Lepton Mixing Patterns from a Scan of Finite Discrete Groups

Kher Sham Lim

Max-Planck-Institut für Kernphysik

Planck 2013, Bonn 21.05.2013

MAX-PLANCK-INSTITUT FÜR KERNPHYSIK INTERNATIONAL MAX PLANCK RESEARCH SCHOOL PTFS FOR PRECISION TESTS OF FUNDAMENTAL SYMMETRIES

based on the work by M. Holthausen, KSL and M. Lindner, hep-ph/1212.2411, Phys.Lett. B721 (2013) 61-67 and

M. Holthausen and KSL, hep-ph/1305.XXXX

Leptonic Mixing Patterns

- Large mixing between leptons (compared to quarks)

U_{PMNS} predicted by TBM

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

U_{PMNS} from experiments (Gonzalez-Garcia et.al. 2012)

$$\begin{pmatrix} 0.795-0.846 & 0.513-0.585 & 0.126-0.178 \\ 0.205-0.543 & 0.416-0.730 & 0.579-0.808 \\ 0.215-0.548 & 0.409-0.725 & 0.567-0.800 \end{pmatrix}$$

- Zero U_{e3} was ruled out by Daya Bay, Reno and Double Chooz collaborations.
- Search for new approach?

Leptonic Mixing Patterns

- Large mixing between leptons (compared to quarks)

U_{PMNS} predicted by TBM

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

U_{PMNS} from experiments (Gonzalez-Garcia et.al. 2012)

$$\begin{pmatrix} 0.795-0.846 & 0.513-0.585 & 0.126-0.178 \\ 0.205-0.543 & 0.416-0.730 & 0.579-0.808 \\ 0.215-0.548 & 0.409-0.725 & 0.567-0.800 \end{pmatrix}$$

- Zero U_{e3} was ruled out by Daya Bay, Reno and Double Chooz collaborations.
- Search for new approach?
  - → Build models which leads to TBM at LO, allow for large NLO corrections. Altarelli et.al.'12, Chen et.al.'13
  - → Look for new groups that predict sizable leptonic mixing patterns at LO.

Leptonic Mixing from Remnant Symmetries

Let the mass matrices of charged lepton and Majorana neutrinos given as:

$$\mathcal{L} = e^T M_e e^c + \frac{1}{2}\nu^T M_\nu \nu$$

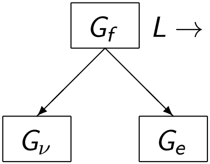
The PMNS matrix defined as:

$$U_{\mathrm{PMNS}} = V_e^\dagger V_\nu$$

can be determined from unitary matrices V_e and V_ν satisfying

$$V_e^T M_e M_e^\dagger V_e^* = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2) \quad \text{and} \quad V_\nu^T M_\nu V_\nu = \mathrm{diag}(m_1, m_2, m_3)$$

Leptonic Mixing from Remnant Symmetries



$L \to \rho(g)L,\ g \in G_f$

$\nu \to \rho(g_\nu)\nu,\ g_\nu \in G_\nu \qquad e \to \rho(g_e)e,\ g_e \in G_e$

Lam '07, '08, R.d.A.Toorop et.al. '11, '12.

The mass matrices have to fulfill:

$$\rho(g_e)^T M_e M_e^\dagger \rho(g_e)^* = M_e M_e^\dagger \quad \text{and} \quad \rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu$$

By diagonalizing $\rho(g_e)$ and $\rho(g_\nu)$ with unitary matrices Ω_e and Ω_ν:

$$\Omega_e^\dagger \rho(g_e)\Omega_e = \rho(g_e)_{diag}, \qquad \Omega_\nu^\dagger \rho(g_\nu)\Omega_\nu = \rho(g_\nu)_{diag},$$

we can obtain the PMNS matrix:

$$U_{\mathrm{PMNS}} = \Omega_e^\dagger \Omega_\nu$$

Requirements of G_ν and G_e

For G_ν

We require 3 distinguishable Majorana neutrinos, this restricts:

- Eigenvalues of $\rho(g_\nu)$ have only real entries.
- G_ν can only be the Klein group $Z_2 \times Z_2$.

For G_e

Charged leptons are distinguishable in 3 generations:

- G_e has to be an abelian group.
- The smallest G_e one can take is Z_3.

In our scan, we will first assume that $G_e = Z_3$ and later generalize it to arbitrary abelian groups.

Examples and Parameterizations

Take $G_e = Z_3$ where the 3d representation of generator T is given by:

$$\rho(T) = T_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and it is diagonalized by Ω_e. As for $G_\nu = Z_2 \times Z_2$, we need two generators S and U with their representations given as:

$$\rho(S) = S_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho(U) = U_3 \equiv - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We need a unitary matrix Ω_ν that simultaneously diagonalizes $\rho(S)$ and $\rho(U)$. The PMNS matrix $U_{\mathrm{PMNS}} = \Omega_e^\dagger \Omega_\nu$ is given as:

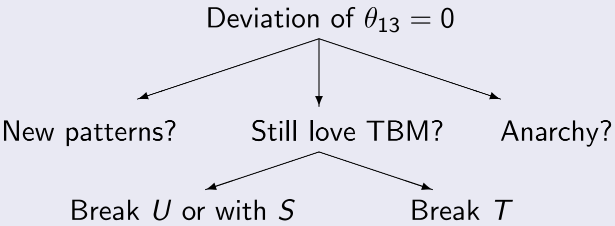
$$U_{\mathrm{PMNS}} = U_{\mathrm{HPS}} \equiv \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The generators S_3, U_3 and T_3 generate the discrete group $\mathcal{S}_4$.

New Starting Points

Until recently, TBM gave a good description of the mixing matrix. However the discovery of non-zero θ_{13} has prompted us to search another new starting point.

Various possibilities to go beyond $\theta_{13} = 0$

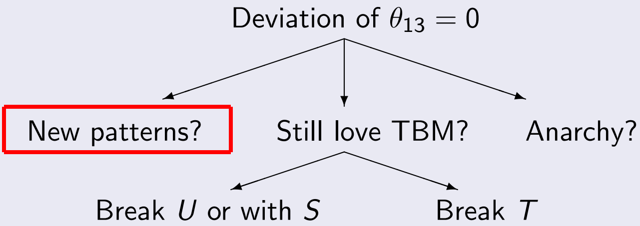


For review see King and Luhn'13

New Starting Points

Until recently, TBM gave a good description of the mixing matrix. However the discovery of non-zero θ_{13} has prompted us to search another new starting point.

Various possibilities to go beyond $\theta_{13} = 0$



For review see King and Luhn'13

Searching for New Starting Patterns

Task

- To survey a large number of discrete groups G_f which contain $G_\nu = Z_2 \times Z_2$ and a chosen G_e as subgroups.
- Find discrete groups that predict experimentally favored leptonic mixing angles.
- If possible, classify and analyze group generators that generate such discrete groups.

How to accomplish that?

Searching for New Starting Patterns

Task

- To survey a large number of discrete groups G_f which contain $G_\nu = Z_2 \times Z_2$ and a chosen G_e as subgroups.
- Find discrete groups that predict experimentally favored leptonic mixing angles.
- If possible, classify and analyze group generators that generate such discrete groups.

How to accomplish that?

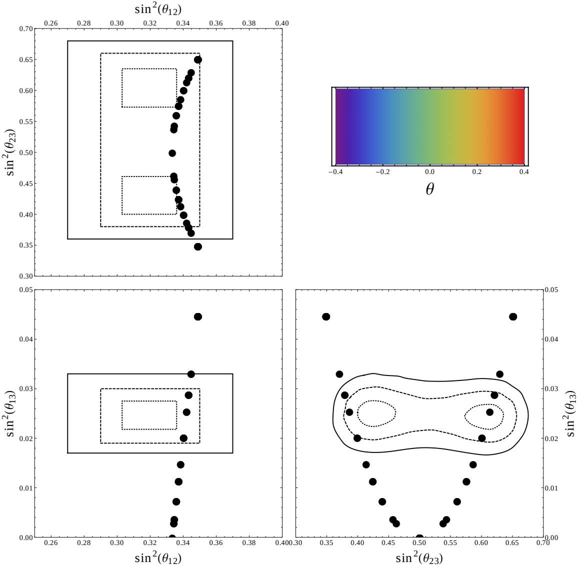
GAP

The case of $G_\nu = Z_2 \times Z_2$, $G_e = Z_3$

Procedure

- We have performed a scan with GAP, considering all discrete groups (more than a million) of size smaller than 1536.
- We utilize the Lagrange theorem to skip over groups with order that is not divisible by 4 and 3. Groups which do not possess at least a faithful 3d irrep will be discarded.
- All the 3d representations of $\rho(U)$, $\rho(S)$ and $\rho(T)$ of a group are recorded.
- Ω_ν and Ω_e that diagonalize the generators above are determined and subsequently the PMNS matrix is found.
- All the permutations of rows and columns of such PMNS matrix is generated. To remove a huge amount of duplicates and junks we demand that the acceptable PMNS matrix has the smallest 13-entry and 11-entry is larger than 12-entry.
- This procedure is repeated for different combinations of Ω_e and Ω_ν.

Result



Characterizing the results

Result

- 3 discrete groups predict experimentally favored mixing angles in 3-sigma range.
- Most of the interesting points lie on a parabola.
- Prediction: Trivial Dirac CP phase!
- Can we categorize them?

Characterizing the results

Result

- 3 discrete groups predict experimentally favored mixing angles in 3-sigma range.
- Most of the interesting points lie on a parabola.
- Prediction: Trivial Dirac CP phase!
- Can we categorize them?

Characterizing the results

Groups that lie on parabola can be presented in a systematic way. Recall:

$$S_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

But U_3 can be generalized to:

$$U_3(n) \equiv -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & z \\ 0 & z^* & 0 \end{pmatrix} \quad \text{with} \quad \langle z \rangle \cong Z_n,\ n \in \mathbb{N}$$

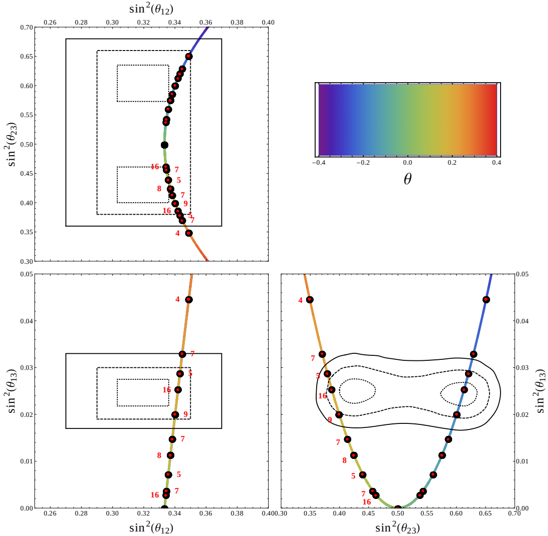
All mixing patterns that lie on the parabola can be written as:

$$U_{\text{PMNS}} = U_{\text{HPS}} U_{13}(\theta = \frac{1}{2}\arg(z), \delta = 0)$$

with

$$U_{13}(\theta, \delta) = \begin{pmatrix} \cos\theta & 0 & e^{i\delta}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\delta}\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The scanned result on parabola agrees with mixing pattern generated by $\langle U_3(n), S_3, T_3 \rangle$



Interesting groups

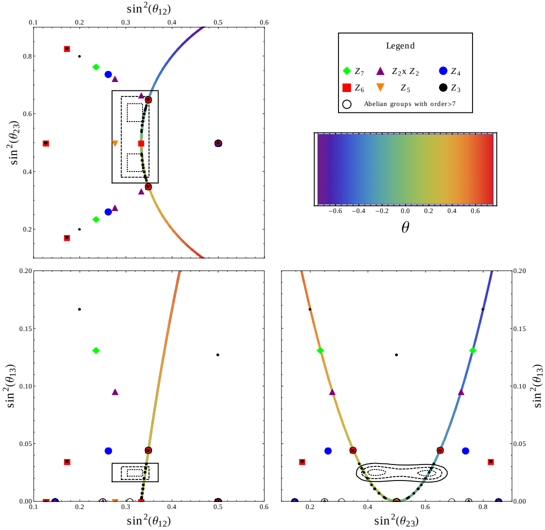
Groups generated by T_3, S_3 and $U_3(n)$, that lead to new starting points

n	G	n	G	n	G
4	$\Delta(6 \cdot 4^2)$	9	$(Z_{18} \times Z_6) \rtimes \mathcal{S}_3$	13	$\Delta(6 \cdot 26^2)$
5	$\Delta(6 \cdot 10^2)$	10	$\Delta(6 \cdot 10^2)$	14	$\Delta(6 \cdot 14^2)$
7	$\Delta(6 \cdot 14^2)$	11	$\Delta(6 \cdot 22^2)$	15	$Z_3 \times \Delta(6 \cdot 10^2)$
8	$\Delta(6 \cdot 8^2)$	12	$Z_3 \times \Delta(6 \cdot 4^2)$	16	$\Delta(6 \cdot 16^2)$

Mixing angles which are compatible with experimental results

n	G	GAP-Id	$\sin^2(\theta_{12})$	$\sin^2(\theta_{13})$	$\sin^2(\theta_{23})$
5	$\Delta(6 \cdot 10^2)$	[600, 179]	0.3432	0.0288	0.3791
			0.3432	0.0288	0.6209
9	$(Z_{18} \times Z_6) \rtimes \mathcal{S}_3$	[648, 259]	0.3402	0.0201	0.3992
			0.3402	0.0201	0.6008
16	$\Delta(6 \cdot 16^2)$	n.a.	0.3420	0.0254	0.3867
			0.3420	0.0254	0.6134

We scanned all the discrete groups up to order 511. Only modular groups and subgroups are found cf. R.d.A.Toorop et.al. '12.



Result for quark mixing patterns

Assume that quark masses also exhibit residual symmetries G_u and G_d from G_f. After searching all the abelian subgroups of G_f in the same representation as the leptonic sector, we obtain the CKM matrix at leading order:

$$U_{\mathrm{CKM}} = \begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the following entries:

n	G	GAP-Id	x	y
5	$\Delta(6 \cdot 10^2)$	$[600, 179]$	0.988	0.156
			0.951	0.309
9	$(Z_{18} \times Z_6) \rtimes S_3$	$[648, 259]$	0.966	0.259
16	$\Delta(6 \cdot 16^2)$	n.a.	0.981	0.195

Group theoretic explanation

All our interesting results come from the group

$$(Z_n \times Z_{n'}) \rtimes \mathcal{S}_3.$$

To obtain sizable LO CKM matrix, the group has to be broken into

$$(Z_m \times Z_{m'}) \rtimes Z_2, \quad n \geq m,\ n' \geq m'.$$

This is a generalized version of D_n, $\Sigma(2 \cdot n^2)$ group.

$\rightarrow$ Cabibbo angle is actually a bonus from the leptonic flavor symmetry!

Dirac Neutrinos and the Mixing Patterns

G_f	GAP-Id	$\{G_e, G_\nu\}$	$\{G_d, G_u\}$	$\sin^2(\theta_{12})$	$\sin^2(\theta_{13})$	$\sin^2(\theta_{23})$	x	y
$\Delta(6 \cdot 5^2)$	[150, 5]	$\{Z_{10}, Z_3\}$	$\{Z_{10}, Z_{10}\}$	0.3428	0.0289	0.6217	0.951	0.309
				0.3428	0.0289	0.3794	0.951	0.309
$\Sigma(3 \cdot 3^3) \rtimes Z_2$	[162, 10]	$\{Z_6, Z_9\}$	$\{Z_6, Z_6\}$	0.3403	0.0202	0.6013	0.866	0.5
$(Z_9 \times Z_3) \rtimes \mathcal{S}_3$	[162, 12]	$\{Z_{18}, Z_9\}$	$\{Z_{18}, Z_{18}\}$	0.3403	0.0202	0.3996	0.866	0.5
	[162, 14]	$\{Z_{18}, Z_3\}$	$\{Z_{18}, Z_{18}\}$					

- However $\Delta(6 \cdot 5^2)$ is a subgroup of $\Delta(6 \cdot 10^2)$ while $\Sigma(3 \cdot 3^3) \rtimes Z_2$ and $(Z_9 \times Z_3) \rtimes \mathcal{S}_3$ are subgroups of $(Z_{18} \times Z_6) \rtimes \mathcal{S}_3$.
- This suggests that in the residual symmetry approach, the leptonic mixing pattern has no correlation with the nature of neutrinos (whether they are Dirac or Majorana).

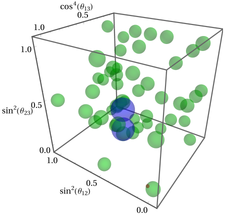
The Ultimate Question... inevitably

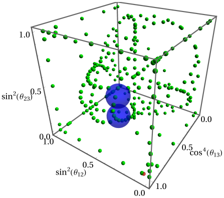
- Groups that predict LO mixing patterns are in general “large”.
- It makes not much difference had we adopted anarchy point of view. Hall et.al.’00, Gouvea and Murayama’12
- But can we quantify this argument?
- What does “large” group mean?

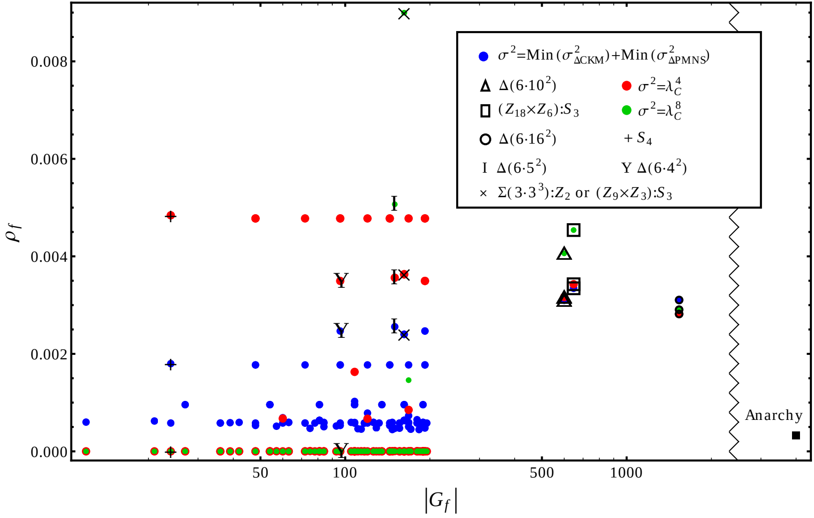
Gentlemen, place your bet

Define a measure:

$$\rho_f \equiv \int_{V_{\rm exp}} p_f(c_{13}^4, s_{12}^2, s_{23}^2)\, \mathrm{d}c_{13}^4 \mathrm{d}s_{12}^2 \mathrm{d}s_{23}^2$$







σ^2=Min($\sigma^2_{\Delta CKM}$)+Min($\sigma^2_{\Delta PMNS}$)
$\Delta(6{\cdot}10^2)$
$\sigma^2=\lambda_C^4$
$(Z_{18}\times Z_6){:}S_3$
$\sigma^2=\lambda_C^8$
$\Delta(6{\cdot}16^2)$
S_4
I $\Delta(6{\cdot}5^2)$
Y $\Delta(6{\cdot}4^2)$
$\Sigma(3{\cdot}3^3){:}Z_2$ or $(Z_9\times Z_3){:}S_3$
0.008
0.006
0.004
0.002
0.000
ρ_f
50
100
500
1000
$|G_f|$
Anarchy

Summary and Conclusion

- We have scanned an extensive range of discrete groups that may used to predict leptonic mixing angles.
- With the assumptions that the remnant symmetries are $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$, a scan of groups with size less than 1536 gives only 3 groups that lead to acceptable mixing patterns, namely $\Delta(6 \cdot 10^2)$, $(Z_{18} \times Z_6) \rtimes S_3$ and $\Delta(6 \cdot 16^2)$. These groups also generate acceptable Cabibbo angle.
- Groups that predict the leptonic mixing patterns which lie on the parabola can be systematically classified by 13-rotation of TBM and all of them predict a trivial Dirac CP phase.
- If $G_e = Z_3$ is relaxed, the scan up to groups with size 511 yields no new interesting groups.
- If neutrinos are Dirac particles, smaller groups can generate also the same mixing patterns obtained by the groups above.
- Groups that predict experimentally favored mixing patterns are large, but have higher goodness of prediction than anarchy.