Scalar cosmology

J.W. van Holten

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How does a universe driven by a (single) scalar field evolve?

Framework: flat FLRW universes

$\rightarrow$ degrees of freedom $(a(t), \varphi(t))$

with $8\pi G = 1$ and $H = \dot{a}/a$:

$$\boxed{\begin{aligned} &\frac{1}{2}\dot{\varphi}^2 + V = 3H^2 \\ &\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \end{aligned}}$$

(assuming $H \neq 0$)

$$\frac{1}{2}\dot{\varphi}^2 + V = 3H^2$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$$

For single scalar-field models

$$H(t) = H[\varphi(t)] \quad \Rightarrow \quad \dot{H} = H'\dot{\varphi}$$

Also

$$\dot{\varphi}\left(\ddot{\varphi} + V'\right) = -3H\dot{\varphi}^2 = 6H\dot{H}$$

Results:

$$\dot{H} = -\tfrac{1}{2}\dot{\varphi}^2 \leq 0, \qquad \dot{\varphi} = -2H'$$

monotonically decreasing H;

potential: $V = 3H^2 - 2H'^2$

$$\boxed{\dot{\varphi} = -2H'}$$

Stationary points of scalar field:

$$\dot{\varphi} = 0 \quad \Rightarrow \quad H' = 0$$

2 kinds:

- end points: $\ddot{\varphi} = 4H''H' = 0 \qquad \Rightarrow |H''| < \infty$

- turning points: $\ddot{\varphi} = 4H''H' \neq 0 \qquad \Rightarrow H'' \propto 1/H' \to \infty$

Example: eternally oscillating field

$$\varphi(t) = \varphi_0 \cos \omega t$$

Turning points:

$$H' = -\frac{1}{2}\dot{\varphi} = \frac{\omega}{2}\varphi_0 \sin \omega t = \frac{\omega}{2}\sqrt{\varphi_0^2 - \varphi^2}$$

$$= 0 \quad \Leftrightarrow \quad \omega t_n = n\pi$$

At these points

$$H'' = -\frac{\omega}{2}\frac{\varphi}{\sqrt{\varphi_0^2 - \varphi^2}} \quad \rightarrow \quad \infty$$

and

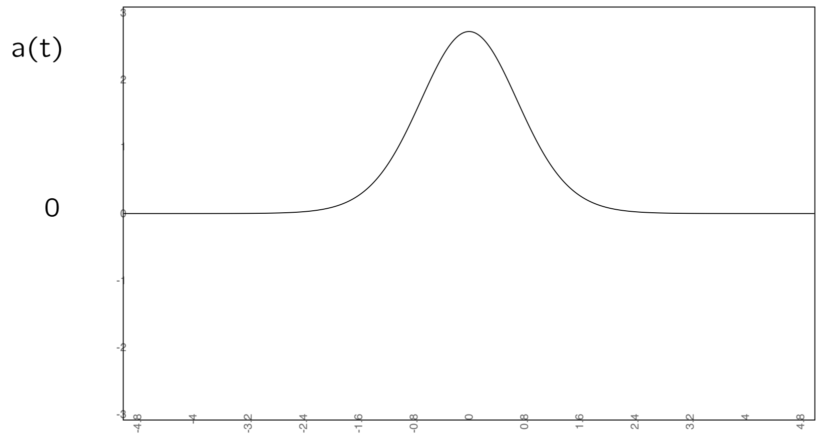
$$\ddot{\varphi} = 4H''H' = -\omega^2\varphi < \infty$$

Solution:

$$
\begin{aligned}
H &= H_0 - \frac{1}{4}\,\omega\,\varphi_0^2 \,\mathrm{arccos}\left(\frac{\varphi}{\varphi_0}\right) + \frac{1}{4}\,\omega\,\varphi\sqrt{\varphi_0^2 - \varphi^2} \\
&= H_0 - \frac{1}{4}\,\varphi_0^2\,\omega^2 t + \frac{1}{8}\,\varphi_0^2\,\omega \sin 2\omega t
\end{aligned}
$$

and

$$
a(t) = a_0 e^{H_0 t - \frac{1}{8}\,\varphi_0^2\,\omega^2 t^2 + \frac{1}{16}\,\varphi_0^2 (1 - \cos 2\omega t)}
$$



Potential

$$
\begin{aligned}
V &= 3H^2 - 2H'^2 \\
&= 3\left(H_0 - \frac{\omega\varphi_0^2}{4}\arccos\frac{\varphi}{\varphi_0} + \frac{1}{4}\omega\varphi\sqrt{\varphi_0^2 - \varphi^2}\right)^2 - \frac{\omega^2}{2}\left(\varphi_0^2 - \varphi^2\right). \\
&= -\frac{\omega^2\varphi_0^2}{4} + \frac{\omega^2\varphi_0^2}{4}\cos 2\omega t + 3\left(H_0 + \frac{1}{8}\varphi_0^2\,\omega\sin 2\omega t - \frac{1}{4}\varphi_0^2\,\omega^2 t\right)^2
\end{aligned}
$$

Potential keeps track of number of oscillations of φ.

$$V = 3H^2 - 2H'^2$$

Final state problem

Stationary scalar field: $\dot{\varphi} = 0$

$$\rightarrow \quad \begin{cases} \ddot{\varphi} = 0, & \text{end point of evolution;} \\ \ddot{\varphi} \neq 0, & \text{turning point.} \end{cases}$$

Can happen only if

$$V = 3H^2 \geq 0.$$

Otherwise

$$V < 0 \quad \rightarrow \quad \dot{\varphi} \neq 0, \quad \dot{H} < 0,$$

implying collapse as soon as $H < 0$.

Collapse can be prevented only if there exists an end point where H is non-negative:

$$H \geq 0, \quad H' = 0, \quad |H''| < \infty.$$

The first 2 conditions imply

$$V = 3H^2 \geq 0.$$

$$\rightarrow \quad \begin{cases} \text{end point at } V, H = 0, & \text{Minkowski space;} \\ \text{end point at } V, H > 0, & \text{De Sitter space.} \end{cases}$$

In addition

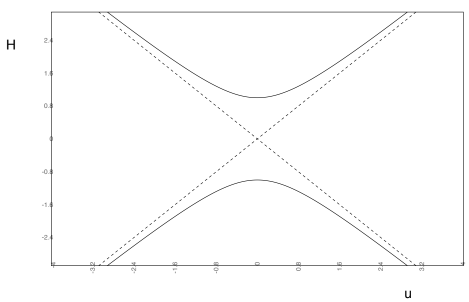
$$V' = 2H'(3H - 2H'')$$

hence end points necessarily require $V' = 0$, an extremum of V.

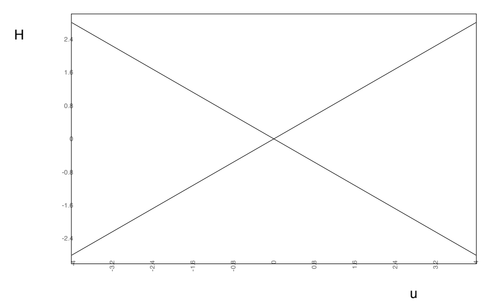
Example: quadratic potentials

$$V = \varepsilon + \frac{m^2}{2}\varphi^2$$

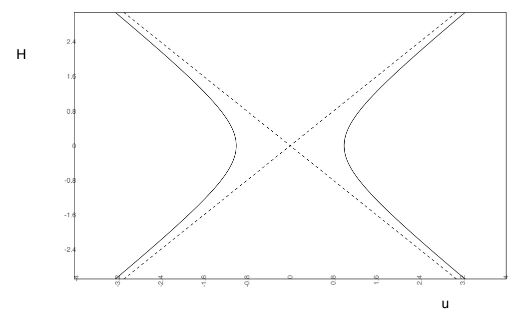
Turning points: $V = 3H^2$



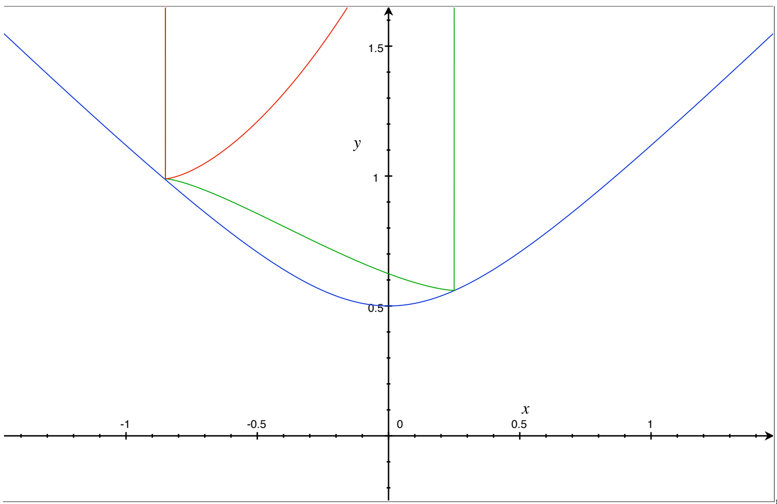
$\varepsilon > 0$



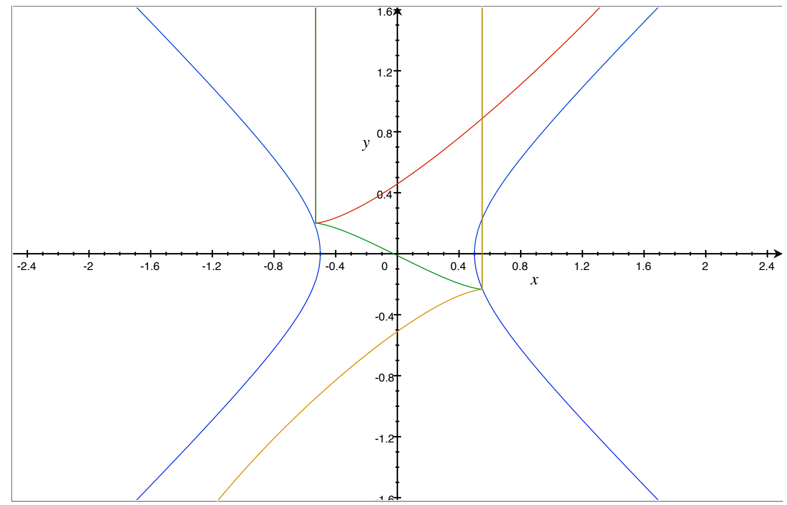
$\varepsilon = 0$



$\varepsilon < 0$



$\epsilon > 0$



$\epsilon < 0$

Constructing solutions

Series expansion

$$H[\varphi] = h_0 + h_1\varphi + h_2\varphi^2 + h_3\varphi^3 + ...$$

with coefficients given by

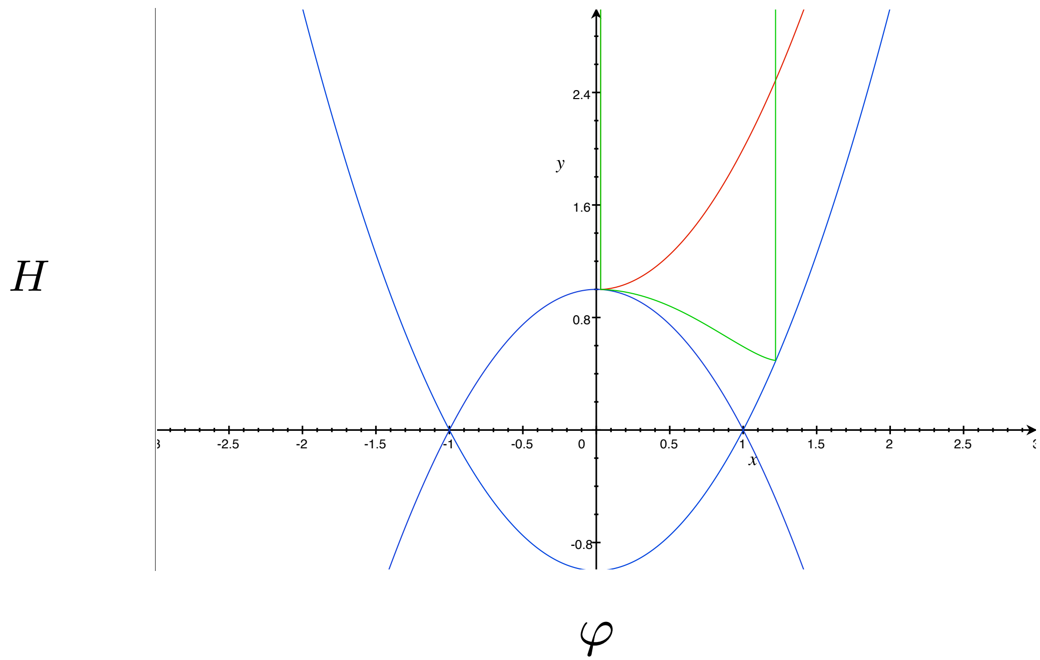
$$3h_0^2 - 2h_1^2 = \varepsilon, \quad h_1(3h_0 - 4h_2) = 0,$$

$$4h_1(h_1 - 4h_3) + \frac{8}{3}h_2(3h_0 - 4h_2) = m^2, \quad ...$$

Expansion factor

$$\begin{aligned} N &= \int_1^2 dt\, H = -\int_1^2 d\varphi\, \frac{H}{2H'} \\ &= -\frac{1}{2}\int_1^2 d\varphi\, \frac{h_0 + h_1\varphi + h_2\varphi^2 + ...}{h_1 + 2h_2\varphi^2 + 3h_3\varphi^2 + ...} \end{aligned}$$

Higgs potential: $V = \varepsilon - \frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$



$$\epsilon = -\frac{\dot{H}}{H^2} = 2\,\frac{H'^2}{H^2} < 1 \quad \Rightarrow \quad \frac{V}{3} < H^2 < \frac{V}{2}$$

For example:

$$\varepsilon = 3h^2, \qquad \lambda = \frac{3\omega^2}{4}, \qquad \mu^2 = \omega(\omega - 3h),$$

gives a solution

$$\varphi(t) = \sqrt{8N}\, e^{-\omega t}, \qquad a(t) = a(0)e^{ht+N(1-e^{-2\omega t})}$$