

CMB  
Anisotropies

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# An Enhanced CMB Power Spectrum from Quantum Gravity

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The fundamental interaction that has not been quantized as yet  
is **Gravitation**

A deeper understanding of the quantum version



Find a unified theory

The real structure of nature

There exist many approaches to a quantum theory of gravity

## nowadays no less than 16 !

Either field-theoretical or of sharply different nature.  
Characteristic scale of the theory: **Planck scale**

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} \text{cm},$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44} \text{s},$$

$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19} \text{GeV}$$

# How can we find a way?

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Very difficult to test the effects in  
laboratory



Possible relevant effects at cosmological  
scale



Cosmic Microwave Background  
Radiation (CMB)

# CMB measurement history

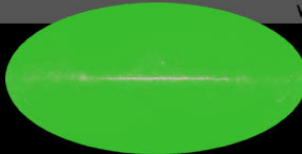
## CMB Anisotropies

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1965



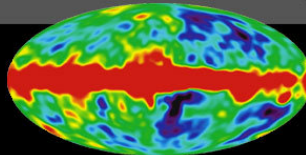
Penzias and  
Wilson



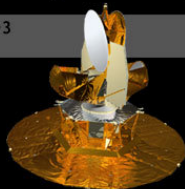
1992



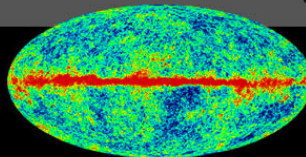
COBE



2003



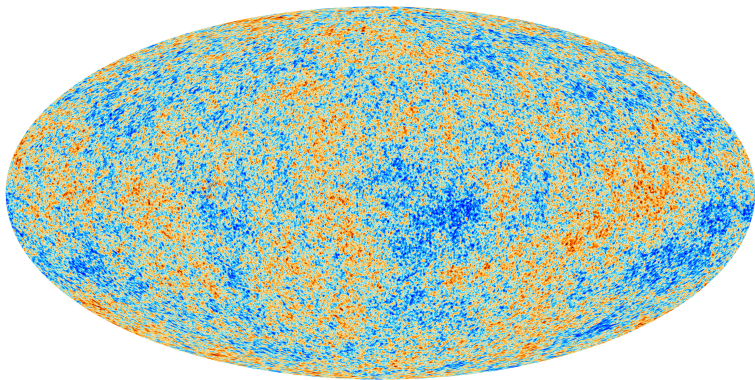
WMAP



## CMB Anisotropies

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## Planck 2013



## Einstein–Hilbert action

$$S = \frac{1}{16\pi} \int_M d^4x \sqrt{-g^{(4)}} R$$



## Arnowitt–Deser–Misner formalism

$$\begin{aligned} \sqrt{-g^{(4)}} R &= N\sqrt{h} \left( {}^{(3)}R + K_{ij}K^{ij} - K^2 \right) + 2\partial_t(\sqrt{h}K) \\ &\quad - 2\partial_i \left[ \sqrt{h}(N^i K - g^{ij} {}^{(3)}\nabla_j N) \right] \end{aligned}$$

$$K_{ij} = \frac{1}{2N} \left( {}^{(3)}\nabla_i N_j + {}^{(3)}\nabla_j N_i - \frac{\partial h_{ij}}{\partial t} \right)$$

## Hamiltonian formalism

$$H = \int d^3x (N\mathcal{H}_t + N_i\chi^i)$$

This formalism enables us to use the Dirac quantization method

$$\hat{h}_{ij}\psi = h_{ij}\psi \qquad \hat{\pi}^{jk}\psi = \frac{\hbar}{i} \frac{\delta\psi}{\delta h_{jk}}$$

↓

## Wheeler–DeWitt equation (WDW)

$$\hat{\mathcal{H}}_t\psi = \left\{ -\hbar^2 G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} - \sqrt{h} {}^{(3)}R \right\} \psi[h_{ij}] = 0$$



Considering a **spatially flat, homogeneous** and **isotropic** Universe, one can describe it by a FLRW metric

$$ds^2 = d\tau^2 - a^2(\tau)\delta_{ij}dx^i dx^j.$$

The Wheeler–DeWitt equation, if one assumes an **inflationary field**  $\phi$ , becomes

$$\left[ \frac{1}{m_P^2} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + e^{6\alpha} m^2 \phi^2 \right] \psi(\alpha, \phi) = 0 \quad \alpha = \ln a$$

We can get a simpler form of the WDW equation if we assume the **slow-roll condition** for the inflationary field  $\phi$

$$\frac{\partial^2 \psi}{\partial \phi^2} \ll e^{6\alpha} m^2 \phi^2 \psi \quad m\phi \rightarrow m_P H$$

Thus the equation becomes

$$\left[ \frac{1}{m_P^2} \frac{\partial^2}{\partial \alpha^2} + e^{6\alpha} m_P^2 H^2 \right] \psi(\alpha, \phi) = 0$$

We now consider the fluctuations of an inhomogeneous inflaton field on top of its homogeneous part

$$\phi \rightarrow \phi(t) + \delta\phi(\mathbf{x}, t) \quad \delta\phi(\mathbf{x}, t) = \sum_{\kappa} f_{\kappa}(t) e^{i\kappa \cdot \mathbf{x}}$$

The smallness of the fluctuations' self-interaction and the **Born–Oppenheimer (BO)** approximation enable us to factorize the wave functional

$$\psi(\alpha, \phi, \{f_{\kappa}\}_{\kappa=1}^{\infty}) = \psi_0(\alpha, \phi) \prod_{\kappa=1}^{\infty} \tilde{\psi}_{\kappa}(\alpha, \phi, f_{\kappa})$$

Thus the WDW equation can be rewritten

$$\left[ \mathcal{H}_0 + \sum_{\kappa=1}^{\infty} \mathcal{H}_{\kappa} \right] \psi(\alpha, \phi, \{f_{\kappa}\}_{\kappa=1}^{\infty}) = 0$$

$$\mathcal{H}_0 = \frac{e^{-3\alpha}}{2} \left[ \frac{1}{m_P^2} \frac{\partial^2}{\partial \alpha^2} + e^{6\alpha} m_P^2 H^2 \right]$$

$$\mathcal{H}_{\kappa} = \frac{e^{-3\alpha}}{2} \left[ -\frac{\partial^2}{\partial f_{\kappa}^2} + W_{\kappa}(\alpha) f_{\kappa}^2 \right]$$

$$W_{\kappa}(\alpha) = \kappa^2 e^{4\alpha} + m^2 e^{6\alpha}$$

Identify the quantum gravitational contributions to the terms of the expansion of the WDW in powers of  $m_P^2$   
**(Effective Theory)**

On writing every single mode in the form

$$\psi_\kappa(\alpha, f_\kappa) = e^{iS(\alpha, f_\kappa)} \quad S(\alpha, f_\kappa) = m_P^2 S_0 + m_P^0 S_1 + m_P^{-2} S_2 + \dots$$

We note that at the zeroth stage of the JWKB approximation one obtains the usual evolution equation for matter

## Schwinger–Tomonaga

$$i \frac{\partial}{\partial t} \psi_{\kappa}^{(0)} = \mathcal{H}_{\kappa} \psi_{\kappa}^{(0)} \quad \psi_{\kappa}^{(0)} \equiv \gamma(\alpha) e^{iS_1(\alpha, f_{\kappa})}$$

Where we have defined the **JWKB time**

$$\frac{\partial}{\partial t} \equiv -e^{-3\alpha} \frac{\partial S_0}{\partial \alpha} \frac{\partial}{\partial \alpha}$$

To second order we obtain the first **quantum-gravitational corrections** to the matter wave functional

$$i \frac{\partial \psi_\kappa^{(1)}}{\partial t} = \mathcal{H}_\kappa \psi_\kappa^{(1)} - \frac{e^{3\alpha}}{2m_P^2 \psi_\kappa^{(0)}} \left[ \frac{(\mathcal{H}_\kappa)^2}{V(\alpha)} \psi_\kappa^{(0)} + i \left( \frac{\psi_\kappa^{(0)}}{V(\alpha)} \frac{\partial \mathcal{H}_\kappa}{\partial t} - \frac{1}{V^2(\alpha)} \frac{\partial V(\alpha)}{\partial t} \mathcal{H}_\kappa \psi_\kappa^{(0)} \right) \right] \psi_\kappa^{(1)}$$

$$\psi_\kappa^{(1)}(\alpha, f_\kappa) \equiv \psi_\kappa^{(0)}(\alpha, f_\kappa) e^{i \frac{\eta_2(\alpha, f_\kappa)}{m_P^2}}$$

By making a Gaussian ansatz

$$\psi_{\kappa}^{(0)}(t, f_{\kappa}) = \mathcal{N}_{\kappa}^{(0)} e^{-\frac{1}{2} \Omega_{\kappa}^{(0)} f_{\kappa}^2}$$

we obtain a coupled system of non-linear differential equations

$$\dot{\mathcal{N}}_{\kappa}^{(0)}(t) = -i \frac{e^{-3\alpha}}{2} \mathcal{N}_{\kappa}^{(0)}(t) \Omega_{\kappa}^{(0)}(t)$$

$$\dot{\Omega}_{\kappa}^{(0)}(t) = i e^{-3\alpha} \left[ -(\Omega_{\kappa}^{(0)}(t))^2 + W_{\kappa}(t) \right]$$



On defining

$$\xi = \frac{\kappa}{Ha(t)} \quad \mu = \frac{m}{H} \quad \nu = \frac{1}{2} \sqrt{9 - 4\mu^2} \quad h = \frac{H^2}{\kappa^3}$$

we get the solution

$$\Omega_{\kappa}^{(0)}(\xi) = \frac{1}{h\xi^2} \frac{1}{(C_1 Y_{\nu}(\xi) + J_{\nu}(\xi))} \\ \times \left[ -iC_1 Y_{\nu+1}(\xi) + \frac{i}{2\xi} (C_1 Y_{\nu}(\xi)(3 + 2\nu) - 2\xi J_{\nu+1}(\xi) + J_{\nu}(\xi)(3 + 2\nu)) \right]$$

We find that the solution of the equation considering  $\mu \ll 1$  coincides, in the limit  $\mu \rightarrow 0$ , i.e.  $\nu \rightarrow \frac{3}{2}$ , with the solution of the complete equation

$$\begin{aligned} \Omega_{\kappa}^{(0)}(\xi) &= \frac{i}{h\xi} \frac{C_1 \cos \xi - \sin \xi}{[(C_1 + \xi) \cos \xi + (C_1 \xi - 1) \sin \xi]} \\ &= \frac{i}{h\xi} \frac{\left( C_1 J_{-\frac{1}{2}} - J_{\frac{1}{2}} \right)}{\left[ (C_1 + \xi) J_{-\frac{1}{2}} + (C_1 \xi - 1) J_{\frac{1}{2}} \right]} \end{aligned}$$

This solution can be re-expressed substituting  $C_1 = \zeta e^{i\beta}$  so that

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2} \frac{i}{\xi} \frac{AB^*}{|B|^2}$$

where

$$A = \rho + i\sigma \quad B = \gamma + i\delta$$

$$\rho = 2(\zeta \cos \beta \cos \xi - \sin \xi) \quad \sigma = 2\zeta \sin \beta \cos \xi$$

$$\gamma = 2\zeta [\cos \beta (\cos \xi + \xi \sin \xi) - (\sin \xi - \xi \cos \xi)]$$

$$\delta = 2\zeta \sin \beta [\cos \xi + \sin \xi]$$

We define the **power spectrum**

$$\mathcal{P}^{(0)}(k) := \frac{k^3}{2\pi^2} |\delta_k(t_{\text{enter}})|^2$$

where

$$\delta_k(t_{\text{enter}}) = \frac{4}{3} \frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)} \Big|_{t=t_{\text{exit}}}$$

and we have set

$$\sigma_{\kappa}^2(t) \equiv \langle \psi_{\kappa} | f_{\kappa}^2 | \psi_{\kappa} \rangle = \frac{1}{2\Re\Omega_{\kappa}(t)}$$

In light of the definition of  $\sigma_\kappa$  one has

$$|\dot{\sigma}_\kappa(t)| = \left| \frac{H\xi}{\sqrt{2}} \frac{d}{d\xi} \left[ (\Re\Omega_\kappa(\xi))^{-\frac{1}{2}} \right] \right|$$

At  $m_P^0$  order we have for the general solution  $\Omega_\kappa^{(0)}$

$$\left| \dot{\sigma}_k^{(0)}(t) \right|_{t_{\text{exit}}} = \frac{2\sqrt{2}\pi^2 H^2}{k^{\frac{3}{2}}} \left| \frac{\sqrt{\zeta}(\zeta + 2\pi \cos \beta)}{\sqrt{\sin \beta} \sqrt{\zeta^2 + 4\pi \cos \beta + 4\pi^2}} \right|$$

At  $m_P^0$  order if we consider the **(Bunch–Davies Vacuum)** boundary condition

$$\Omega_\kappa^{(0)}(\infty) = \frac{1}{h\xi^2}$$

we obtain at the  $\xi(t_{exit}) = 2\pi$  time

$$\left| \dot{\sigma}_\kappa^{(0)} \right| = \frac{H^2}{\kappa^{\frac{3}{2}}} \frac{2\sqrt{2}\pi^2}{\sqrt{4\pi^2 + 1}}$$

At  $m_P^2$  order, making the same Gaussian ansatz, we can write the wave functional in the form

$$\psi_{\kappa}^{(1)}(t, f_{\kappa}) = \left( \mathcal{N}_{\kappa}^{(0)}(t) + \frac{1}{m_P^2} \mathcal{N}_{\kappa}^{(1)}(t) \right) \exp \left[ -\frac{1}{2} \left( \Omega_{\kappa}^{(0)}(t) + \frac{1}{m_P^2} \Omega_{\kappa}^{(1)}(t) \right) f_{\kappa}^2 \right]$$

and inserting it into the  $m_P^2$  order equation

$$\begin{aligned} & i \frac{d}{dt} \log \left( \mathcal{N}_k^{(0)} + \frac{\mathcal{N}_k^{(1)}}{m_P^2} \right) - \frac{i}{2} \left( \dot{\Omega}_k^{(0)} + \frac{\dot{\Omega}_k^{(1)}}{m_P^2} \right) f_k^2 = \\ & \frac{1}{2} e^{-3\alpha} \left\{ \Omega_k^{(0)} + \frac{1}{m_P^2} \left[ \Omega_k^{(1)} - \frac{3}{4V} \left( (\Omega_k^{(0)})^2 - \frac{2}{3} W_k \right) \right] + \right. \\ & \left. \left[ W_k - \left( \Omega_k^{(0)} + \frac{\Omega_k^{(1)}}{m_P^2} \right)^2 - \frac{3\Omega_k^{(0)}(W_k - (\Omega_k^{(0)})^2)}{2Vm_P^2} \right] f_k^2 + \mathcal{O}(f_k^4) \right\} \end{aligned}$$

The equation for  $\Omega_k^{(1)}$  is

$$\dot{\Omega}_\kappa^{(1)}(t) = -2ie^{-3\alpha}\Omega_\kappa^{(0)}(t) \left[ \Omega_\kappa^{(1)}(t) - \frac{3}{4V(t)} \left( (\Omega_\kappa^{(0)}(t))^2 - W_\kappa \right) \right]$$

if we substitute the massless expression for  $\Omega_k^{(0)}$  with the Bunch-Davies boundary condition

$$\frac{d\Omega_k^{(1)}}{d\xi} = \frac{2i\xi}{(\xi - i)}\Omega_k^{(1)} + \frac{3}{2}\xi^3 \frac{(2\xi - i)}{(\xi - i)^3}$$



One can factorize the correction contributions

$$\left| \dot{\sigma}_{\kappa}^{(1)}(t) \right| = \left| \sigma_{\kappa}^{(0)} \right| |C_{\kappa}|$$

$$C_k(\xi) \equiv \left( 1 + \frac{\xi^2 + 1}{\kappa^3} \frac{H^2}{m_p^2} \Re \Omega_{\kappa}^{(1)}(\xi) \right)^{-\frac{3}{2}} \left( 1 - \frac{(\xi^2 + 1)^2}{2\xi\kappa^3} \Re \left[ \frac{d}{d\xi} \Omega_{\kappa}^{(1)}(\xi) \right] \frac{H^2}{m_p^2} \right)$$

In order to evaluate this quantity we have to find the function  $\Omega_{\kappa}^{(1)}$ , that is, for the **massless form** of  $\Omega_{\kappa}^{(0)}$ , and considering the boundary condition  $\Omega_{\kappa}^{(1)}(0) = 0$

$$\Omega_{\kappa}^{(1)}(\xi) = \frac{-3e^{2i\xi}}{8} \frac{1 + \text{Ei}(1, 2)e^2}{(1 + i\xi)^2} + \frac{3}{8} \frac{1 + 6i\xi + 4\text{Ei}(1, 2i\xi + 2)e^{2i\xi+2} - 4\xi^2 - 4i\xi^3}{(1 + i\xi)^2}$$

$$\text{Ei}(a, z) \equiv \int_1^{\infty} \frac{e^{-tz}}{t^a} dt \quad a \in \mathbb{R} \text{ and } \Re(z) > 0$$

At  $t_{exit} \rightarrow \xi = 2\pi$  time

$$C_k \equiv \left(1 - \frac{54.37 H^2}{\kappa^3 m_P^2}\right)^{-\frac{3}{2}} \left(1 + \frac{7.98 H^2}{\kappa^3 m_P^2}\right)$$

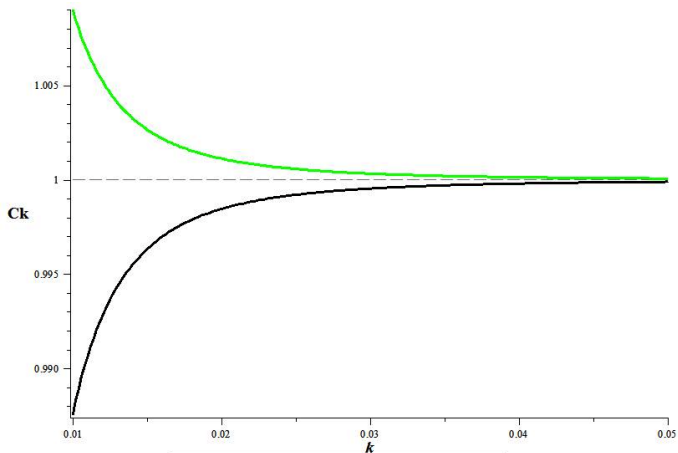
and for what concerns the power spectrum

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k) C_k^2 \sim \mathcal{P}^{(0)}(k) \left[1 + \frac{89.54 H^2}{k^3 m_P^2} + \frac{1}{k^6} \mathcal{O}\left(\frac{H^4}{m_P^4}\right)\right]^2$$

# The $C_k$ Behavior at Various Scales

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Remarkably, by passing to the new variable

$$z = 1 + i\xi$$

the  $m_p^2$  order equation can be written

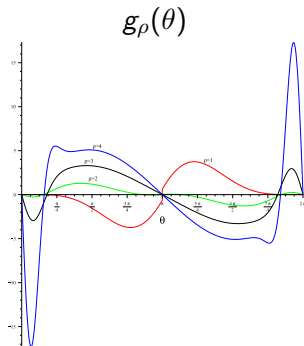
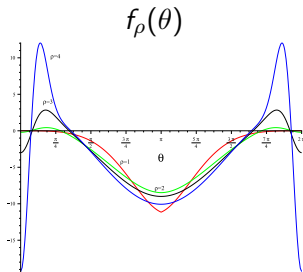
$$\frac{d\Omega_k^{(1)}}{dz} = 2 \left( 1 - \frac{1}{z} \right) \Omega_k^{(1)} + \frac{3}{2} \left( 7 - 2z - \frac{9}{z} + \frac{5}{z^2} - \frac{1}{z^3} \right)$$

that leads to the solution

$$\Omega_k^{(1)}(z) = P_1 \frac{e^{2z}}{z^2} + \frac{3}{8z^2} [4z^3 - 8z^2 + 10z - 5 + 4e^{2z} \text{Ei}(1, 2z)]$$

Such a solution can be studied graphically by introducing the complex polar representation for  $z = \rho e^{i\theta}$  and defining the functions

$$f_\rho(\theta) = \text{Re} \left[ \Omega_k^{(1)}(\rho e^{i\theta}) \right] \quad g_\rho(\theta) = \text{Im} \left[ \Omega_k^{(1)}(\rho e^{i\theta}) \right]$$



We note that the uncorrected power spectrum is proportional to

$$\mathcal{P}^{(0)}(k) \propto \frac{H^4}{\left|\dot{\phi}(t)\right|_{t_{\text{exit}}}^2}$$

this corresponds, apart from a dimensionless constant, to the standard power spectrum of scalar cosmological perturbations

$$\mathcal{P}_s^{(0)}(k) = \frac{G}{\epsilon\pi} H^2$$

where we have introduced the first slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G \left|\dot{\phi}\right|_{t_{\text{exit}}}^2}{H^2}$$

The quantum correction takes the approximate form

$$C_k^2 = 1 + \delta_{\text{WDW}}^{\pm}(k) + \frac{1}{k^6} \mathcal{O}\left(\left(\frac{H}{m_{\text{P}}}\right)^4\right)$$

where  $\delta_{\text{WDW}}^{\pm}(k)$  could take the values

$$\delta_{\text{WDW}}^{+}(k) = \frac{179.09}{k^3} \left(\frac{H}{m_{\text{P}}}\right)^2 \quad \delta_{\text{WDW}}^{-}(k) = -\frac{247.68}{k^3} \left(\frac{H}{m_{\text{P}}}\right)^2$$

The basic equations in the theory of the spectral index  $n_s$  and its running  $\alpha_s$  are

$$n_s - 1 := \frac{d \log \mathcal{P}_s}{d \log k} \approx \frac{1}{H} \frac{d \log \mathcal{P}_s}{dt} \approx 2\eta - 4\epsilon - 3\delta_{\text{WDW}}^{\pm}$$

$$\alpha_s := \frac{dn_s}{d \log k} \approx 2(5\epsilon\eta - 4\epsilon^2 - \Xi^2) + 9\delta_{\text{WDW}}^{\pm}$$

where we have defined the slow-roll parameters

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \quad \Xi^2 := \frac{1}{H^2} \frac{d}{dt} \frac{\ddot{\phi}}{\dot{\phi}}$$



Reinserting a reference wave number which can either correspond

to  $k_{min} \approx 1.4 \times 10^{-4} \text{ Mpc}^{-1}$  largest observable scale or  
to  $k_0 = 0.002 \text{ Mpc}^{-1}$  pivot scale

we find the corrections for  $k \rightarrow \frac{k}{k_0}$

$$|\delta_{\text{WDW}}^+(k_0)| \lesssim 2.9 \times 10^{-9}, \quad |\delta_{\text{WDW}}^-(k_0)| \lesssim 4.0 \times 10^{-9}$$

and for  $k \rightarrow \frac{k}{k_{min}}$

$$|\delta_{\text{WDW}}^+(k_0)| \lesssim 9.8 \times 10^{-13}, \quad |\delta_{\text{WDW}}^-(k_0)| \lesssim 1.4 \times 10^{-12}$$

the resulting upper bounds for  $H$  are

$$H \lesssim 1.67 \times 10^{-2} m_{\text{P}} \approx 4.43 \times 10^{17} \text{ GeV}$$

$$H \lesssim 1.42 \times 10^{-2} m_{\text{P}} \approx 3.76 \times 10^{17} \text{ GeV}$$

- ★ Exact form of the functions  $\Omega_{\kappa}^{(0)}$ ,  $\Omega_{\kappa}^{(1)}$  and  $\dot{\sigma}_{\kappa}^{(0)}$ .  
Enhancement/Suppression of quantum gravitational corrections, hard to discriminate on observational ground.
- ★ Unobservable corrections to CMB anisotropy spectrum; nevertheless, their size is bigger than QG corrections in laboratory situations.
- ★ Other choices of vacuum besides Bunch–Davies allowed by the general integral of our non linear equation?

- ★ Calculation of an upper limit for  $H$  in an inflationary model.
- ★ Gauge-invariant Mukhanov variables instead of a scalar field.
- ★ More complicated quantum state (instead of ground state) to see how the results depend on this choice.
- ★ We have found a way of dealing with unitarity violating terms.

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