Worldsheet Description of the Refined Topological String

Part I

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Overview

* Review of unrefined topological string
  - worldsheet description
  - relation to BPS saturated effective string couplings
  - relation to gauge theory partition functions

* Proposal for worldsheet description of refined topological string
  - generalized class of BPS saturated couplings
  - connection to gauge theory partition function
  - worldsheet description of refined topological string

* Conclusions

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Genus g partition function through integration over Beltrami differentials

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{a=1}^{3g-3} G^{-}(\mu_a) \right|^2 \right\rangle$$

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  i.e. are there refined objects, which accomodate ϵ_+

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discuss one proposal in the remainder of the talk

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discuss one proposal in the remainder of the talk

for technical details: see following talk by **A. Zein Assi**

Omega-background and refined deformation

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Questions:

* How to include self-dual field strength tensors in the topological amplitudes consistent with supersymmetry?

* Which type of fields exactly should be included?

Refinement from the effective action point

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General idea is to generalize the 'unrefined' topological couplings

[Antoniadis, SH, Narain, Taylor 2010]

$$\mathcal{I}_g = \int d^4x \int d^4\theta \mathcal{F}_g(X)(W^{ij}_{\mu\nu} W^{\mu\nu}_{ij})^g$$

Notation:

$$(x^\mu, \theta^i_\alpha, \bar{\theta}^{\dot{\alpha}}_i) \in \mathbb{R}^{4|4,4}, \qquad g \geq 1$$

$$W^{ij}_{\mu\nu} = F^{G,ij}_{(-),\mu\nu} + \theta^{[i} B^{j]}_{(-),\mu\nu} - (\theta^i \sigma^{\rho\tau} \theta^j) R_{(-),\mu\nu\rho\tau}$$

$F^{G,ij}_{\mu\nu} \ldots$ gravitphoton field strength tensor

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$R_{\mu\nu\rho\tau} \ldots$ Riemann tensor

$$X = \varphi + \theta^i \lambda_i + \tfrac{1}{2} F_{(-)\,\mu\nu} \epsilon_{ij} (\theta^i \sigma^{\mu\nu} \theta^j)$$

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question: which vector fields to choose?

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  $F^{\bar{U}}$ vector partner of complex structure modulus of T^2

* exact match with the partition function requires to choose the complex structure modulus

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(Z^1, Z^2, X, Z^4, Z^5) = bosonic coords.

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These amplitudes can be computed using a generating function

$$G(\epsilon_\pm) = \sum_{g=1}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_-^{2g-2} \epsilon_+^{2n}}{n!^2 (g-1)!^2} \mathcal{F}_{g,n}$$

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This generating functional can be computed **exactly** at 1-loop order in heterotic

$$G(\epsilon_\pm) = \int_{\mathbb{F}} d^2\tau \, G^{\text{bos}}(\epsilon_-, \epsilon_+) \frac{1}{\eta^4 \bar{\eta}^{24}} \frac{1}{2} \sum_{h,g=0}^{1} G^{\text{ferm}}[^h_g](\check{\epsilon}_+) Z[^h_g] \; \Gamma_{(2,2+8)}(T, U, Y)$$

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$$= \frac{(2\pi)^2 (\epsilon_-^2 - \epsilon_+^2) \, \bar{\eta}(\bar{\tau})^6}{\bar{\theta}_1(\tilde{\epsilon}_- - \tilde{\epsilon}_+; \bar{\tau}) \, \bar{\theta}_1(\tilde{\epsilon}_- + \tilde{\epsilon}_+; \bar{\tau})} \; e^{-\frac{\pi}{\tau_2}(\tilde{\epsilon}_-^2 + \tilde{\epsilon}_+^2)} \times G_{\text{non-hol}}(\epsilon_\pm)$$

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the **moduli** dependence is localized in the Narain-lattice $\Gamma_{(2,2+8)}(T,U,Y)$

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* corresponds precisely to (pert. part of) Nekrasov's partition function upon expansion in $\epsilon_\pm$
* the amplitude $\mathcal{F}_{g,n}$ is accompanied by a factor $\mu^{2-2g-2n}$ which is precisely the expected behaviour in the vicinity of a point of enhanced gauge symmetry

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The dual amplitudes appear at g-loops in type II

[Antoniadis, SH, Narain, Taylor 2010]

$$\mathcal{F}_{g,n}^{(II)} = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{k=1}^{3g-3+n} |\mu_k \cdot G^-|^2 \prod_{k=1}^{n} \int \psi(z_k) \prod_{\ell=1}^{n} \hat{\psi} \right\rangle_{\text{top}}$$

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This expression can serve as a worldsheet description of the refined topological string

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Thank you for your attention!