The low lying spectrum of the Dirac operator: chiral condensate

Elena García Ramos
in collaboration with
K. Cichy, V. Drach, K. Jansen

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Spontaneous Breaking of Chiral Symmetry

- Massless Lagrangian:

$$\mathcal{L}_F = \sum_a \bar{\psi}^a \gamma^\mu D_\mu \psi^a$$

- Introducing the projector operators:

$$\psi_L = \frac{(1-\gamma_5)}{2}\psi, \qquad \psi_R = \frac{(1+\gamma_5)}{2}\psi,$$

- We can rewrite this into

$$\mathcal{L}_F = \bar{\psi}^a_R \gamma^\mu D_\mu \psi^a_R + i\bar{\psi}^a_L \gamma^\mu D_\mu \psi^a_L$$

- Mass term breaks chiral symmetry explicitly

$$m_a\bar{\psi}\psi = m_a\bar{\psi}_L\psi_R + m_a\bar{\psi}_R\psi_L$$

- Chiral Condensate is not invariant under chiral transformations

$$\langle\bar{\psi}\psi\rangle = \langle\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\rangle \neq 0$$

  - **Spontaneously broken symmetry**
  - $N_f = 2 \rightsquigarrow$ 3 Generators $\rightsquigarrow$ 3 Goldstone Bosons $\pi^+,\ \pi^-,\ \pi^0$
  - not massless because the symmetry is explicitly broken by the quark masses.

Chiral Condensate and Banks-Casher Relation

- In the continuum:

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty} \rho(\lambda, m)$$

$$\rho(\lambda, m) = \frac{1}{V}\sum_{k=1}^{\infty} \langle \delta(\lambda - \lambda_k)\rangle \ , \qquad \Sigma = -\lim_{m\to 0}\lim_{V\to\infty} \langle \bar{u}u\rangle$$

- mode number $\nu \rightsquigarrow$ average number of eigenmodes of $D_m^\dagger D_m$ with $\lambda \leq M^2$

$$\nu(M, m) = V\int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda, m), \qquad \Lambda = \sqrt{M^2 - m^2}$$

$\nu(M, m) = \nu_R(M_R, m_R)$ $\rightsquigarrow$ renormalization-group invariant (Giusti & Lüscher, 2008)

- $$\Sigma_R \propto \frac{\partial}{\partial M_R}\nu_R$$

for non-vanishing mass and finite volume

Mode number and Spectral Projectors

- Spectral Projector $\mathbb{P}_M$ to compute $\nu(M, m)$ (Giusti & Lüscher, 2008)

$$\nu(M, m_q) = \langle \mathrm{Tr}\{\mathbb{P}_M\} \rangle$$

- Approximation of $\mathbb{P}_M$:

$$\mathbb{P}_M \approx h(\mathbb{X})^4, \qquad \mathbb{X} = 1 - \frac{2M_\star^2}{D_m^\dagger D_m + M_\star^2}, \qquad M_\star \approx M$$

$\rightsquigarrow$ $h(x)$ is an approximation to the step function $\theta(-x)$ in the interval $[-1, 1]$.

$$h(x) = \frac{1}{2}\{1 - xP(x^2)\}$$

where $P(x)$ is the polynomial which minimizes

$$\delta = \max_{\epsilon \leq y \leq 1} \|1 - \sqrt{y}P(y)\|$$

- $$\nu(M, m_q) = \langle \mathcal{O}_N \rangle \,, \quad \mathcal{O}_N = \frac{1}{N}\sum_{k=1}^{N} (\eta_k, \mathbb{P}_M \eta_k)$$

η_k sources generated randomly.

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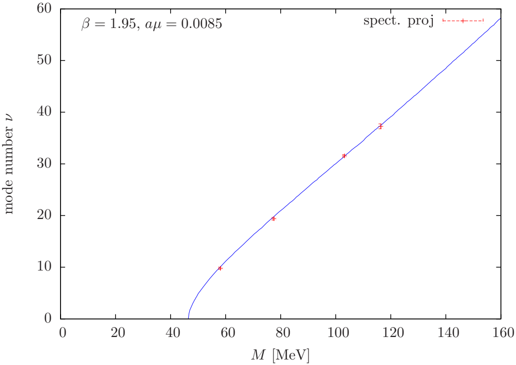
$M^\star$ for chiral condensate

We want to compute the mode number in the linear region to extract the chiral condensate.

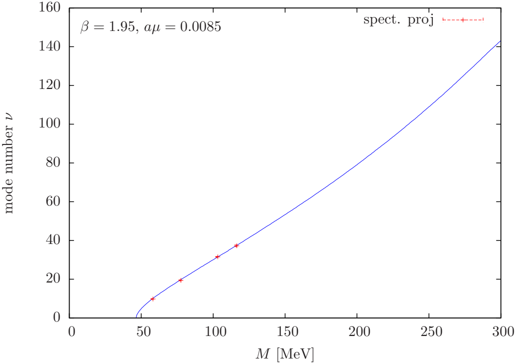
$$\Sigma_R = \frac{\pi}{2V}\sqrt{1-\left(\frac{m_R}{M_R}\right)^2}\,\frac{\partial}{\partial M_R}\nu_R$$



avoid values $\approx m_q$ and $>> m_q$

M⋆ for chiral condensate

We want to compute the mode number in the linear region.



avoid values $\approx m_q$ and $>> m_q$

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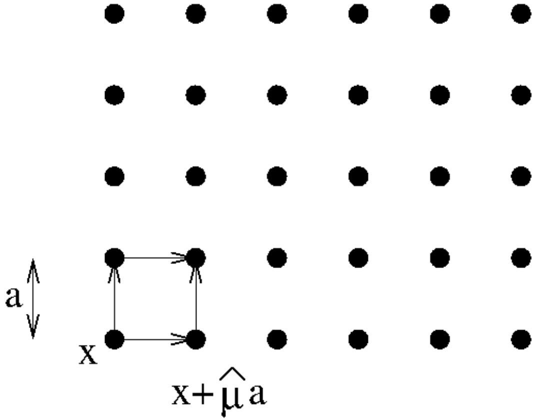
Lattice QCD



Configurations setup

- Wilson Twisted Mass Action at maximal twist (Frezotti & Rossi, 2004)
- Tree-Level Symanzik Gauge Action
- $N_\mathrm{f} = 2$ dynamical fermions

Ensemble	β	lattice	$a\mu$	μ_R (MeV)	κ_C	L (fm)
b30.32	3.90	$32^3 \times 64$	0.003	16	0.160856	2.7
b40.16	3.90	$16^3 \times 32$	0.004	21	0.160856	1.4
b40.20	3.90	$20^3 \times 40$	0.004	21	0.160856	1.7
b40.24	3.90	$24^3 \times 48$	0.004	21	0.160856	2.0
b40.32	3.90	$32^3 \times 64$	0.004	21	0.160856	2.7
b64.24	3.90	$24^3 \times 48$	0.0064	34	0.160856	2.0
b85.24	3.90	$24^3 \times 48$	0.0085	45	0.160856	2.0
c30.32	4.05	$32^3 \times 64$	0.003	19	0.157010	2.1
c60.32	4.05	$32^3 \times 64$	0.006	37	0.157010	2.1
c80.32	4.05	$32^3 \times 64$	0.008	49	0.157010	2.1
d20.48	4.20	$48^3 \times 96$	0.002	15	0.154073	2.6
d65.32	4.20	$32^3 \times 64$	0.0065	47	0.154073	1.7

Configurations setup

- Wilson Twisted Mass Action at maximal twist
- Iwasaki Gauge Action
- $N_f = 2 + 1 + 1$ dynamical fermions

Ensemble	β	lattice	$a\mu_l$	$\mu_{l,R}$ (MeV)	κ_c	L (fm)
A30.32	1.90	$32^3 \times 64$	0.0030	13	0.163272	2.8
A40.20	1.90	$20^3 \times 40$	0.0040	17	0.163270	1.7
A40.24	1.90	$24^3 \times 48$	0.0040	17	0.163270	2.1
A40.32	1.90	$32^3 \times 64$	0.0040	17	0.163270	2.8
A50.32	1.90	$32^3 \times 64$	0.0050	22	0.163267	2.8
A60.24	1.90	$24^3 \times 48$	0.0060	26	0.163265	2.1
A80.24	1.90	$24^3 \times 48$	0.0080	35	0.163260	2.1
B25.32	1.95	$32^3 \times 64$	0.0025	13	0.161240	2.5
B35.32	1.95	$32^3 \times 64$	0.0035	18	0.161240	2.5
B55.32	1.95	$32^3 \times 64$	0.0055	28	0.161236	2.5
B75.32	1.95	$32^3 \times 64$	0.0075	38	0.161232	2.5
B85.24	1.95	$24^3 \times 48$	0.0085	45	0.161231	1.9
D15.48	2.10	$48^3 \times 96$	0.0015	9	0.156361	2.9
D20.48	2.10	$48^3 \times 96$	0.0020	12	0.156357	2.9
D30.48	2.10	$48^3 \times 96$	0.0030	19	0.156355	2.9

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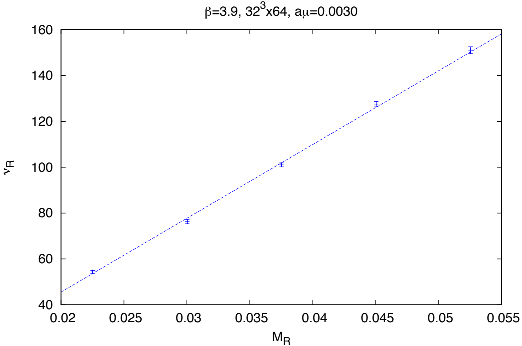
Extracting Σ_R from ν_R

(Giusti & Lüscher, 2008)

$$\Sigma_R = \frac{\pi}{2V}\sqrt{1-\left(\frac{m_R}{M_R}\right)^2}\,\frac{\partial}{\partial M_R}\nu_R$$

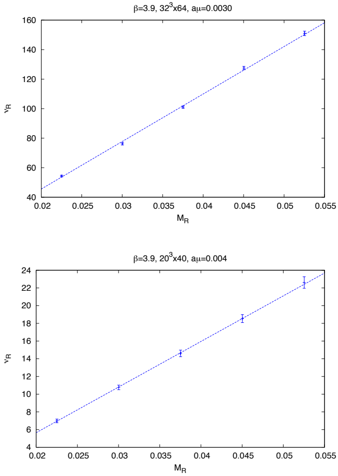
$\nu(M,m) = \nu_R(M_R,m_R)$ $\rightsquigarrow$ renormalization-group invariant

- We extract the term $\frac{\partial}{\partial M_R}\nu_R$ through the slope of a linear fit.



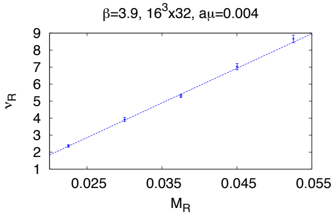
Extracting Σ_R from ν_R

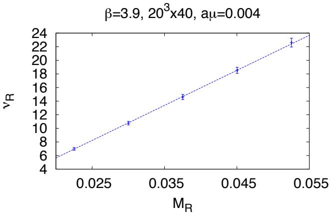
- Thus we get a value of the Σ_R for each value of the m_q

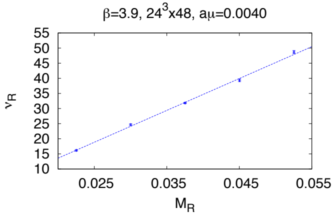


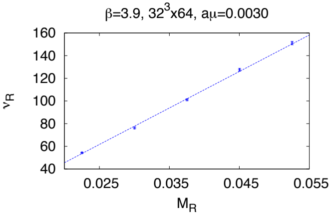
Extracting Σ_R from ν_R $N_f = 2$

- Thus we get a value of the Σ_R for each value of the m_q



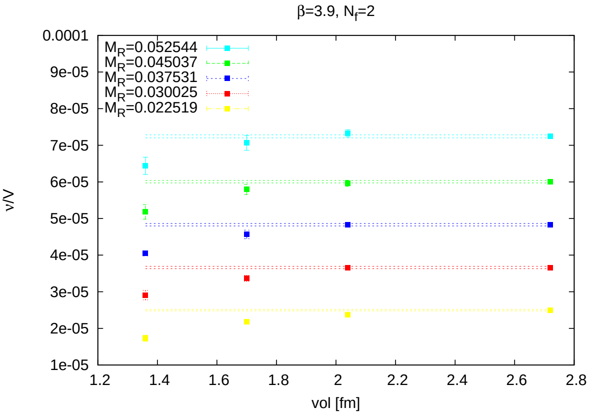






Finite Volume Effects ($\frac{\nu}{vol}$) $N_f = 2$

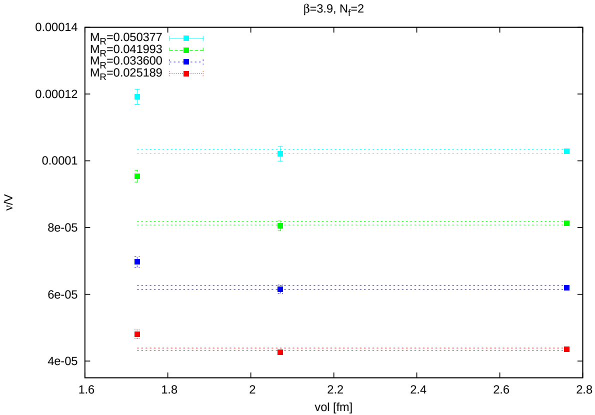
Study of the finite size effects for the chiral condensate Σ. $\frac{\nu}{vol} = const$



Finite Volume Effects ($\frac{\nu}{vol}$) $N_f = 2 + 1 + 1$

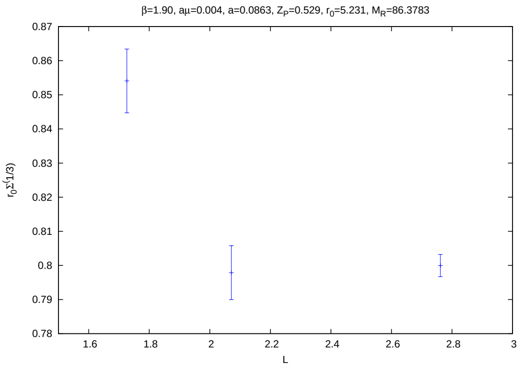
Study of the finite size effects for the chiral condensate Σ.

$\frac{\nu}{vol} = const$

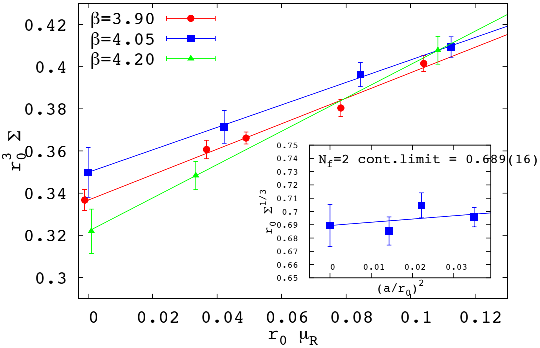


Finite Volume Effects (Σ) $N_f = 2 + 1 + 1$

Study of the finite size effects for the chiral condensate Σ. $\frac{\nu}{vol} = const$



Chiral and Continuum Limit of Σ $N_f = 2$



⋆ Chiral limit

β	$r_0\Sigma^{1/3}$
3.9	0.6957(35)(37)(52)(186)
4.05	0.7046(78)(30)(42)(206)
4.2	0.6853(73)(59)(49)(265)

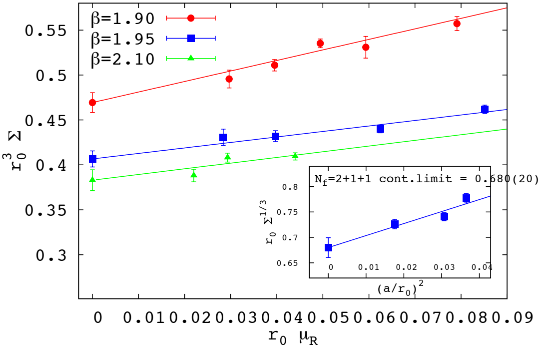
errors→ (stat)(Z_P)(r_0)(fit)

⋆ Continuum limit

$$r_0\Sigma^{1/3} = 0.689(16)(29)$$

errors→ (combined)(fit)

Chiral and Continuum Limit of Σ $N_f = 2$



★ Chiral limit

β	$r_0\Sigma^{1/3}$
1.9	0.7772(61)(44)(56)(157)
1.95	0.7408(55)(25)(53)(112)
2.1	0.7262(72)(14)(56)(75)

errors$\rightarrow$ (stat)(Z_P)(r_0)(fit)

★ Continuum limit

$$r_0\Sigma^{1/3} = 0.680(20)(21)$$

errors$\rightarrow$ (combined)(fit)

Comparison of continuum results for $r_0\Sigma^{1/3}$

Result	method	N_f	fermions	$r_0\Sigma^{1/3}$
this work	spectral proj.	2	twisted mass	0.689(16)(29)
this work	spectral proj.	2+1+1	twisted mass	0.680(20)(21)
RBC-UKQCD (1)	chiral fits	2+1	domain wall	0.632(19)
MILC (2)	chiral fits	2+1	staggered	0.654(22)
MILC (3)	chiral fits	2+1	staggered	0.652(22)
ETMC (4)	chiral fits	2	twisted mass	0.575(54)
ETMC (5)	quark propagator	2	twisted mass	0.676(59)(66)

(1) Aoki et al., 2010
(2) Bazavov et al., 2009
(3) Bazavov et al. ,2010
(4) Baron et al. , 2009
(5) Burger et al. ,2012

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Spectral Projectors: Other Applications

- Smart way of computing it:
  - ⋆ Mode Number (Giusti & Lüscher, 2008)

$$\nu(M, m_q) = \langle \mathrm{Tr}\{\mathbb{P}_M\}\rangle$$

  - ⋆ Topological Susceptibility (Lüscher & Palombi, 2010)

$$\chi_{top} = \frac{Z_S^2}{Z_P^2}\frac{1}{V}\left\langle \mathrm{Tr}\left\{\gamma 5\mathbb{P}_M\right\} \mathrm{Tr}\left\{\gamma 5\mathbb{P}_M\right\}\right\rangle$$

  - ⋆ Renormalization Constants (Giusti & Lüscher, 2008)

$$\frac{Z_S^2}{Z_P^2} = \frac{\langle \mathrm{Tr}\left\{\mathbb{P}_M\right\}\rangle}{\langle \mathrm{Tr}\left\{\gamma_5\mathbb{P}_M\gamma_5\mathbb{P}_M\right\}\rangle}$$

Spectral Projectors: Other Applications

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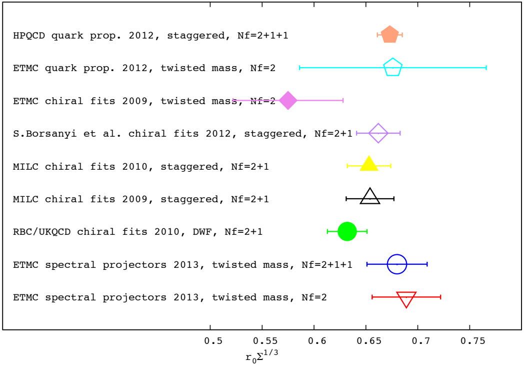
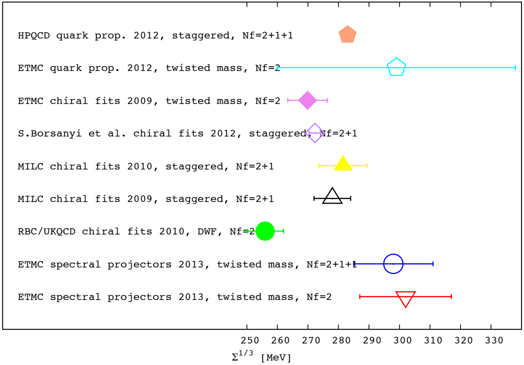
Conclusions and outlook

- We have applied the spectral projector method using $N_\mathrm{f} = 2$ and $N_\mathrm{f} = 2+1+1$ twisted mass ensembles generated by ETMC.
- We have analyzed the finite volume effects for the mode number and the chiral condensate.
- We have computed the continuum limit of the chirally extrapolated chiral condensate.
  - ⋆ $N_\mathrm{f} = 2$ using different quark masses for 3 different lattice spacings.
  - ⋆ $N_\mathrm{f} = 2+1+1$ using different quark masses for 3 different lattice spacings.
  - ⋆ Our results are consistent with the results of other groups.
- We have computed the χ_top in the quenched for different values of the lattice spacing. Work in progress.
- We have computed χ_top for dynamical case. Work in progress.
- We have computed the ratio of renormalization constants $\frac{Z_P}{Z_S}$ for different ensembles. Work in progress.

Thank you for your attention!


HPQCD quark prop. 2012, staggered, Nf=2+1+1
ETMC quark prop. 2012, twisted mass, Nf=2
ETMC chiral fits 2009, twisted mass, Nf=2
S.Borsanyi et al. chiral fits 2012, staggered, Nf=2+1
MILC chiral fits 2010, staggered, Nf=2+1
MILC chiral fits 2009, staggered, Nf=2+1
RBC/UKQCD chiral fits 2010, DWF, Nf=2+1
ETMC spectral projectors 2013, twisted mass, Nf=2+1+1
ETMC spectral projectors 2013, twisted mass, Nf=2
0.5
0.55
0.6
0.65
0.7
0.75
$r_0\Sigma^{1/3}$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ETMC spectral projectors 2013, twisted mass, Nf=2
250 260 270 280 290 300 310 320 330
Σ^{1/3} [MeV]