Basics of Event Generators I

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Outline of Lectures

- Lecture I: Basics of Monte Carlo, the event generator strategy, matrix elements, LO/NLO, . . .
- Lecture II: Parton showers, Sudakov formfactors, initial/final state, angular ordering, $k_\perp$-factorization, . . .
- Lecture III: Underlying events, multiple interactions, minimum bias, pile-up, hadronization, decays, . . .

Outline of Lecture I

How do we calculate an integral of an arbitrary function $f(\boldsymbol{x})$?

$$I = \int_{\Omega} d^n\boldsymbol{x}\, f(\boldsymbol{x})$$

Simple discretization (Simpsons rule, Gaussian quadrature) can be extremely inefficient if

- n is large
- Ω is complicated
- $f(\boldsymbol{x})$ has peaks and divergencies.

Importance sampling

Assume we are able to generate random variables $\boldsymbol{X}_i$ such that

$$P\left(x^{(j)} < X_i^{(j)} < x^{(j)} + dx^{(j)}\right) = p_X(\boldsymbol{x})$$

if $p(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \Omega$ and zero outside, we can rewrite our integral

$$I = \int_\Omega d^n\boldsymbol{x} \frac{f(\boldsymbol{x})}{p_X(\boldsymbol{x})} p_X(\boldsymbol{x}).$$

Now, for any random variable Y, we know that

$$\frac{1}{N}\sum_{i=1}^{N} g(Y_i) \approx \langle g(Y) \rangle = \int_{-\infty}^{\infty} dy\, p_Y(y) g(y)$$

Hence

$$\left\langle \frac{f(\mathbf{X})}{p_X(\mathbf{X})} \right\rangle = \int_\Omega d^n\mathbf{x} \frac{f(\mathbf{x})}{p_X(\mathbf{x})} p_X(\mathbf{x}) = I$$

So, we can numerically estimate our integral by generating N points $\mathbf{X}_i$ and take the average of $f(\mathbf{X})/p_X(\mathbf{X})$.

In doing so we will get an error which we can estimate by

$$\delta \approx \sigma\left(\frac{f(\mathbf{X})}{p_X(\mathbf{X})}\right)/\sqrt{N}$$

where the variance is given by $\sigma^2(Y) = \langle Y^2 \rangle - \langle Y \rangle^2$.

Clearly if $p_X(\boldsymbol{x}) = C|f(\boldsymbol{x})|$, we get the smallest possible error (if $f(x) > 0$ the error is zero).

However, with a bad choice of p_X, the variance and the error need not even be finite.

Numerically generating points directly according to $p_X(\boldsymbol{x}) = C|f(\boldsymbol{x})|$ is in general difficult, and typically involves analytically solving the integral we want to estimate. But there are some tricks…

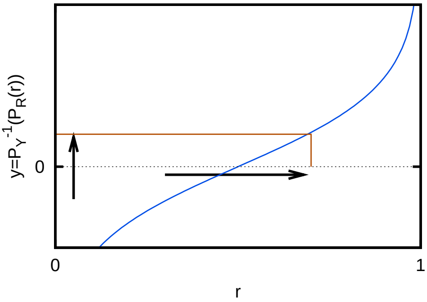
Normally we only have uniformly distributed (flat) random numbers available on the computer

$$p_R(r) = \begin{cases} 1 & 0 < r < 1 \\ 0 & \text{otherwise} \end{cases}$$

We can transform any distribution into any other by the a transformation using the cumulative distributions

$$P_Y(y) = \int_{-\infty}^{y} dt\, p_Y(t) = \int_{-\infty}^{r} dt\, p_R(t) = P_R(r)$$

as long as $P_Y^{-1}(P_R(r))$ is a monotonically increasing function.



If $P_Y^{-1}(P_R(r))$ is not monotonous, we can divide up in intervals.

What if P_Y^{-1} is hard to find . . .

The Accept/Reject Method

Assume we want to generate random variables, Y_i, according to some difficult distribution $p_Y(y)$. We already know how to generate according to some other distribution, $p_{Y'}(y)$ such that $Cp_{Y'}(y) \geq p_Y(y)$ everywhere.

1. Generate Y' according to $p_{Y'}(y)$
2. Generate R according to a flat distribution
3. - If $\frac{p_Y(Y')}{Cp_{Y'}(Y')} > R$ then accept $Y = Y'$
   - otherwise reject Y' and goto 1

The accepted Y will be distributed according to $p_Y(y)$.

We need $2C$ random numbers to get one Y.

Predicting an Observable

To calculate the expectation value of an observable, $\mathcal{O}$, in a $pp \to X$ collision we need to evaluate an integral looking like

$$\langle \mathcal{O} \rangle = \sum_n \sum_{\boldsymbol{Q}} \int d^{4n}\boldsymbol{p}\ \left|\mathcal{M}_n(\boldsymbol{Q},\boldsymbol{p})\right|^2 \mathcal{O}_n(\boldsymbol{Q},\boldsymbol{p})\,\boldsymbol{\Phi}_n(\boldsymbol{p})$$

- $\boldsymbol{p}$ is the momenta of the n particles
- $\boldsymbol{Q}$ is their quantum numbers
- $\mathcal{M}$ is the matrix element
- $\boldsymbol{\Phi}_n$ is the phase space density etc.

So now, all we need to do is to find a probability distribution $p(n, \boldsymbol{Q}, \boldsymbol{p})$ such that

$$C\,p(n, \boldsymbol{Q}, \boldsymbol{p}) = |\mathcal{M}_n(\boldsymbol{Q}, \boldsymbol{p})|^2\ \boldsymbol{\Phi}_n(\boldsymbol{p})$$

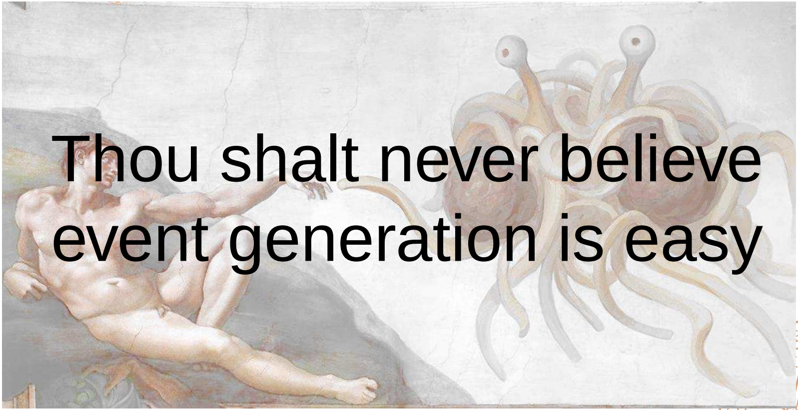
Then we generate N points, $(n_i, \boldsymbol{Q}_i, \boldsymbol{p}_i)$ according to this and get

$$\langle\mathcal{O}\rangle = \frac{C}{N}\sum_i^N \mathcal{O}_n(\boldsymbol{Q}_i, \boldsymbol{p}_i)$$

In the same way as we do when measuring the observable experimentally.

We are generating events. And we can measure several observables in one go. Life is simple!

The First Commandment of Event Generation



Thou shalt never believe event generation is easy

There are no free lunches

- $\mathcal{M}$ can typically only be calculated perturbatively to leading and maybe next-to-leading order for a small number of particles.
- $\mathbf{\Phi}_n$ is not trivial
- finding $p(n, \boldsymbol{Q}, \boldsymbol{p})$ may be very difficult

Weighted vs. Unweighted Events

We can, of course use any probability distribution and get

$$\langle\mathcal{O}\rangle = \frac{C}{N}\sum_{i}^{N}\frac{|\mathcal{M}_n(\boldsymbol{Q}_i,\boldsymbol{p}_i)|^2\;\boldsymbol{\Phi}_n(\boldsymbol{p}_i)}{p(n_i,\boldsymbol{Q}_i,\boldsymbol{p}_i)}\mathcal{O}_n(\boldsymbol{Q}_i,\boldsymbol{p}_i)$$

which means we get weighted events.

This is OK as long as the variance is not too big.

$$
\begin{aligned}
\langle \mathcal{O} \rangle \quad = \quad & \sum_{n_1, \boldsymbol{Q}_q} \int d^{4n_q}\boldsymbol{q} \, \left| \mathcal{M}_{n_q}(\boldsymbol{Q}_q, \boldsymbol{q}) \right|^2 \boldsymbol{\Phi}_{n_q}(\boldsymbol{q}) \times \\
& \left[\sum_{n_k, \boldsymbol{Q}_k} \int d^{4n_k}\boldsymbol{k} \, PS(\boldsymbol{Q}_q, \boldsymbol{q}; \boldsymbol{Q}_k, \boldsymbol{k}) \times \right. \\
& \left. \left\{ \sum_{n_p, \boldsymbol{Q}_p} \int d^{4n_p}\boldsymbol{p} \, H(\boldsymbol{Q}_k, \boldsymbol{k}; \boldsymbol{Q}_p, \boldsymbol{p}) \mathcal{O}_{n_p}(\boldsymbol{Q}_p, \boldsymbol{p}) \right\} \right]
\end{aligned}
$$

- $\mathcal{M}$ now only gives a few partons
- PS is a parton shower giving more partons with unit probability
- H is hadronization and decays giving final state hadrons with unit probability

Factorization

Relies on the factorization ansatz.

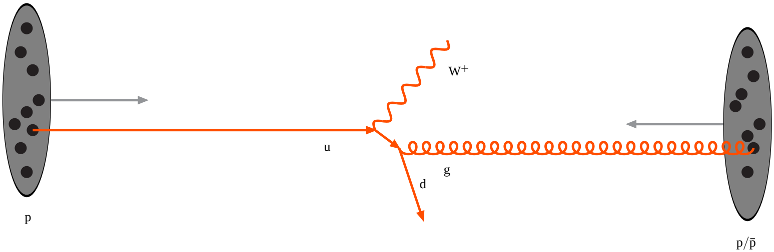
The cross section and main structure of the event is determined by the hard partonic sub process.

Parton showers and hadronization happens at lower (softer) scales and *dresses* the events without influencing the cross section.

The structure of an event

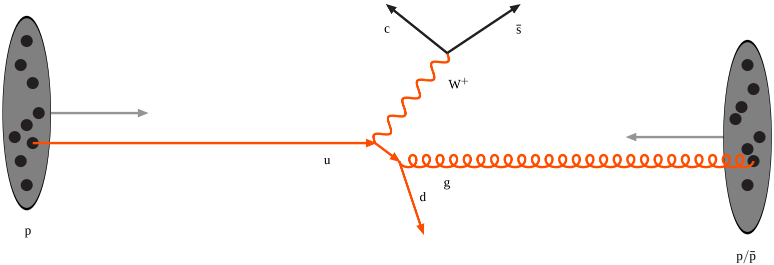
Incoming beams, parton densities

The hard sub-process



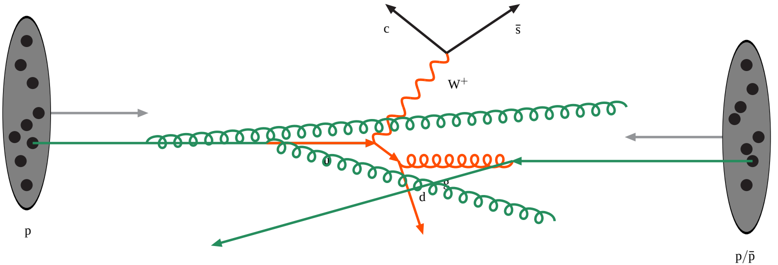
Matrix elements

Resonance decays



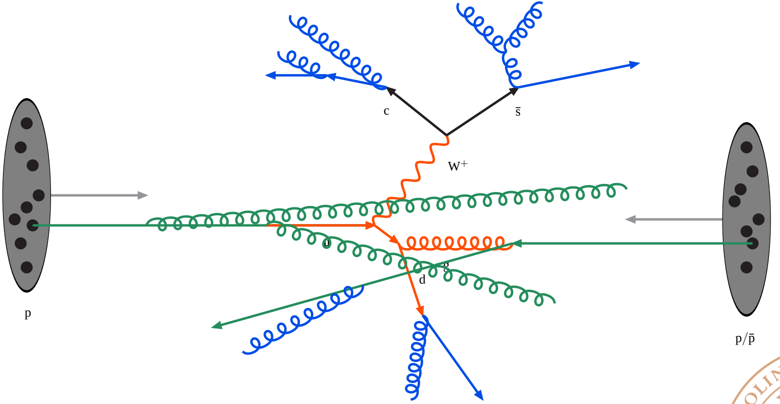
Correlated with the hard sub-process

Initial-state radiation



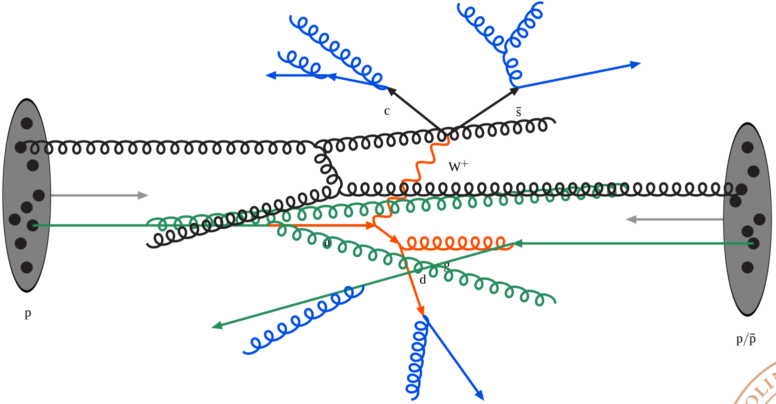
Space-like shower, *backward* evolution

Final-state radiation



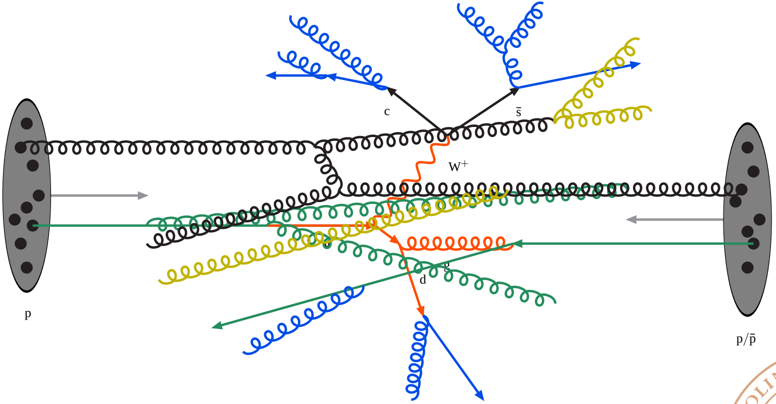
Time-like shower

Multiple parton–parton interactions



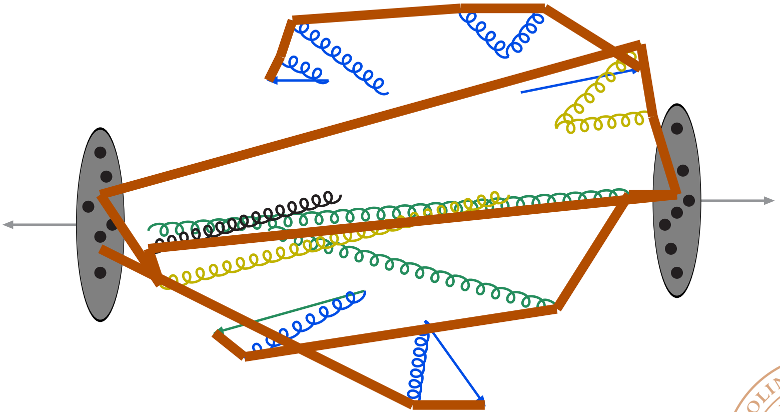
Soft, semi-hard or even hard scatterings

Initial/final-state shower from MI



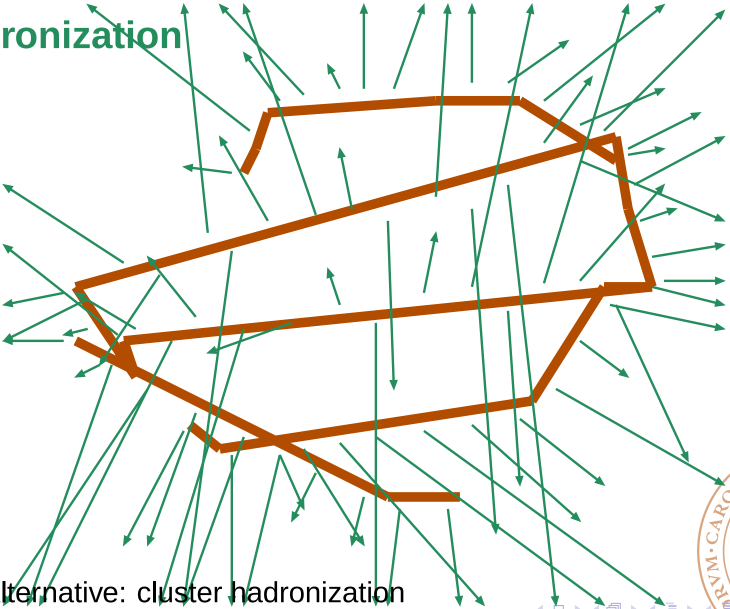
May be *interleaved* with the shower from hard sub-process

Formation of colour strings



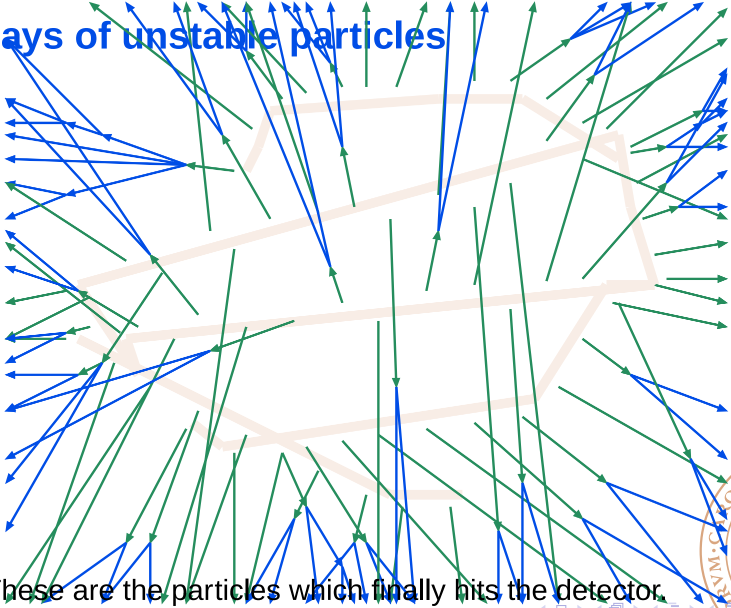
All outgoing partons and beam remnants

Hadronization



Alternative: cluster hadronization

Decays of unstable particles



These are the particles which finally hits the detector.

Everything at the LHC is QCD

- Any measurement at the LHC requires understanding of QCD
- Electro-weak processes or BSM processes are easy (although sometimes tedious)
- Even golden signals such as $H \to 4\mu$ are influenced by QCD
- Any observable will have QCD corrections $\langle\mathcal{O}\rangle = \sigma_0(1 + \alpha_s C_1 + \alpha_s^2 C_2 + \ldots)$
- Any signal will have a QCD background
- QCD is difficult

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Event Generators are all about QCD.

Why is QCD difficult?

- α_s is not very small ($\gtrsim$ 0.1)
- The gluon has a self-coupling and we get alot of gluons
- Even if α_s is small the phase space for emitting gluons is large. In any α_s expansion the coeficients may be large.
- In the end we need hadrons, which are produced in a non-perturbative process.

We need models for parton showers and hadronization

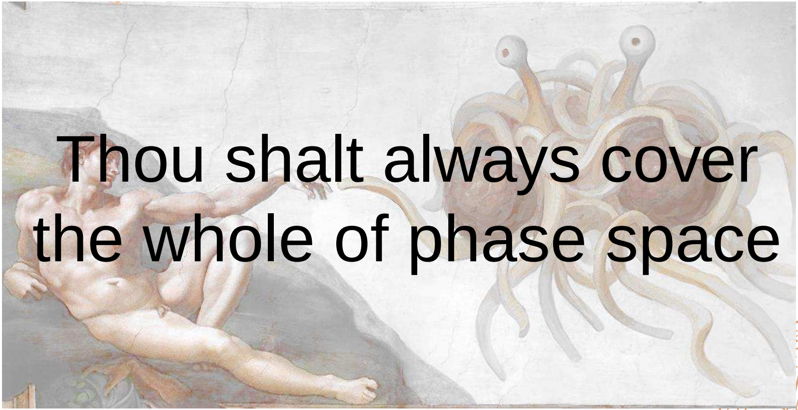
Matrix Element Generation

We always need to start with a $2 \to n$ matrix element. This can in principle be obtained from the standard model (or BSM) Lagrange density in a straight-forward manor.

However,

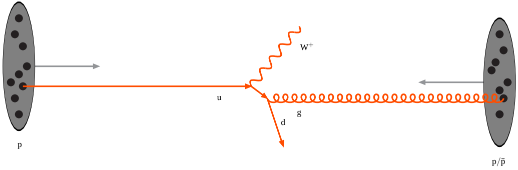
- On tree-level we have divergencies if the scale ($\sim p_\perp$) is small
- Beyond leading order we get nasty loops and infinities
- If n is large, the number of diagrams grows factorially
- If n is large, it is difficult to find a suitable probability distribution for the momenta

The Second Commandment of Event Generation



Thou shalt always cover the whole of phase space

Simple $2 \to 2$ Matrix Elements



Can in principle be written down by hand from relevant Feynman diagrams.

$$\sigma = \int dx_1 dx_2 d\hat{t}\, f_i(x_1, Q^2) f_j(x_2, Q^2) \frac{d\hat{\sigma}_{ij}}{d\hat{t}}$$

With the parton densities sampled at a scale $Q^2 \sim |\hat{t}| \sim p_\perp^2$.

Also fairly easy to generate as the integrand is fairly flat in

$$d\ln(x_1)\, d\ln(x_2)\, d\ln(p_\perp^2)$$

Note however that $\hat{\sigma}$ is may be divergent as $\hat{t} \to 0$.

Eg. Standard QCD ME:

$$\frac{\hat{\sigma}_{gg\to gg}}{d\hat{t}} = \frac{9\pi\alpha_s^2}{4\hat{s}^2}\left(\frac{\hat{s}^2}{\hat{t}^2} + 2\frac{\hat{s}}{\hat{t}} + 3 + 2\frac{\hat{t}}{\hat{s}} + \frac{\hat{t}^2}{\hat{s}^2} + \frac{\hat{u}^2}{\hat{s}^2} + 2\frac{\hat{u}}{\hat{s}} + 3 + 2\frac{\hat{s}}{\hat{u}} + \frac{\hat{s}^2}{\hat{u}^2} + \frac{\hat{t}^2}{\hat{u}^2} + 2\frac{\hat{t}}{\hat{u}} + 3 + 2\frac{\hat{u}}{\hat{t}} + \frac{\hat{u}^2}{\hat{t}^2}\right)$$

We clearly need a cutoff.

Typically this is given as a jet resolution scale, which for this simple process typically means a $p_\perp$-cut.

Eg. the $k_\perp$-algorithm:

Find the pair of particles with smallest

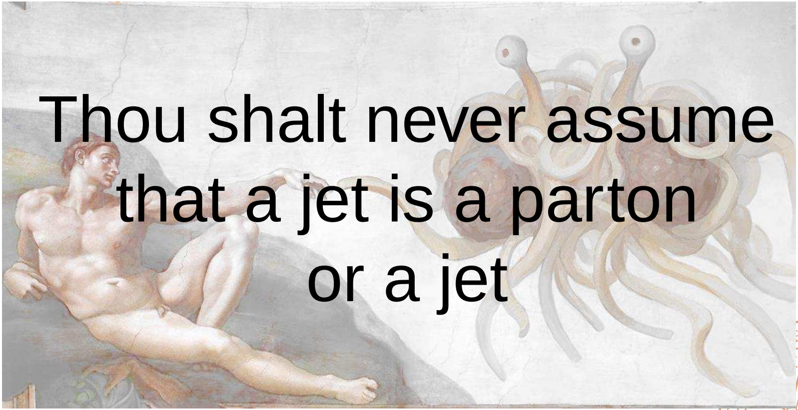
$$k_{\perp ij} = \min(k_{\perp i}, k_{\perp j})\left(\Delta\phi_{ij}^2 + \Delta\eta_{ij}^2\right)$$

and cluster them together into one. Or if any $k_{\perp i}$ is smaller cluster it to the beam.

Continue until all *clusters* have $k_{\perp ij}$ and $k_{\perp i}$ above some cut.

These remaining jets are then close to the original partons.

The Third Commandment of Event Generation



Thou shalt never assume that a jet is a parton or a jet

Higher Order Tree-Level Matrix Elements

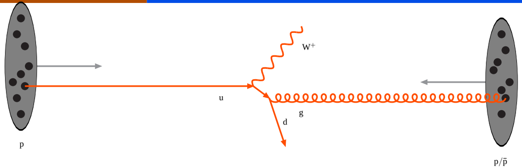
We can go on to higher order $2 \to n$ Matrix Elements. This is in principle straight forward and can even be automated. However the number of diagrams grows $\propto n!$ which makes generation of events forbiddingly slow for $n \gtrsim 6$.

[Recent superstring inspired developments with MHV and twistors reduces the number of diagrams, but faster generation has yet to be delivered.]

Remember also the difficulty in constructing a reasonable probability distribution for the momenta to sample the phase space, especially since there are divergencies everywhere.

Available Tree-Level Generators

- AlpGen
  `http://mlm.web.cern.ch/mlm/alpgen`
- AMEGIC++
  `http://projects.hepforge.org/sherpa`
- CompHep
  `http://comphep.sinp.msu.ru`
- Helac/Phegas
  `http://helac-phegas.web.cern.ch/helac-phegas`
- MadGraph/MadEvent
  `http://madgraph.hep.uiuc.edu`
- ...



We can use a tree-level $2 \to 2$ ME to predict an observable such as the rapidity distribution of a jet in a W-event.

We can try to get a better estimate by going to higher order tree-level MEs

$$\begin{aligned}
\langle\mathcal{O}\rangle_{1j} &= \sigma_{\to W+1j}(\mu) \otimes \mathcal{O}(W+j) \\
\langle\mathcal{O}\rangle_{2j} &= \sigma_{\to W+2j}(\mu) \otimes \mathcal{O}(W+j) \\
\langle\mathcal{O}\rangle_{3j} &= \sigma_{\to W+3j}(\mu) \otimes \mathcal{O}(W+j) \\
&\vdots
\end{aligned}$$

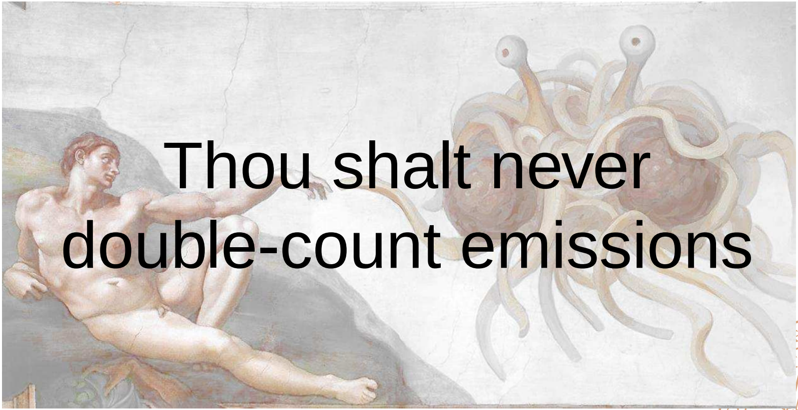
Where we use some jet-resolution scale μ to cut off divergencies.

But we cannot simply add these together, since each cross section is inclusive

The tree-level $ab \to W + 1j$ matrix element gives the cross section for the production of a W plus at least one jet.

Hence it includes also a part of the tree-level $ab \to W + 2j$ matrix element.

The Fourth Commandment of Event Generation



Thou shalt never double-count emissions

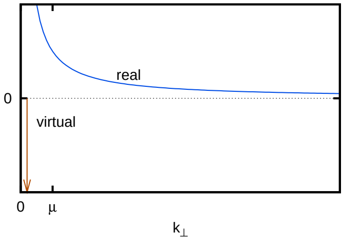
Next-to-Leading Order

To correctly sum $W + 1j$ and $W + 2j$ contributions to an observable, we need to add virtual contributions to the generated $W + 1j$ states. In that way we get a consistent expansion of the observable.

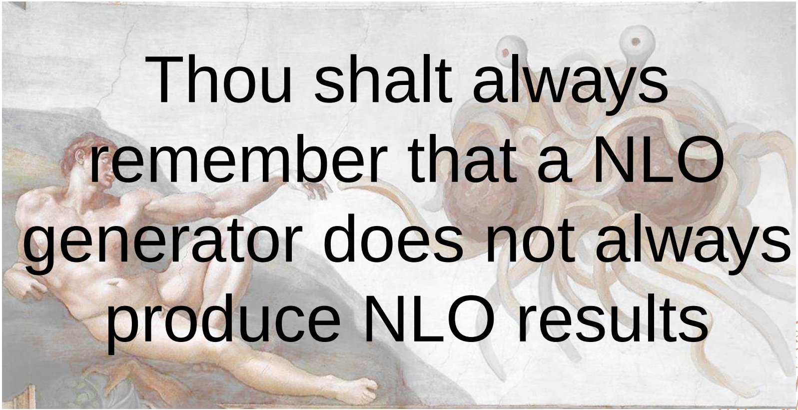
$$\begin{aligned} \langle\mathcal{O}\rangle_{1j} &= \alpha_{\mathrm{s}} C_{11}(\mu) + \alpha_{\mathrm{s}}^2 C_{12}(\mu) \\ \langle\mathcal{O}\rangle_{2j} &= \alpha_{\mathrm{s}} C_{22}(\mu) \end{aligned}$$

$$\langle\mathcal{O}\rangle_{\mathrm{NLO}} = \langle\mathcal{O}\rangle_{1j} + \langle\mathcal{O}\rangle_{2j}$$

Here the jet resolution scale μ is essential, since the virtual corrections are infinite and negative. But if we add together the $1j$ virtual terms and the unresolved $2j$ contributions, (the contributions below μ) the sum, $\alpha_s^2 C_{12}(\mu)$ is finite.



The Fifth Commandment of Event Generation



Thou shalt always remember that a NLO generator does not always produce NLO results

Today there are several NLO generators aviablable (cf. next lecture).

They produce few-parton events and you can measure jet observables

Clearly if you have a generator producing $W+1j$ to NLO, any observable you measure which depends on two jets will only be predicted to leading order.

This can sometimes be tricky…

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Today there are several NLO generators aviablable (cf. next lecture).

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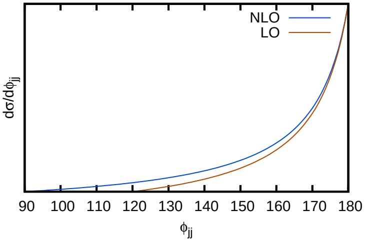
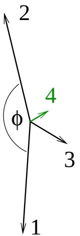
Di-jet decorrelation





Measure the azimuthal angle between the two hardest jets. Clearly the 2-jet matrix element will only give back-to back jets, so the three-jet matrix element will give the leading order. And an NLO 3-jet generator will give us NLO.

But for $\phi_{jj} < 120^\circ$, the two hardest jets needs at least two softer jets to balance. So the NLO becomes LO here.

- Leading order is the first order in α_s which gives a non-zero result for a given observable.
- If NLO corrections are large, we need NNLO.
- However, chances are that we have a poorly converging series in α_s.
- This means we need to resum.

All-Order Resummation

Rather than calculating a few terms in the α_s expansion exactly, we can try to approximate all terms.

It turns out that if we just consider the divergent part of the cross section, everything exponentiates

$$\begin{aligned}
\sigma_{0j} &= C_{00} + \alpha_s C_{01} + \alpha_s^2 C_{02} + \ldots \approx C_{00} \exp(\alpha_s C'_{01}/C_{00}) \\
\sigma_{1j} &= \alpha_s C_{11} + \alpha_s^2 C_{12} + \alpha_s^3 C_{13} + \ldots \approx \alpha_s C_{11} \exp(\alpha_s C'_{12}/C_{11}) \\
&\vdots
\end{aligned}$$

Even if the coefficients diverge as $\mu \to 0$ the exponentiation is finite.

The resummation corresponds to obtaining the leading logarithmic contributions to the coefficients

$$\propto \alpha_s^n \ln(\mu)^{2n}$$

This can be done analytically
(even to next-to-leading log $\propto \alpha_s^n \ln(\mu)^{2n-1}$).

Or by using parton showers...

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