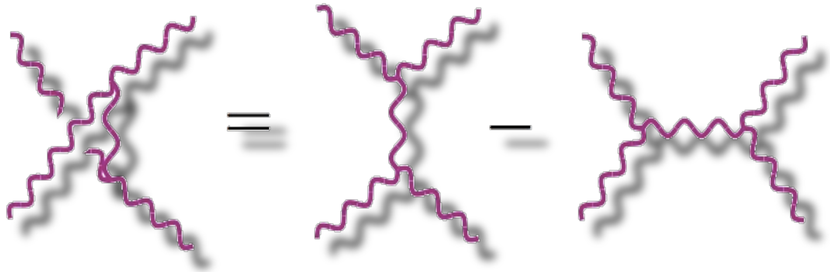
Recent progress on gauge and gravity scattering amplitudes



Henrik Johansson

CERN

March 19, 2013

XXV Workshop Beyond the Standard Model

Physikzentrum Bad Honnef

Based on work in collaboration with:

Z.Bern, J.J.Carrasco, L.Dixon, Y-t. Huang, R.Roiban

This talk:

1) Overview of Color—Kinematics Duality

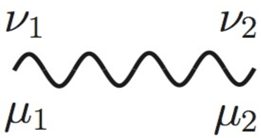
- Yang-Mills Theory = Kinematical Lie 2-Algebra
- Chern-Simons Matter Theory = Kinematical Lie 3-Algebra
- Gravity = Double Copy of Gauge Theories.
- What is the Lie Algebra? (partial results)

2) An exercise in calculating 3-loop 4-pt supergravity ampl.

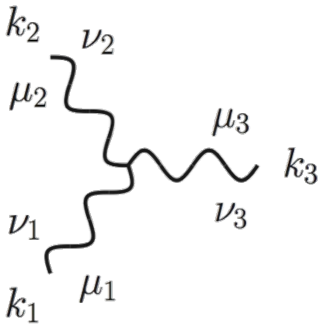
Text-Book: Perturbative Gravity is Complicated !

de Donder gauge:

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$



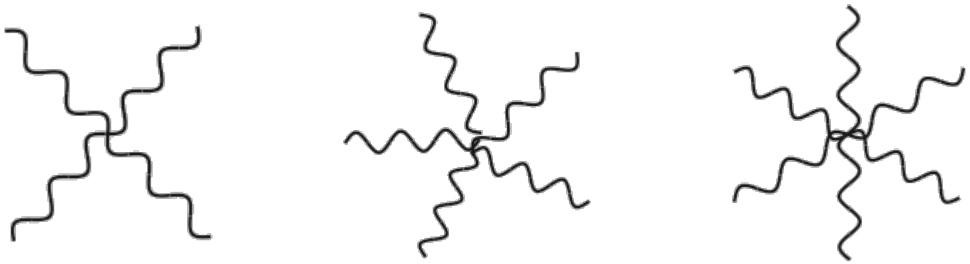
$$= \frac{1}{2}\left[\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right]\frac{i}{p^2+i\epsilon}$$



$$\begin{aligned} = \text{sym}[&-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}) - \frac{1}{2}P_6(k_{1\mu_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_3\nu_3}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3}) \\ &+P_6(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}) + 2P_3(k_{1\mu_2}k_{1\nu_3}\eta_{\mu_1\nu_1}\eta_{\nu_2\mu_3}) - P_3(k_{1\nu_2}k_{2\mu_1}\eta_{\nu_1\mu_1}\eta_{\mu_3\nu_3}) \\ &+P_3(k_{1\mu_3}k_{2\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + P_6(k_{1\mu_3}k_{1\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + 2P_6(k_{1\mu_2}k_{2\nu_3}\eta_{\nu_2\mu_1}\eta_{\nu_1\mu_3}) \\ &+2P_3(k_{1\mu_2}k_{2\mu_1}\eta_{\nu_2\mu_3}\eta_{\nu_3\nu_1}) - 2P_3(k_1\cdot k_2\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}\eta_{\nu_3\mu_1})]\end{aligned}$$

After symmetrization
~ 100 terms !

higher order
vertices…



. . .

~10^3 terms

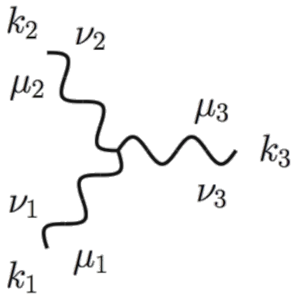
H. Johansson, Bad Honnef 2013

On-shell simplifications

Graviton plane wave: $\varepsilon^{\mu}(p)\varepsilon^{\nu}(p)\, e^{ip\cdot x}$

↑ Yang-Mills polarization

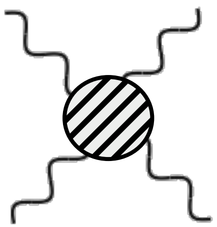
On-shell 3-graviton vertex:



$$= \; i\kappa\Big(\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3}+\text{cyclic}\Big)\Big(\eta_{\nu_1\nu_2}(k_1-k_2)_{\nu_3}+\text{cyclic}\Big)$$

↑ Yang-Mills vertex

Gravity scattering amplitude:



$$M_4^{\text{tree}}(1,2,3,4) = -i\frac{st}{u}A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,3,4)$$

↑ Yang-Mills amplitude

On-shell, gravity is the "square" of Yang-Mills – Kawai, Lewellen, Tye
holds for the entire S-matrix – Bern, Carrasco, HJ

Color-Kinematics Duality

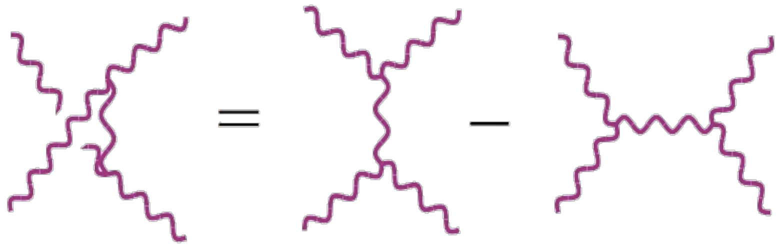
Yang-Mills theories are controlled by a kinematic Lie algebra

- Amplitude represented by cubic graphs:

$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

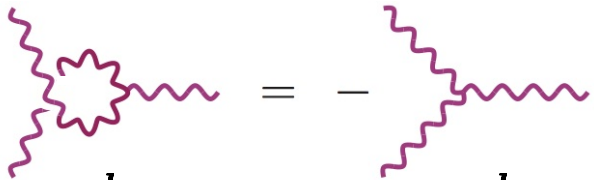
numerators (n_i), color factors (c_i), propagators ($p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2$)

Color & kinematic numerators satisfy same relations:



$$f^{adc} f^{ceb} = f^{eac} f^{cbd} - f^{abc} f^{cde}$$

Jacobi identity



$$f^{bac} = - f^{abc}$$

antisymmetry

Duality: color ↔ kinematics

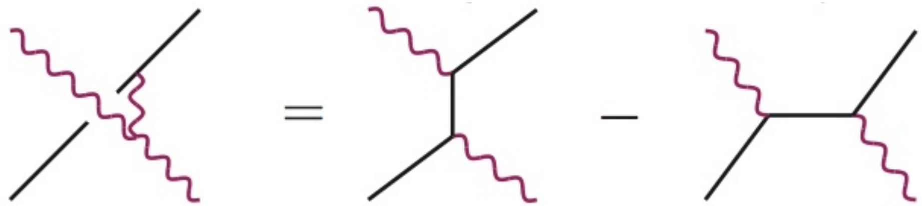
Bern, Carrasco, HJ

Some details of color-kinematics duality

Bern, Carrasco, HJ

can be checked for 4pt on-shell ampl. using Feynman rules

Example with two quarks:



$$\varepsilon_2 \cdot (\bar{u}_1 \not{V} u_3) \cdot \varepsilon_4 = \bar{u}_1 \not{\varepsilon}_4 \not{p}_t \not{\varepsilon}_2 u_3 - \bar{u}_1 \not{\varepsilon}_2 \not{p}_s \not{\varepsilon}_4 u_3$$

$$f^{cba} T^c_{ik} = T^b_{ij} T^a_{jk} - T^a_{ij} T^b_{jk}$$

1. $(A^\mu)^4$ contact interactions absorbed into cubic graphs
   - by hand 1=*s*/*s*
   - or by auxiliary field $B \sim (A^\mu)^2$
2. Beyond 4-pts duality not automatic ➔ Lagrangian reorganization
3. Known to work at tree level: all-*n* example Kiermaier; Bjerrum-Bohr et al.
4. Enforces (BCJ) relations on partial amplitudes ➔ (*n*-3)! basis

Duality gives new amplitude relations

In color ordered tree amplitudes 3 legs can be fixed: $(n\text{-}3)!$ basis BCJ

4 points:

$$A_4^{\text{tree}}(1,2,4,3) = \frac{A_4^{\text{tree}}(1,2,3,4)s_{14}}{s_{24}}$$

5 points:

$$A_5^{\text{tree}}(1,2,4,3,5) = \frac{A_5^{\text{tree}}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\text{tree}}(1,2,3,5,4)s_{14}}{s_{24}}$$

$$A_5^{\text{tree}}(1,2,4,5,3) = -\frac{A_5^{\text{tree}}(1,2,3,4,5)s_{34}s_{15} + A_5^{\text{tree}}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}}$$

...relations obtained for any multiplicity

Similar relations found in string theory: monodromy relations on the open string worldsheet Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

Used to solve string theory at tree level: Mafra, Schlotterer, Stieberger

See talk by Schlotterer, Brödel

Gravity is a double copy

- Gravity amplitudes obtained by replacing color with kinematics

$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

BCJ

$$\mathcal{M}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

- The two numerators can belong to different theories:

n_i	$\tilde{n}_i$		
$(\mathcal{N}=4)$	$\times\ (\mathcal{N}=4)$	$\rightarrow$	$\mathcal{N}=8$ sugra
$(\mathcal{N}=4)$	$\times\ (\mathcal{N}=2)$	$\rightarrow$	$\mathcal{N}=6$ sugra
$(\mathcal{N}=4)$	$\times\ (\mathcal{N}=0)$	$\rightarrow$	$\mathcal{N}=4$ sugra
$(\mathcal{N}=0)$	$\times\ (\mathcal{N}=0)$	$\rightarrow$	Einstein gravity + axion+ dilaton

similar to Kawai-Lewellen-Tye but works at loop level

see talk by Isermann

Kawai-Lewellen-Tye Relations

String theory
tree-level identity:

closed string ~ (left open string) × (right open string)

$$A_n \sim \int \frac{dx_1 \cdots dx_n}{\mathcal{V}_{abc}} \prod_{1\leq i<j\leq n} |x_i - x_j|^{k_i\cdot k_j} \exp\Bigg[\sum_{i<j}\Big(\frac{\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(x_i-x_j)}\Big)\Bigg]\Bigg|_{\text{multi-linear}}$$

KLT relations emerge after nontrivial world-sheet integral identities

Field theory limit $\Rightarrow$ gravity theory ~ (gauge theory) × (gauge theory)

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\,\widetilde{A}_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\,\widetilde{A}_5^{\text{tree}}(2,1,4,3,5)$$
$$+is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\,\widetilde{A}_5^{\text{tree}}(3,1,4,2,5)$$

gravity states are products of gauge theory states:

$$|1\rangle_{\text{grav}} = |1\rangle_{\text{gauge}} \otimes |1\rangle_{\text{gauge}}$$

See talk by Schlotterer

What is the Kinematic Lie Algebra?

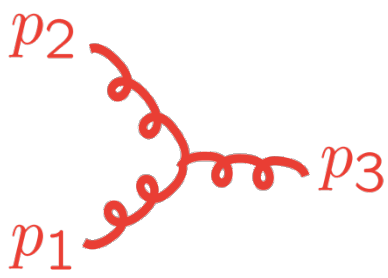
Self-Dual Kinematic Algebra

Self dual YM in light-cone gauge:

Monteiro and O'Connell

Generators of diffeomorphism invariance:

$$L_k = e^{-ik\cdot x}(-k_w \partial_u + k_u \partial_w)$$



Lie Algebra:

YM vertex

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1 p_2}{}^{k} L_k$$

The $X(p_1, p_2)$ are YM vertices of type ++– helicity.

Diffeomorphism symmetry hidden in YM theory!

Self dual sector gives +++…+ amplitudes: only one-loop S-matrix.

Boels, Isermann, Monteiro, O'Connell

We need to find the algebra beyond that.

Order-by-order Lagrangian

- First attempt at Lagrangian with manifest duality

1004.0693 [hep-th]
Bern, Dennen, Huang, Kiermaier

YM Lagrangian receives corrections at 5 points and higher

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \ldots$$

corrections proportional to the Jacobi identity (thus equal to zero)

$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}\big([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]\big)$$

Introduction of auxiliary "dynamical" fields gives local cubic Lagrangian

$$\mathcal{L}_{YM} = \tfrac{1}{2} A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - g f^{abc}\underbrace{(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu})A^{b\mu}A^{c\nu} \quad + \ldots}_{\text{kinematical structure constants}}$$

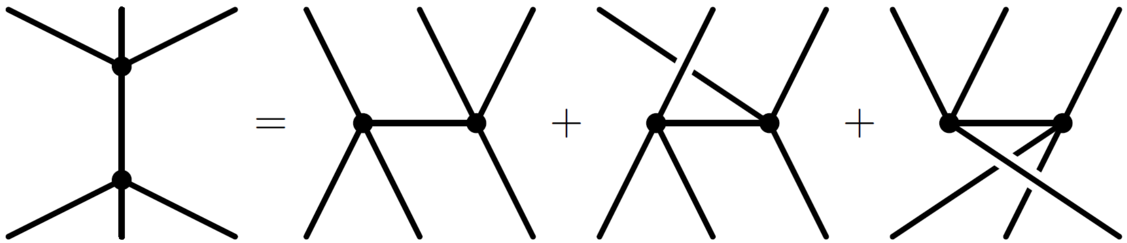
3-Algebra Color-Kinematics in *D*=3

BLG color-kinematics

D=3 Chern-Simons matter (CSM) theories obey color-kinematics duality.

3-algebra gauge group $[T^a, T^b, T^c] = f^{abc}{}_d T^d$ Bagger, Lambert, Gustavsson

Fundamental identity (Jacobi identity):



Bargheer, He, and McLoughlin

$$c_s = c_t + c_u + c_v \Leftrightarrow n_s = n_t + n_u + n_v$$

4 and 6 point checks shows that the double copy of BLG
Is *N* = 16 $E_{8(8)}$ SUGRA of Marcus and Schwarz

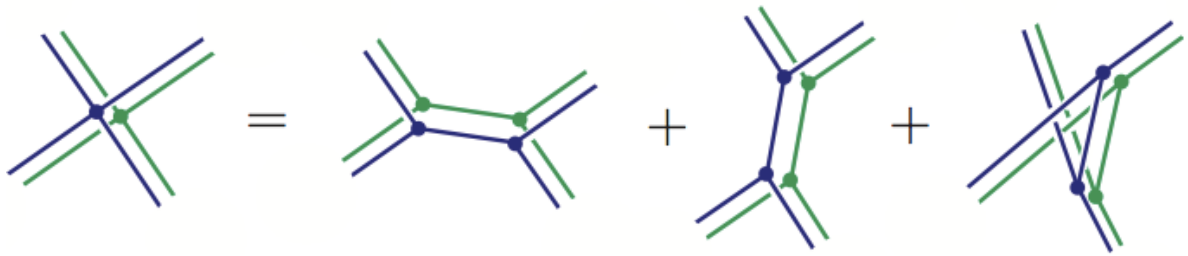
BLG ='square root' of N=16 SG $A_4^{\mathrm{BLG}} = \sqrt{M_4^{\mathcal{N}=16}} = \sqrt{\dfrac{\delta^{16}(Q)}{stu}}$

Same D=3 Supergravity Either Way !

In *D*=3, supergravity obtained from in two different double copies:

$$\mathcal{M}_m = \sum_{\substack{j \in \text{cubic} \\ N_j \in \text{2-algebra}}} \frac{N_j \tilde{N}_j}{\prod_{\beta_j} s_{\beta_j}} = \sum_{\substack{i \in \text{quartic} \\ n_i \in \text{3-algebra}}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}$$

Huang, H.J.



CSM ⊗ CSM = SYM ⊗ SYM

- Dimension mismatch? → propagators in SYM ⊗ SYM compensates!
- Odd matrix element mismatch? → double copy enhances R symmety!

SYM: $SO(7) \otimes SO(7) \rightarrow SO(16)$

CSM: $SO(8) \otimes SO(8) \rightarrow SO(16)$

For N=16 SG: all states are SO(16) spinors → no odd S-matrix elements

Marcus and Schwarz

Different D=3 Supergravity Theories

Verified double copy constructions:

Huang, H.J.

SG theory	$\text{CSm}_\text{L}\times\text{CSm}_\text{R}$ = supergravity	$\text{sYm}_\text{L}\times\text{sYM}_\text{R}$ = supergravity	coset
$\mathcal{N}=16$	$16^2=256$	$16^2=256$	$\text{E}_{8(8)}/\text{SO}(16)$
$\mathcal{N}=12$	$8^2+\bar{8}^2 = 16\times(4+\bar{4}) = 128$	$16\times 8=128$	$\text{E}_{7(-5)}/\text{SO}(12)\otimes\text{SO}(3)$
$\mathcal{N}=10$	$8\times 4+\bar{8}\times\bar{4} = 16\times(2+\bar{2}) = 64$	$16\times 4=64$	$\text{E}_{6(-14)}/\text{SO}(10)\otimes\text{SO}(2)$
$\mathcal{N}=8,\ n=2$	$4^2+\bar{4}^2 = 8\times 2+\bar{8}\times\bar{2} = 32$	$16\times 2=32$	$\text{SO}(8{,}2)/\text{SO}(8)\otimes\text{SO}(2)$
$\mathcal{N}=8,\ n=1$	$16\times 1=16$	$16\times 1=16$	$\text{SO}(8{,}1)/\text{SO}(8)$

Examples 4pts:

$$\mathcal{M}_4^{\mathcal{N}=12}(\bar{1},2,\bar{3},4) = (A_4^{\mathcal{N}=6})^2 = \left(\frac{\delta^{(6)}(\sum_i \lambda^\alpha \eta_i^I)}{\langle 1\,2\rangle\,\langle 2\,3\rangle}\right)^2$$

$$\mathcal{M}_{4,n=2}^{\mathcal{N}=8}(\bar{1},2,\bar{3},4) = (A_4^{\mathcal{N}=4})^2 = \left(\frac{\delta^{(4)}(\sum_i \lambda^\alpha \eta_i^I)\,\langle 1\,3\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle}\right)^2$$

$$\mathcal{M}_{4,n=1}^{\mathcal{N}=8} = \frac{1}{2}\,\frac{\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^I)(s^2+t^2+u^2)}{\langle 1\,2\rangle^2\,\langle 2\,3\rangle^2\,\langle 1\,3\rangle^2}$$

checked double copy up to 6pts!

Example calculation

Goal: Calculate 3-loop 4-pt N=8 SG ampl.

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- First find a duality-satisfying N=4 SYM ampl.
- Square each kinematic numerator → N=8 SG.
- See: 1201.5366 [hep-th] for this example.

Step 1. List diagram topologies

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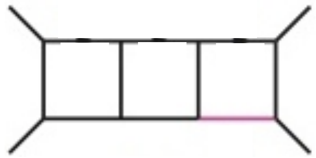

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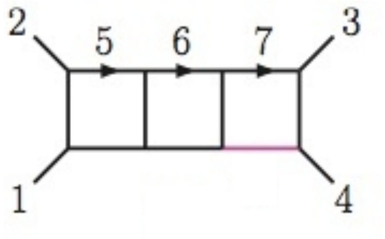

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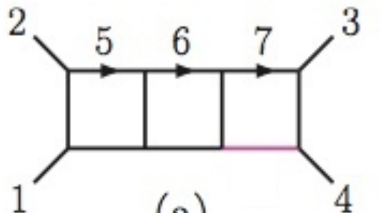

(a)

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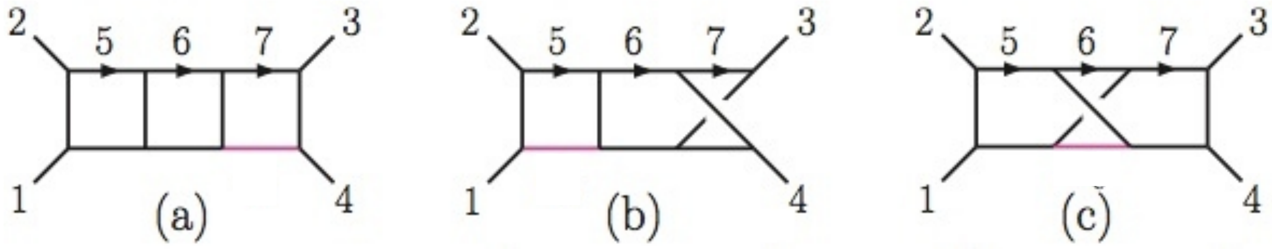

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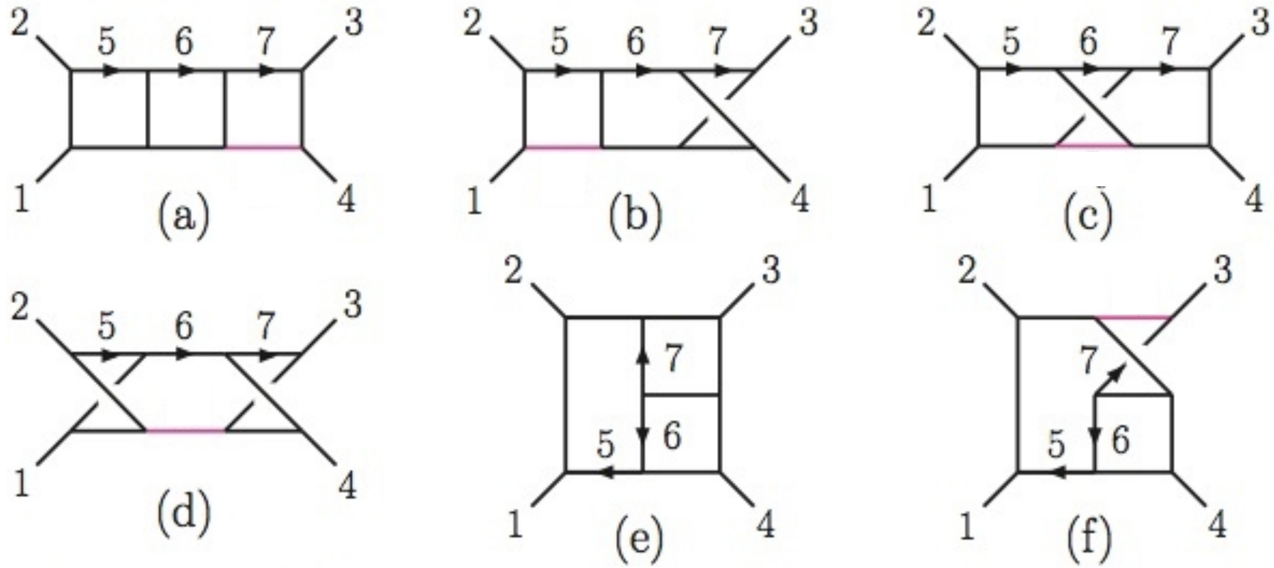

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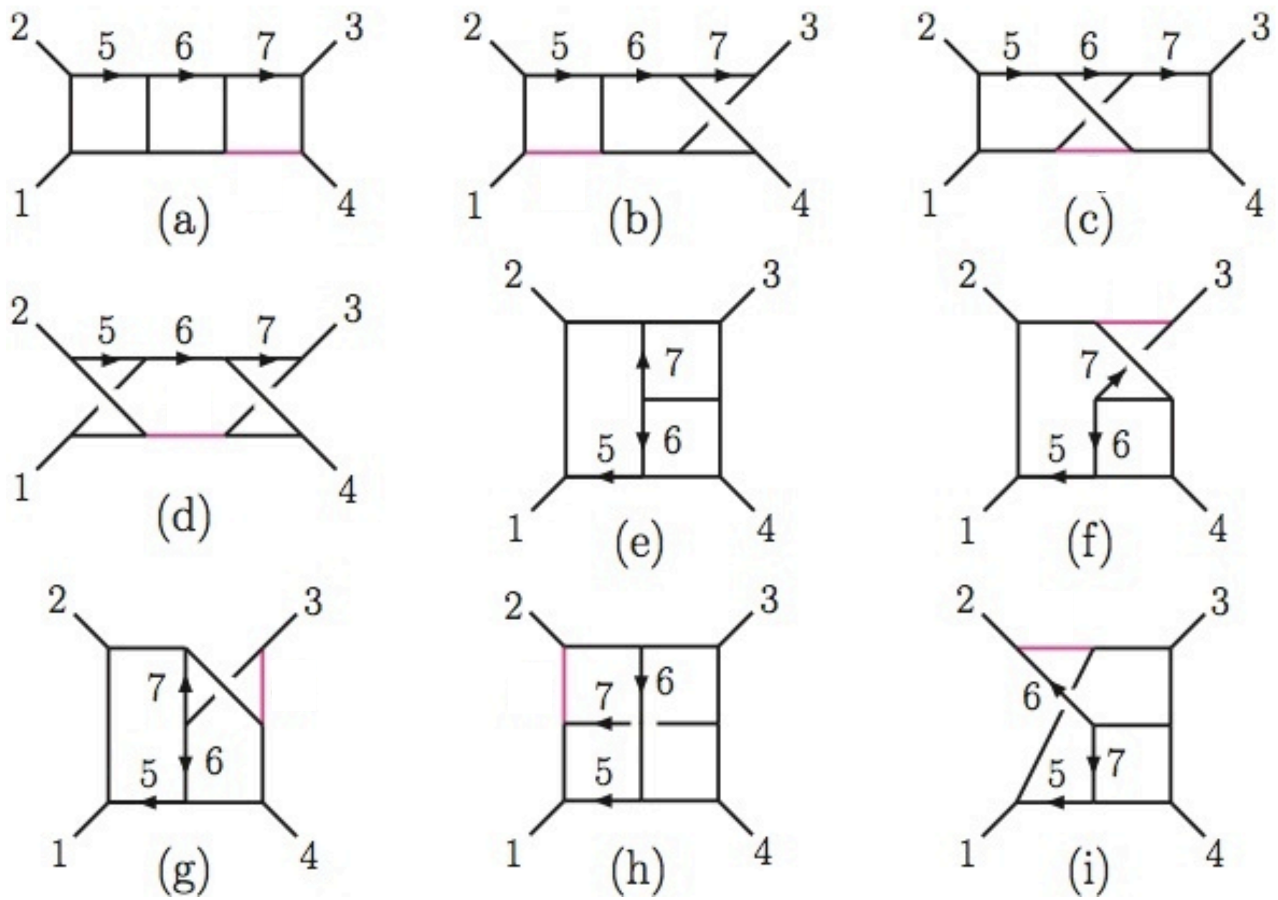

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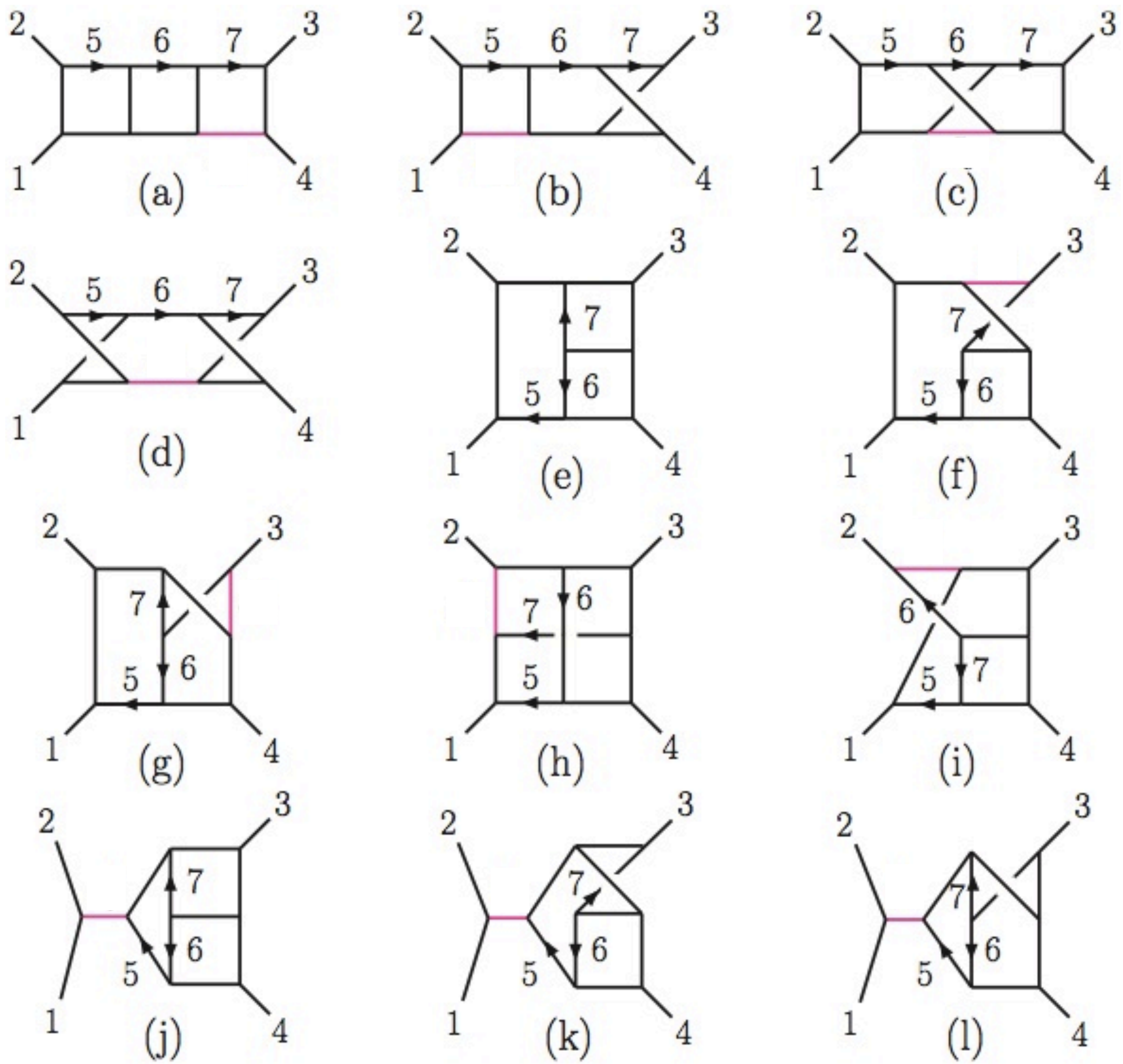
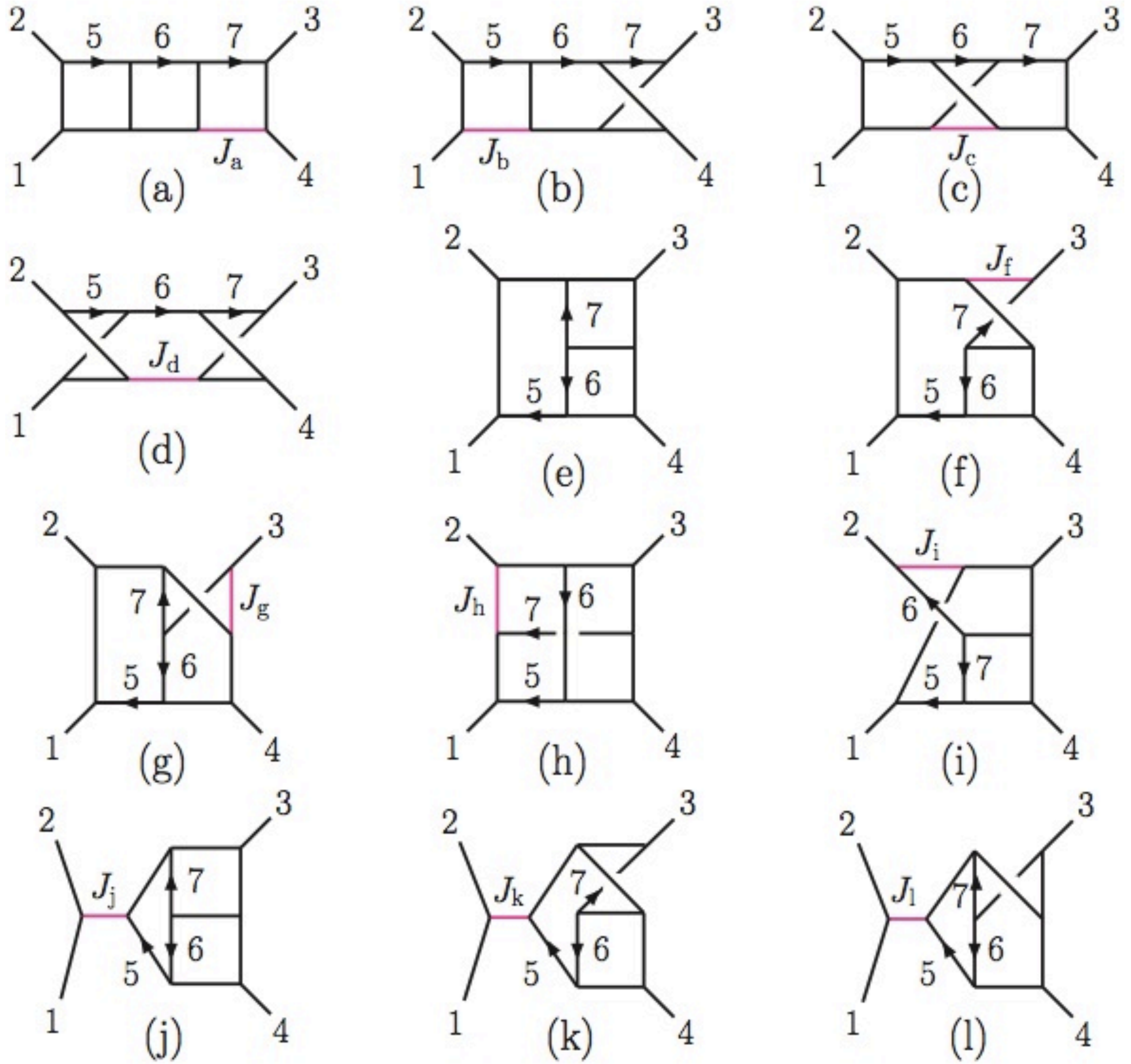


Step 2. Identify Jacobi relations



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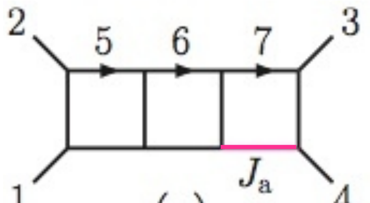

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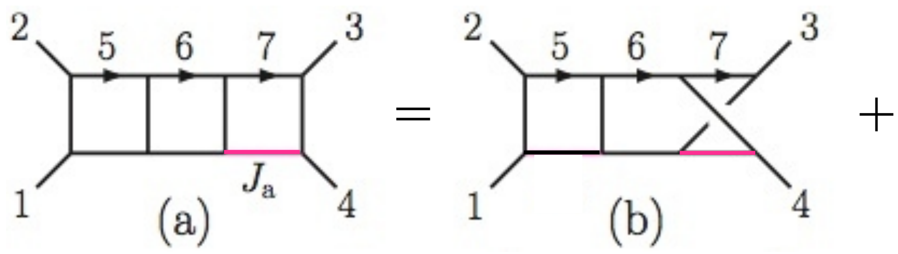

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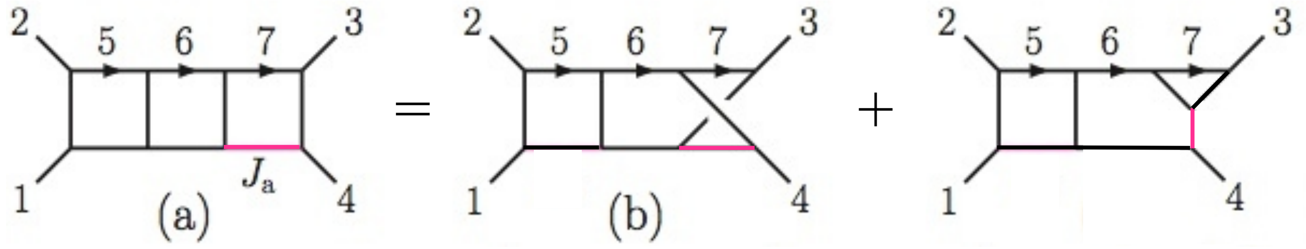

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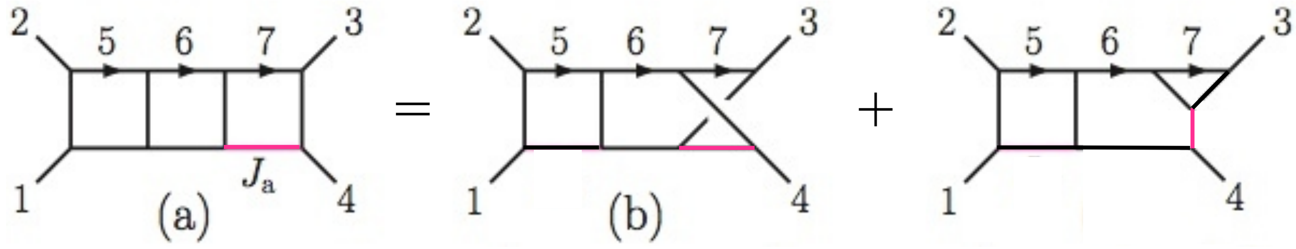
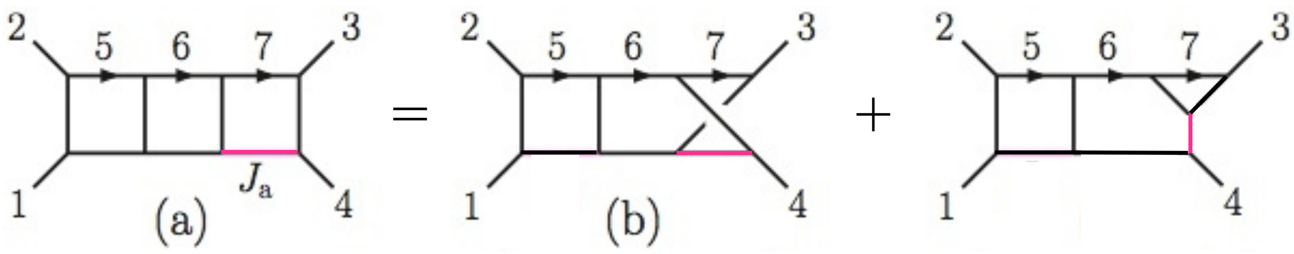


Some simplifications due to N=4 susy:

$$n^{(x)} = stA_4^{\text{tree}}(1,2,3,4)\,N^{(x)}\,,$$
$$N^{(x)} \equiv N^{(x)}(k_1,k_2,k_3,l_5,l_6,l_7)$$

Step 2. Identify Jacobi relations



Some simplifications due to N=4 susy:

crossing symmetric

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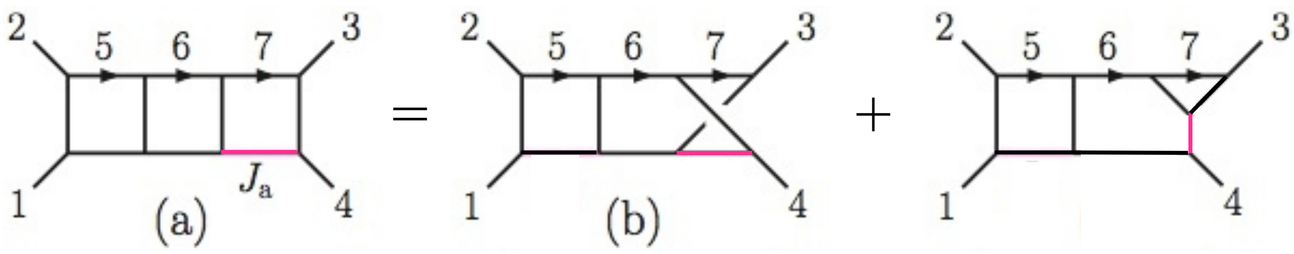
functions of external & internal momenta

Step 2. Identify Jacobi relations



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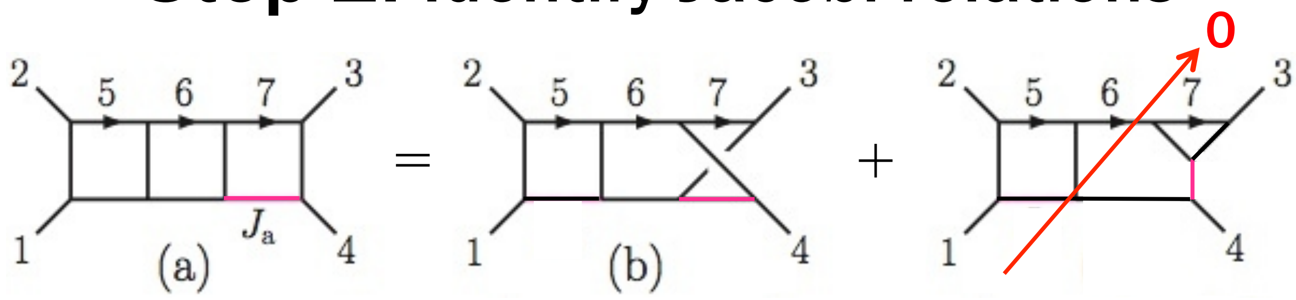
functions of external & internal momenta

Kinematic Jacobi Id.

$$N^{(\text{a})}(k_1,k_2,k_3,l_5,l_6,l_7) = N^{(\text{b})}(k_1,k_2,k_3,l_5,l_6,l_7) + N^{(\text{tri})}(k_1,k_2,k_3,l_5,l_6,l_7)$$

Step 2. Identify Jacobi relations



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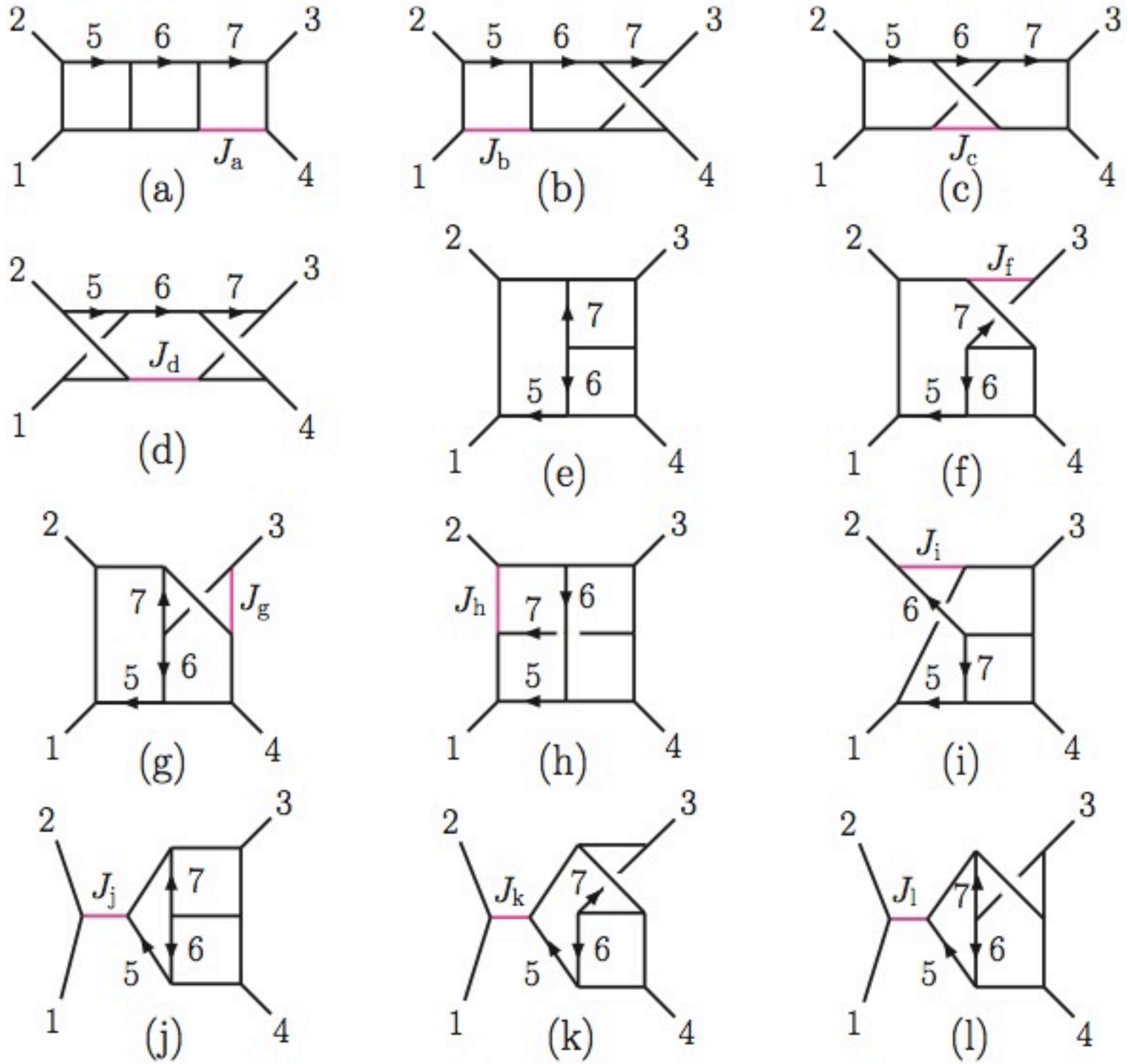
functions of external & internal momenta

Kinematic Jacobi Id.

$$N^{(a)}(k_1,k_2,k_3,l_5,l_6,l_7) = N^{(b)}(k_1,k_2,k_3,l_5,l_6,l_7) + N^{(\text{tri})}(k_1,k_2,k_3,l_5,l_6,l_7)$$

0

Step 2. Identify Jacobi relations



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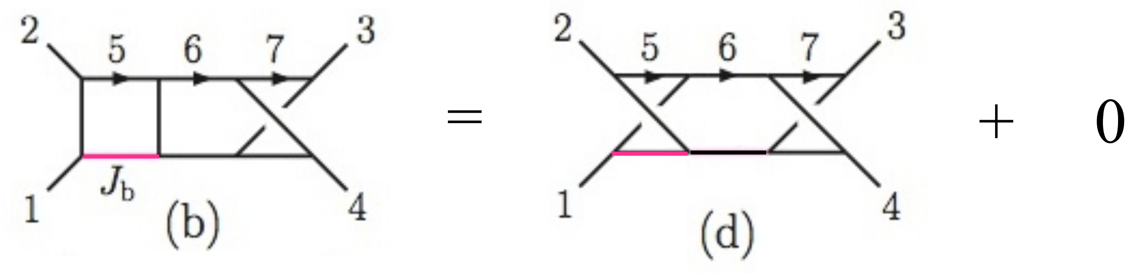

$$N^{(b)}(k_1, k_2, k_3, l_5, l_6, l_7) = N^{(d)}(k_1, k_2, k_3, l_5, l_6, l_7) \; + \; 0$$

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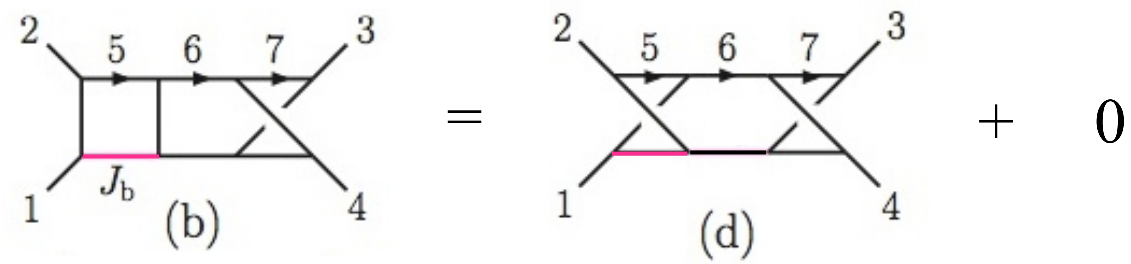
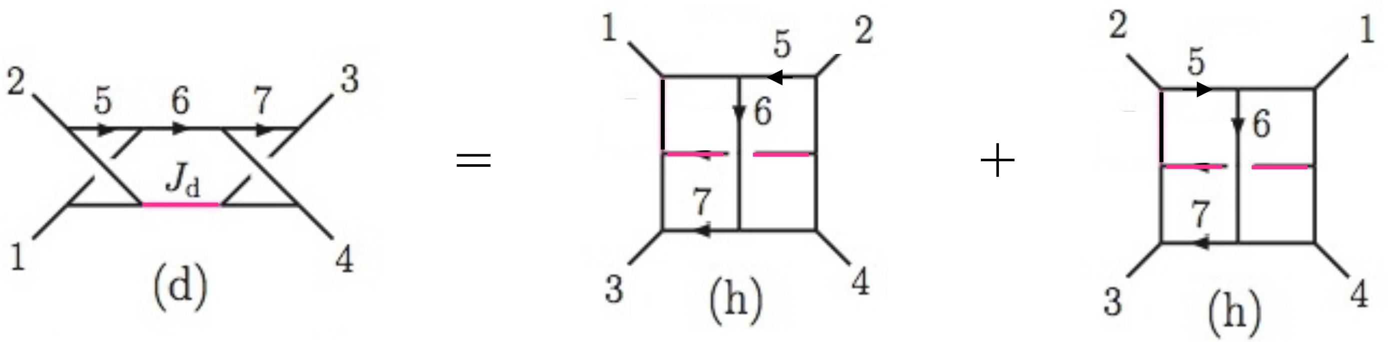


$$N^{(\mathrm{b})}(k_1, k_2, k_3, l_5, l_6, l_7) = N^{(\mathrm{d})}(k_1, k_2, k_3, l_5, l_6, l_7) \;+\; 0$$



$$N^{(\mathrm{d})} \;=\; N^{(\mathrm{h})}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7) + N^{(\mathrm{h})}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7)$$

Step 3. Reduce to master numerators

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The marked Jacobi relations $J_{\rm a}, J_{\rm b}, \ldots, J_{\rm k}, J_{\rm l}$ gives functional eqns

$$
\begin{aligned}
N^{(\rm a)} &= N^{(\rm b)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\rm b)} &= N^{(\rm d)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
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N^{(\rm d)} &= N^{(\rm h)}(k_3,k_1,k_2,l_7,l_6,k_{1,3}-l_5+l_6-l_7)+N^{(\rm h)}(k_3,k_2,k_1,l_7,l_6,k_{2,3}+l_5-l_7)\,,\\
N^{(\rm f)} &= N^{(\rm e)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\rm g)} &= N^{(\rm e)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\rm h)} &= -N^{(\rm g)}(k_1,k_2,k_3,l_5,l_6,k_{1,2}-l_5-l_7)-N^{(\rm i)}(k_4,k_3,k_2,l_6-l_5,l_5-l_6+l_7-k_{1,2},l_6)\,,\\
N^{(\rm i)} &= N^{(\rm e)}(k_1,k_2,k_3,l_5,l_7,l_6)-N^{(\rm e)}(k_3,k_2,k_1,-k_4-l_5-l_6,-l_6-l_7,l_6)\,,\\
N^{(\rm j)} &= N^{(\rm e)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(\rm e)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\rm k)} &= N^{(\rm f)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(\rm f)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\rm l)} &= N^{(\rm g)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(\rm g)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,
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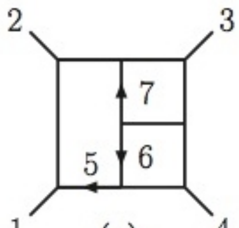
\end{aligned}
$$

Note: all numerators can be reduced to linear combinations of $N^{(\rm e)}$

$N^{(\rm e)}(k_1, k_2, k_3, l_5, l_6, l_7)$ is a “master numerator”



(e)

Step 4. Use Ansatz for master(s)

To simplify the ansatz we use auxiliary constraints (specific to *N*=4):

1) n-gon subgraphs carries at most n - 4 powers of loop momenta

2) $N^{(x)}$ are polynomials in Lorents products of momenta.

3) $N^{(x)}$ have the (crossing) symmetries of theirs graphs.

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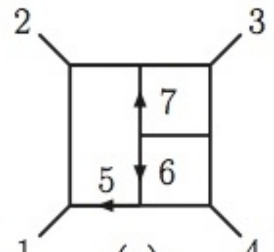
➔ $$N^{(\mathrm{e})} = N^{(\mathrm{e})}(k_1, k_2, k_3, l_5)$$



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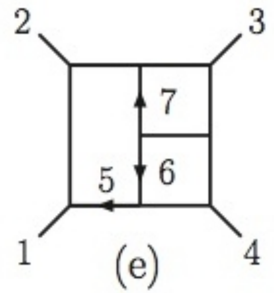

Physical constraint: on the maximal unitarity cut $N^{(\mathrm{e})} \to s(l_5 + k_4)^2$

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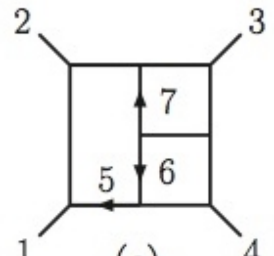

(e)

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This gives a four-parameter ansatz:

$$N^{(\mathrm{e})} = s(l_5 + k_4)^2 + (\alpha s + \beta t) l_5^2 + (\gamma s + \delta t)(l_5 - k_1)^2 + (\alpha s + \beta t)(l_5 - k_1 - k_2)^2$$

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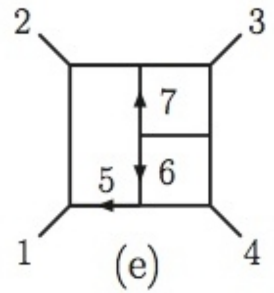

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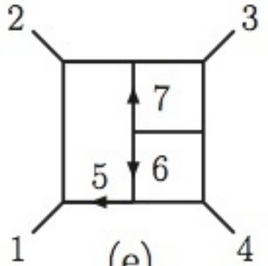
Enforcing linearity in l_5 : $\gamma = -1 - 2\alpha$ $\delta = -2\beta$

Step 5. Impose constraints on derived numerators

$$N^{(\mathrm{e})} = s(\tau_{45} + \tau_{15}) + (\alpha s + \beta t)(s + \tau_{15} - \tau_{25})$$



(e)

$$\tau_{ij} = 2k_i \cdot l_j$$

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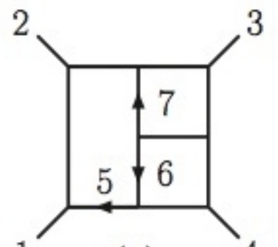
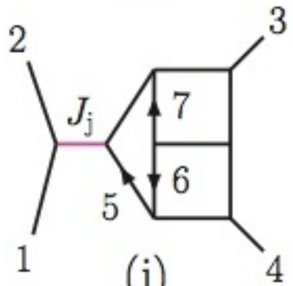


(e)

$$\tau_{ij} = 2k_i \cdot l_j$$

$$\begin{aligned} N^{(j)} &= N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(e)}(k_2, k_1, k_3, l_5, l_6, l_7) \\ &= s(1 + 2\alpha - \beta)(\tau_{15} - \tau_{25}) + \beta s(t - u) \end{aligned}$$

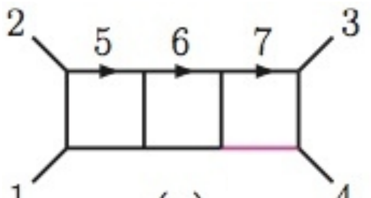


(j)

Only boxes (4-gons): $\beta = 1 + 2\alpha$

Step 5. Impose constraints on derived numerators

$$\begin{aligned} N^{(a)} &= N^{(e)}(k_1, k_2, k_4, -k_3 + l_5 - l_6 + l_7, l_5 - l_6, -l_5) \\ &\quad + N^{(e)}(k_2, k_1, k_4, -k_3 - l_5 + l_7, -l_5, l_5 - l_6) \\ &\quad - N^{(e)}(k_4, k_1, k_2, l_6 - l_7, l_6, l_5 - l_6) - N^{(e)}(k_4, k_2, k_1, l_6 - l_7, l_6, -l_5) \\ &\quad - N^{(e)}(k_3, k_1, k_2, l_7, l_6, l_5 - l_6) - N^{(e)}(k_3, k_2, k_1, l_7, l_6, -l_5)\,. \\ &= s^2 + (1+3\alpha)\Big((\tau_{16} - \tau_{46})s - 2(\tau_{17} + \tau_{37})s + (\tau_{16} - 2\tau_{17} - \tau_{26} + 2\tau_{27})t + 4ut\Big) \end{aligned}$$

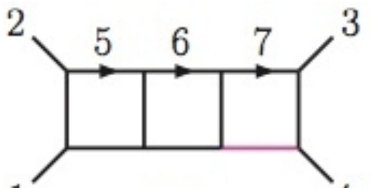


(a)

Only boxes (4-gons): $\alpha = -\dfrac{1}{3}$

Step 5. Impose constraints on derived numerators

$$
\begin{aligned}
N^{(a)} &= N^{(e)}(k_1, k_2, k_4, -k_3 + l_5 - l_6 + l_7, l_5 - l_6, -l_5) \\
&\quad + N^{(e)}(k_2, k_1, k_4, -k_3 - l_5 + l_7, -l_5, l_5 - l_6) \\
&\quad - N^{(e)}(k_4, k_1, k_2, l_6 - l_7, l_6, l_5 - l_6) - N^{(e)}(k_4, k_2, k_1, l_6 - l_7, l_6, -l_5) \\
&\quad - N^{(e)}(k_3, k_1, k_2, l_7, l_6, l_5 - l_6) - N^{(e)}(k_3, k_2, k_1, l_7, l_6, -l_5)\,. \\
&= s^2 + (1 + 3\alpha)\Big((\tau_{16} - \tau_{46})s - 2(\tau_{17} + \tau_{37})s + (\tau_{16} - 2\tau_{17} - \tau_{26} + 2\tau_{27})t + 4ut\Big)
\end{aligned}
$$

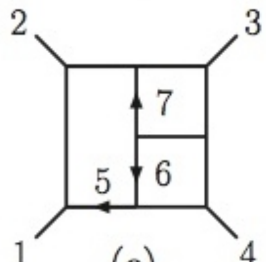


(a)

Only boxes (4-gons): $\alpha = -\frac{1}{3}$

Final solution for master:

$$
N^{(e)} = s(\tau_{45} + \tau_{15}) + \frac{1}{3}(t - s)(s + \tau_{15} - \tau_{25})
$$

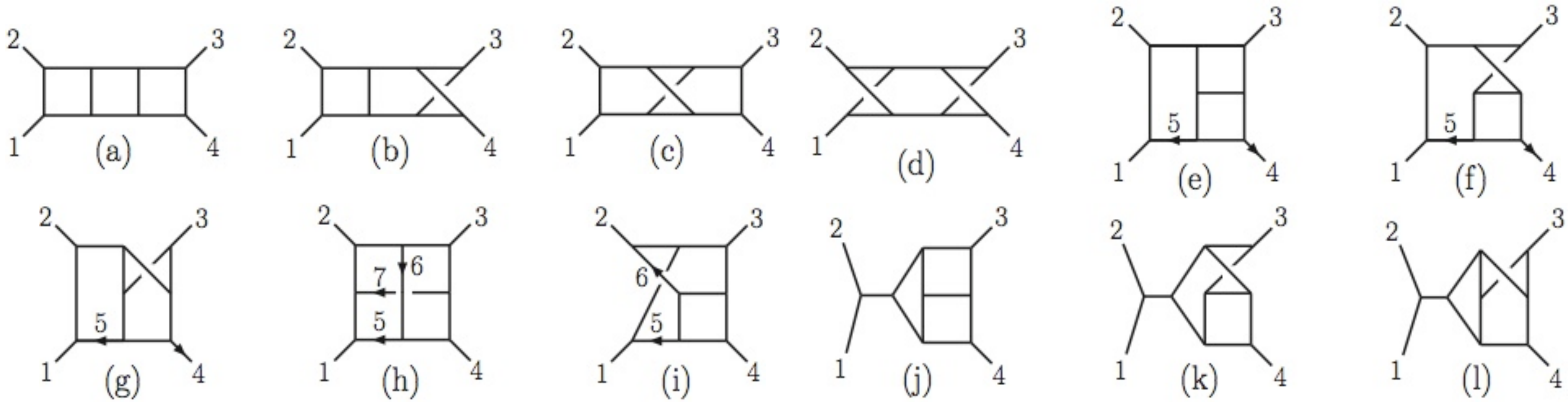


(e)

➔ N=4 SYM and N=8 SUGRA amplitude integrands fully determined

Collecting the result

1004.0476 [hep-th] Bern, Carrasco, HJ



Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t)-t\,(\tau_{25}+\tau_{45})+u\,(\tau_{25}+\tau_{35})-s^2\right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)\right.$ $\left.+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)\right.$ $\left.+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

Used to show absence of N=4 SG divergence   Bern, Davies, Dennen, Huang

Summary

- Yang-Mills theories are controlled by a kinematic Lie 2-algebra
- Chern-Simons-matter theories controlled by a kinematic Lie 3-algebra
- With duality manifest: Gravity becomes double copy of Yang-Mills theory for any dim., or, in D=3, of Chern-Simons-matter theory
- A complete representation of the kinematic algebra is still missing for all but the simplest case of self-dual Yang-Mills.
- Constructing CK-amplitude representations is nonetheless possible, case by case. Double-copy formula gives gravity integrands for free.
- Duality is a key tool for nonplanar gauge and gravity calculations.
  - $\mathcal{N}$=8 supergravity UV behavior at five (seven) loops ?
  - D=4 UV divergence 3,4 loops $\mathcal{N}$=4 supergravity ?

THANK YOU!