Instantons on Special Geometries

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Contents

Instantons in String Theory

- Higher-dimensional Super-Yang-Mills theory appears in the low-energy limit of the heterotic superstring
- String compactification: Spacetime is decomposed as

$$M^{10} = M^{10-n} \times X^n$$

- Instanton equations on X^n arise as conditions of supersymmetry preservation in heterotic superstring compactifications
- Instantons on special geometries can be lifted to solutions of heterotic supergravity ($\rightarrow$ brane interpretation)

Aim of my work:
Better understanding of instantons on special geometries

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Instantons in Higher Dimensions

- $\mathcal{A}$: connection on a bundle over a manifold X with curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$.
- Anti-self-duality equation in four dimensions ($\dim(X) = 4$):

$$*\mathcal{F} = -\mathcal{F} \tag{1}$$

- In higher dimensions ($\dim(X) = n$): Assume there exists a 4-form $Q \in \Omega^4(X)$
- Instanton equation (generalized anti-self-duality equation):

$$*\mathcal{F} = - * Q \wedge \mathcal{F} \tag{2}$$

- Consider a coset space G/H, where G is a compact semisimple Lie group and H a closed Lie subgroup
- We work with the cone over this coset space:

$$X := \mathcal{C}(G/H) = (\mathbb{R} \times G/H, g_{cone})$$
$$g_{cone} = e^{2\tau}\left(d\tau^2 + g_{G/H}\right) \tag{3}$$

τ: parameter in $\mathbb{R}$-direction

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Special Geometries

Why do we consider cones over coset spaces?

- A family of explicit higher-dimensional instanton solutions exists on Euclidean spaces and on certain conical manifolds.
- Cones over special holonomy manifolds are part of the manifolds on which certain branes are defined:

$$M = \mathbb{R}^{1,1} \times \mathbb{T}^{7-k} \times \mathcal{C}(G/H)$$

- Instantons on cones over 5-dimensional Sasaki, 6-dimensional nearly Kähler, 7-dimensional 3-Sasakian and nearly parallel G_2-manifolds can be lifted to solutions of heterotic supergravity.

Lechtenfeld et.al [arxiv 1108.3951, 1202.5046]

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Reduction to Matrix Equations

- Instanton equation $*\mathcal{F} = - * Q \wedge \mathcal{F}$ in components:

$$\mathcal{F}_{0a} = -\frac{1}{2} Q_{0acd} \mathcal{F}_{cd} \qquad \mathcal{F}_{ab} = \frac{1}{2} (Q_{0eab} Q_{0ecd} - Q_{abcd}) \mathcal{F}_{cd} \tag{4}$$

- Ansatz for the gauge potential:

$$\mathcal{A} = e^i I_i + e^a X_a(\tau) \tag{5}$$

$X_a(\tau)$	matrix-valued function that satisfies the equivariance condition $[I_i, X_a] = f_{ia}^b X_b$
I_A	generators of G
e^A	coframe on $\mathcal{C}(G/H)$
f_{AB}^C	structure constants of G

- Curvature:

$$\mathcal{F} = \frac{dX_a}{d\tau} e^0 \wedge e^a - \frac{1}{2} \left(f_{ab}^i I_i + f_{ab}^e X_e - [X_a, X_b] \right) e^a \wedge e^b \tag{6}$$

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Reduction to Matrix Equations

- With this ansatz, the instanton equations (4) reduce to matrix equations:

$$\frac{dX_a}{d\tau} = \frac{1}{2} Q_{0acd} \left(f^i_{cd} I_i + f^a_{cd} X_a - [X_c, X_d] \right) \tag{7}$$

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- Explicit expression for Q needed! Assumptions:
  - $g \propto \delta_{AB} e^A e^B$ (induced by Cartan-Killing form)
  - $f = \frac{1}{3!} f_{abc} e^a \wedge e^b \wedge e^c$
  - $Q = \beta_1 d\tau \wedge f + \beta_2 df$
- Equations can be further specified (expanding X_a in a basis and comparing coefficients)
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Special Example

- Suppose $\mathcal{A}$ is parametrized by only one function $\phi(\tau)$:

$$\mathcal{A} = e^i I_i + e^a X_a(\tau) = e^i I_i + \phi(\tau) e^a I_a \tag{9}$$

with $X_a(\tau) = \phi(\tau) I_a$

- The geometrical condition (8) is identically satisfied
- The differential equation (7) takes the form

$$\frac{d\phi}{d\tau} = \frac{1}{2}(\phi^2 - \phi) \tag{10}$$

- This is known as the Kink equation.

Interpretation: Lechtenfeld, Rahn et.al [arxiv 0904.0654]

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- Interpretation of instantons:
  brane solutions of heterotic supergravity
- Such branes are defined on manifolds of the form
  $M = \mathbb{R}^{1,1} \times \mathbb{T}^{7-k} \times \mathcal{C}(G/H)$
- We have constructed instanton conditions on $\mathcal{C}(G/H) \subset M$
  and solved them under special assumptions
- These solutions are building blocks for heterotic string vacua with explicitly non-trivial gauge field
- **Outlook:**
  Interpretation of instanton conditions in more general setups
  Lifting of the building blocks to heterotic supergravity solutions

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