Exercises to *Lectures on Baryogenesis*

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1. **Baryon number operator $\hat{B}$**

   Let $q(\mathbf{x},t)$ be the Dirac field operator that describes a quark of flavor $q = u, ..., t$, $q^\dagger(\mathbf{x},t)$ denotes its Hermitean adjoint, and $\bar{q} = q^\dagger\gamma^0$. The baryon number operator is

   $$\hat{B} = \frac{1}{3}\sum_q \int d^3x : q^\dagger(\mathbf{x},t)q(\mathbf{x},t) : ,$$

   and the colons denote normal ordering. Let C, P denote the unitary and T the anti-unitary operator which implement the charge conjugation, parity, and time reversal transformations, respectively, in the space of states.

   a) Show that $\hat{B}$ is even under P and odd under C and CP.

   b) How does $\hat{B}(t)$, respectively $\hat{B}(0)$ transform under $\Theta \equiv CPT$?

   Use that the action of P, C, T on the quark fields is, adopting standard phase conventions,

   $$\begin{aligned} Pq(\mathbf{x},t)P^{-1} &= \gamma^0 q(-\mathbf{x},t)\,, \\ Cq(\mathbf{x},t)C^{-1} &= i\gamma^2 q^\dagger(\mathbf{x},t)\,, \\ Tq(\mathbf{x},t)T^\dagger &= \gamma_5\gamma^0\gamma^2 q(\mathbf{x},-t)\,, \end{aligned}$$

   where γ^0, γ^2, and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ denote Dirac matrices.

2. **The 3. Sakharov condition**

   A system which is in thermal equilibrium is described in quantum theory by a density operator $\rho = \exp(-H/T)$, where H is the Hamiltion operator of the system. The thermal average of an observable $\mathcal{O}$ is given by $\langle\mathcal{O}\rangle_T = \mathrm{tr}(\rho\mathcal{O})$.
   Show that $\langle\hat{B}(t)\rangle_T = 0$ if the system is in thermal equilibrium and H ist CPT-invariant.

3. **Weyl and Majorana fields**

   Consider a Dirac field

   $$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix},$$

   where ξ, η are 2-component spinor fields. In the chiral representation of the γ matrices, using the convention where $\gamma_5 = \mathrm{diag}(\mathrm{I}_2, -\mathrm{I}_2)$, we have $\xi = \psi_R, \eta = \psi_L$, where ψ_R, ψ_L are the right-chiral and left-chiral Weyl fields. In the chiral representation the charge conjugated spinor field ψ^c reads

   $$\psi^c \equiv i\gamma^2\psi^\dagger = \begin{pmatrix} i\sigma_2\eta^\dagger \\ -i\sigma_2\xi^\dagger \end{pmatrix}, \tag{1}$$

   and σ_2 is the second Pauli matrix.
   a) Use the Weyl fields in 4-component form, $\psi_R = (\xi, 0)^T, \psi_L = (0, \eta)^T$, and determine, using (1), their charge-conjugates:

   $$\psi_L^c \equiv (\psi_L)^c \quad \text{and} \quad \psi_R^c \equiv (\psi_R)^c\,.$$

b) Interpret the Weyl fields $\psi_L, \psi_R, \psi_L^c, \psi_R^c$; that is, which $L-$ and R-chiral states are created/annihilated by these fields?

c) Obtain $\overline{\psi_L^c}$ and $\overline{\psi_L^c}$ in terms of the fields ξ and η.

d) A Majorana field is defined by the condition

$$\psi^c \stackrel{!}{=} r\psi \,,$$

where $|r| = 1$ is a phase chosen by convention. Determine, for $r = +1$, the two solutions of this equation in terms of Weyl fields.

4. **Lepton number violation, seesaw mechanism**

The mass term for neutrino fields ν_L and ν_R with a Dirac term and a Majorana term for ν_R is in the 1-flavor case:

$$-\mathcal{L}_{D+M} = m_D \bar{\nu}_R \nu_L + \frac{M}{2} \overline{\nu_R^c} \nu_R \, + \, \text{h.c.}$$

We use real masses m_D and M.

a) Show that this Lagrangian violates lepton number.

b) Compute the eigenvalues and the eigen-fields of the mass matrix for $M \gg m_D$.

Solution to problem 1:
a) From the above P, C, T transformations of q we get

$$\begin{aligned} Pq^\dagger(\mathbf{x},t)P^{-1} &= q^\dagger(-\mathbf{x},t)\gamma^0\,, \\ Cq^\dagger(\mathbf{x},t)C^{-1} &= iq(\mathbf{x},t)\gamma^2\,, \\ Tq^\dagger(\mathbf{x},t)T^\dagger &= -q^\dagger(\mathbf{x},-t)\gamma^2\gamma^0\gamma_5\,. \end{aligned}$$

Then

$$\begin{aligned} P : q^\dagger(\mathbf{x},t)q(\mathbf{x},t) : P^{-1} &=: q^\dagger(-\mathbf{x},t)q(-\mathbf{x},t) :, \\ C : q^\dagger(\mathbf{x},t)q(\mathbf{x},t) : C^{-1} =: q(\mathbf{x},t)q^\dagger(\mathbf{x},t) : &= - : q^\dagger(\mathbf{x},t)q(\mathbf{x},t) :, \\ T : q^\dagger(\mathbf{x},t)q(\mathbf{x},t) : T^{-1} &=: q^\dagger(\mathbf{x},-t)q(\mathbf{x},-t) : \,. \end{aligned}$$

With these relations we immediately obtain:

$$\begin{aligned} P\hat{B}P^{-1} &= \hat{B}\,, \\ C\hat{B}C^{-1} &= -\hat{B}\,. \end{aligned}$$

b) As shown in the lectures the baryon number operator is time-dependent due to non-perturbative effects. Using translation invariance we have $\hat{B}(t) = e^{iHt}\hat{B}(0)e^{-iHt}$, where H is the Hamiltonian of the system. The operator $\hat{B}(0)$ is even with respect to T and odd with respect to $\Theta \equiv CPT$:

$$\Theta\hat{B}(0)\Theta^\dagger = -\hat{B}(0)\,.$$

Solution to problem 2:
Recall that a system which is in thermal equilibrium is stationary and is described by a density operator $\rho = \exp(-H/T)$. Using $\hat{B}(t) = e^{iHt}\hat{B}(0)e^{-iHt}$ we have

$$< \hat{B}(t) >_T = tr(e^{-H/T}e^{iHt}\hat{B}(0)e^{-iHt}) = tr(e^{-iHt}e^{-H/T}e^{iHt}\hat{B}(0)) =< \hat{B}(0) >_T\,,$$

If the Hamiltonian H is $\Theta \equiv CPT$ invariant, $\Theta^\dagger H\Theta = H$, we get for the equilibrium average of $\hat{B} \equiv \hat{B}(0)$:

$$\begin{aligned} < \hat{B} >_T &= tr(e^{-H/T}\hat{B}) = tr(\Theta^\dagger\Theta e^{-H/T}\hat{B}) \\ &= tr(e^{-H/T}\Theta\hat{B}\Theta^\dagger) = - < \hat{B} >_T\,, \end{aligned}$$

where we used that $\hat{B}$ is odd under CPT. Thus $< \hat{B} >_T = 0$ in thermal equilibrium.

Solution to problem 3:
Consider a Dirac field

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \tag{2}$$

where ξ, η are 2-component spinor fields. In the chiral representation of the γ matrices, using the convention where $\gamma_5 = \text{diag}(\text{I}_2, -\text{I}_2)$, we have $\xi = \psi_R, \eta = \psi_L$, where ψ_R, ψ_L are the right-handed and left-handed Weyl fields. In the chiral representation the charge conjugated spinor field ψ^c reads

$$\psi^c \equiv i\gamma^2\psi^\dagger = \begin{pmatrix} i\sigma_2\eta^\dagger \\ -i\sigma_2\xi^\dagger \end{pmatrix}, \tag{3}$$

and σ_2 is the second Pauli matrix.

a) Let's use the Weyl fields in 4-component form, $\psi_R = (\xi, 0)^T, \psi_L = (0, \eta)^T$, and determine, using (3), their charge-conjugates:

$$\psi_L^c \equiv (\psi_L)^c = \begin{pmatrix} i\sigma_2\eta^\dagger \\ 0 \end{pmatrix}, \tag{4}$$

$$\psi_R^c \equiv (\psi_R)^c = \begin{pmatrix} 0 \\ -i\sigma_2\xi^\dagger \end{pmatrix}. \tag{5}$$

b) From this equation we can also read off the relation between the 2-component Weyl fields and their charge conjugates. Eq. (5) tells us that $\psi_L^c(\psi_R^c)$ is a right-handed (left-handed) Weyl field. Thus the Weyl field operator

$\psi_L(\psi_R)$ annihilates a fermion state $|\psi>$ having L (R) chirality

and creates an antifermion state $|\bar{\psi}>$ with R (L) chirality.

$\psi_L^c(\psi_R^c)$ annihilates $|\bar{\psi}>$ having R (L) chirality

and creates a state $|\psi>$ with L (R) chirality.

c) Moreover, we immediately obtain that

$$\overline{\psi_L^c} \equiv (\psi_L^c)^\dagger\gamma^0 = (0,\, i\eta^T\sigma_2)\,, \tag{6}$$

$$\overline{\psi_R^c} \equiv (\psi_R^c)^\dagger\gamma^0 = (-i\xi^T\sigma_2,\, 0)\,. \tag{7}$$

d) A Majorana field is defined by the condition

$$\psi^c \stackrel{!}{=} r\psi\,, \tag{8}$$

where $|r| = 1$ is a phase chosen by convention. For $r = +1$ the four-component field $\psi_1 = (i\sigma_2\eta^\dagger,\, \eta)^T$ is a solution of this equation. In terms of Weyl fields this solution reads

$$\psi_1 = \psi_L + \psi_L^c\,. \tag{9}$$

The other solution of eq. (8) with $r = 1$ is

$$\psi_2 = \psi_R + \psi_R^c\,. \tag{10}$$

Solution to problem 4:

a) Recalling the connection between symmetries and conservation laws we see that the non-conservation of L-number is related to the fact that $\mathcal{L}_{D+M}$ is not invariant under the global $U(1)$ transformation $\nu_{L,R} \to e^{i\omega}\nu_{L,R}$, $\bar{\nu}_{L,R} \to e^{-i\omega}\bar{\nu}_{L,R}$. The Majorana mass term violates the L-number by 2 units, $|\Delta \mathrm{L}| = 2$. For instance $< \bar{\nu}_R|\overline{\nu_L^c}\nu_L|\nu_L > \neq 0$; i.e., the Majorana term flips a left-handed $|\nu_L>$ into a right-handed $|\bar{\nu}_R>$.

b) It is useful to put the mass matrix into the following form:

$$\begin{aligned} -\mathcal{L}_{D+M} &= \frac{M}{2}\overline{\nu_R^c}\nu_R + m_D\bar{\nu}_R\nu_L + \text{h.c.} \\ &= \frac{1}{2}(\bar{\psi}_1,\, \bar{\psi}_2)\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \end{aligned}$$

where

$$\psi_1 = \nu_L + \nu_L^c\,,$$
$$\psi_2 = \nu_R + \nu_R^c$$

are Majorana fields. The mass parameters are taken to be real.
In order to obtain this representation of the mass matrix, one uses that
$\bar{\psi}_A \psi_A = \overline{\psi_A^c} \psi_A^c = 0$ for A=L,R,
$\overline{\nu_R^c} \nu_L^c = \bar{\nu}_R \nu_L$,
$\bar{\nu}_R \nu_L^c + \overline{\nu_R^c} \nu_L = 0$.
Let's diagonalize the mass matrix for the case $M \gg m_D$. We obtain in the mass basis

$$-\mathcal{L}_{D+M} = \frac{m_\nu}{2} \bar{\nu} \nu + \frac{m_N}{2} \bar{N} N\,,$$

where

$$-m_\nu \simeq \frac{m_D^2}{M} \ll m_D\,,$$
$$m_N \simeq M + \frac{m_D^2}{M}\,,$$

and the eigen-fields are, up to terms of order m_D/M:

$$\nu \simeq \psi_1\,, \quad N \simeq \psi_2\,,$$

The eigenvalue m_ν can be made positive by an appropriate change of phase of the field ν. For $M \gg m_D$ the neutrino mass eigenstates consist of a very light left-handed state $|\nu >$ and a very heavy right-handed state $|N >$. This constitutes the **seesaw mechanism** for generating a very small mass for a left-handed neutrino from $m_D = \mathcal{O}(h_\ell v)$ and from a large M.