a_i}\right|^{\frac{1}{3}}\right), \quad F^{(1,0)} = \frac{1}{24}\log\left(\Delta u^{b_0} m_i^{b_i}\right)$$

Holomorphic ambiguity fixed by the gap condition

$$\begin{aligned} F(s, g_s, t) &= \int_0^\infty \frac{ds}{s}\frac{\exp(-st)}{4\sinh(s\epsilon_1/2)\sinh(s\epsilon_2/2)} + \mathcal{O}(t_0) \\ &= \left(-\frac{1}{12} + \frac{1}{24}s g_s^{-2}\right)\log(t) \\ &\quad + \underbrace{\left(-\frac{1}{240}g_s^2 + \frac{7}{1440} - \frac{7}{5760}s^2 g_s^{-2}\right)\frac{1}{t^2} + \mathcal{O}(t^0)}_{\text{Gap structure for } g=2} \end{aligned}$$

$$s = (\epsilon_1 + \epsilon_2)^2, \quad g_s^2 = \epsilon_1\epsilon_2$$

$\Longrightarrow$ Compute refined BPS invariants

Huang, Klemm, M.P. ’13

Modular invariance

$2j_L \backslash 2j_R$	0
0	27

$d = 1$

$2j_L \backslash 2j_R$	0	1
0		27

$d = 2$

The GV invariants $n^d_{j_L, j_R}$ for $d = 1, 2$, for the local E_6 model

$2j_L \backslash 2j_R$	0
0	56

$d = 1$

$2j_L \backslash 2j_R$	0	1	2
0		133	
1			1

$d = 2$

The GV invariants $n^d_{j_L, j_R}$ for $d = 1, 2$ for the local E_7 model

$2j_L \backslash 2j_R$	0	1
0	248	
1		1

$d = 1$

$2j_L \backslash 2j_R$	0	1	2	3
0		3876		
1			248	
2				1

$d = 2$

The GV invariants $n^d_{j_L, j_R}$ for $d = 1, 2$ for the local E_8 model

Huang, Klemm, M.P. '13

Conclusions

We have presented a framework to compute refined BPS numbers that

- works everywhere in the moduli space
- works for toric del Pezzo surfaces
- works for non-toric del Pezzo surfaces

Outlook:

- Compute refined BPS invariants for compact Calabi-Yau manifolds
- Improve the development of the corresponding mathematical theory