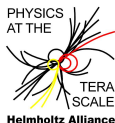


Monte Carlos — Part II

Stefan Gieseke

Institut für Theoretische Physik
KIT

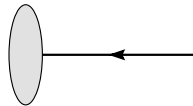
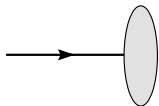
GK Mass Spectra Symmetry
Spring Block Course 2013, 7-11 Apr 2013



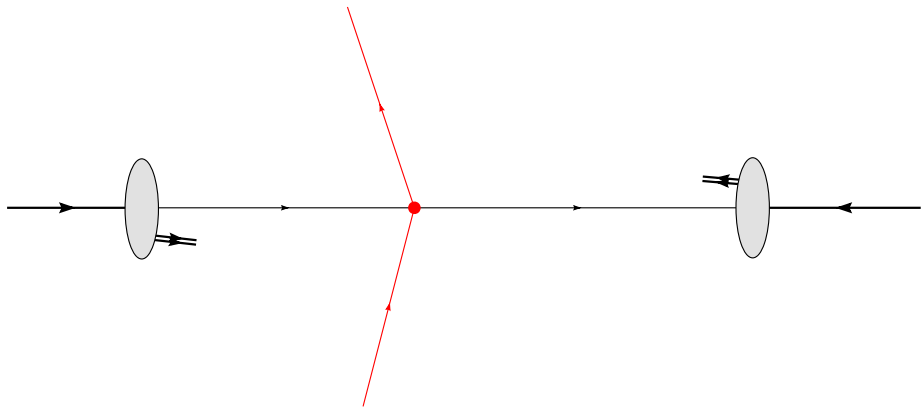
- ▶ Part I — Basics
 - ▶ Introduction
 - ▶ Monte Carlo techniques
- ▶ Part II — Perturbative physics
 - ▶ Hard scattering
 - ▶ Parton showers
- ▶ Part III — Merging/Matching
 - ▶ Matrix element corrections
 - ▶ Merging multiple tree level MEs with parton showers
 - ▶ Matching NLO and parton showers
- ▶ Part IV — Non-perturbative physics
 - ▶ Hadronization
 - ▶ Hadronic decays
 - ▶ Comparison to data
- ▶ Part V — Multiple Partonic Interactions
 - ▶ Minimum Bias/Underlying Event in data
 - ▶ Modelling

- ▶ Hard scattering
 - ▶ Matrix elements and phase space
 - ▶ Mini event generator
- ▶ Parton showers
 - ▶ e^+e^- annihilation and collinear limits
 - ▶ Multiple emissions
 - ▶ Sudakov form factor
 - ▶ Parton cascades
 - ▶ Coherence/angular ordering
 - ▶ Misc aspects

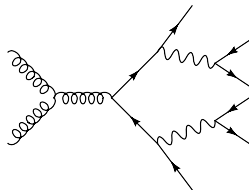
Hard scattering



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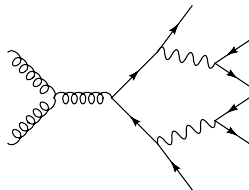


- ▶ Perturbation theory/Feynman diagrams give us (fairly accurate) final states for a few number of legs ($O(1)$).



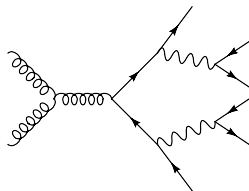
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- ▶ Starting point for further simulation.
- ▶ Want exclusive final state at the LHC ($O(100)$).

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- ▶ OK for very inclusive observables.
- ▶ Starting point for further simulation.
- ▶ Want exclusive final state at the LHC ($O(100)$).
- ▶ Want arbitrary cuts.
- ▶ → use Monte Carlo methods.

Where do we get (LO) $|M|^2$ from?

- ▶ Most/important simple processes (SM) are 'built in'.
- ▶ Calculate yourself (≤ 3 particles in final state).
- ▶ Matrix element generators:
 - ▶ MadGraph/MadEvent.
 - ▶ Comix/AMEGIC (part of Sherpa).
 - ▶ HELAC/PHEGAS.
 - ▶ Whizard.
 - ▶ CalcHEP/CompHEP.

generate code or event files that can be further processed.

- ▶ → FeynRules interface to ME generators.

Cross section formula

From Matrix element, we calculate

$$\sigma = \int f_i(x_1, \mu^2) f_j(x_2, \mu^2) \frac{1}{F} \sum \overline{|M|^2} \, dx_1 dx_2 d\Phi_n ,$$

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now,

$$\frac{1}{F} dx_1 dx_2 d\Phi_n = J(\vec{x}) \prod_{i=1}^{3n-2} dx_i \quad \left(d\Phi_n = (2\pi)^4 \delta^{(4)}(\dots) \prod_{i=1}^n \frac{d^3\vec{p}}{(2\pi)^3 2E_i} \right)$$

such that

$$\begin{aligned} \sigma &= \int g(\vec{x}) d^{3n-2}\vec{x} , & \left(g(\vec{x}) = J(\vec{x}) f_i f_j \overline{\sum} |M|^2 \Theta(\text{cuts}) \right) \\ &= \frac{1}{N} \sum_{i=1}^N \frac{g(\vec{x}_i)}{p(\vec{x}_i)} = \frac{1}{N} \sum_{i=1}^N w_i . \end{aligned}$$

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We generate **events** \vec{x}_i with **weights** w_i .

- ▶ We generate pairs (\vec{x}_i, w_i) .

Mini event generator

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Generate events with same frequency as in nature!

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$$P_i = \frac{w_i}{w_{\max}},$$

where w_{\max} has to be chosen sensibly.

→ reweighting, when $\max(w_i) = \bar{w}_{\max} > w_{\max}$, as

$$P_i = \frac{w_i}{\bar{w}_{\max}} = \frac{w_i}{w_{\max}} \cdot \frac{w_{\max}}{\bar{w}_{\max}},$$

i.e. reject events with probability $(w_{\max}/\bar{w}_{\max})$ afterwards.
(can be ignored when #(events with $w_i > \bar{w}_{\max}$) small.)

- ▶ We generate pairs (\vec{x}_i, w_i) .
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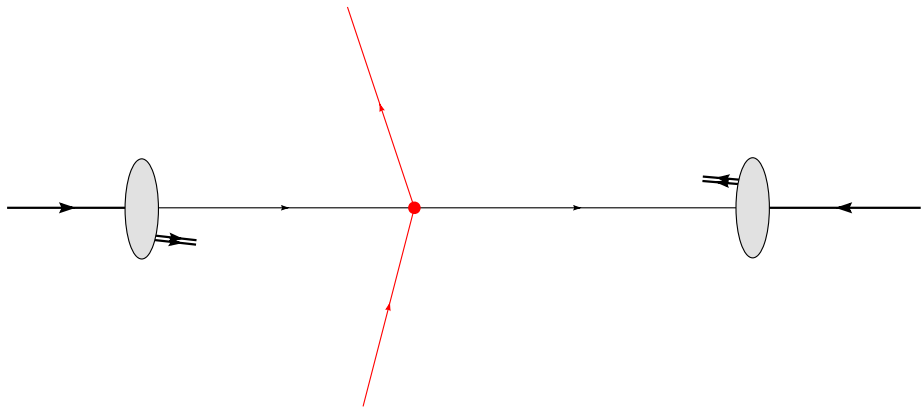
Some comments:

- ▶ Use techniques from lecture 1 to generate events efficiently.
Goal: small variance in w_i distribution!

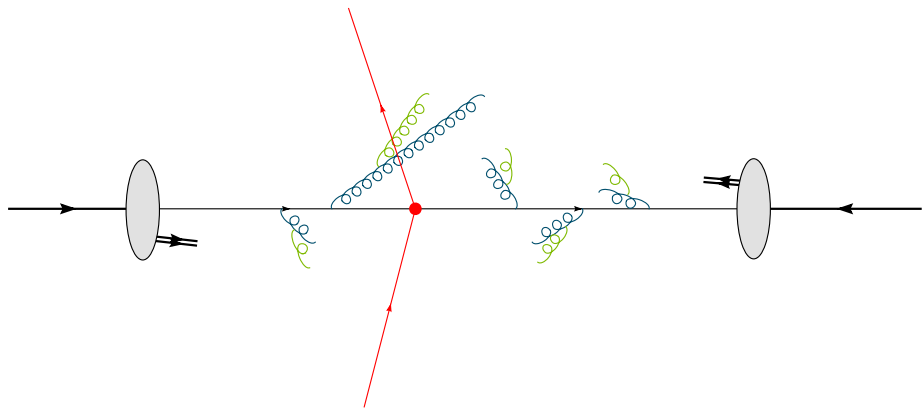
Some comments:

- ▶ Use techniques from lecture 1 to generate events efficiently.
Goal: small variance in w_i distribution!
- ▶ Clear from lecture 1: efficient generation closely tied to knowledge of $f(\vec{x}_i)$, *i.e.* the matrix element's propagator structure.
→ build phase space generator already while generating ME's automatically.

Hard matrix element



Hard matrix element \rightarrow parton showers



Quarks and gluons in final state, pointlike.

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- ▶ Know short distance (short time) fluctuations from matrix element/Feynman diagrams: $Q \sim \text{few GeV to } O(\text{TeV})$.

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Dominated by large logs, terms

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Generated from emissions *ordered* in Q .

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Generated from emissions *ordered* in Q .
Soft and/or collinear emissions.

Good starting point: $e^+e^- \rightarrow q\bar{q}g$:

Final state momenta in one plane (orientation usually averaged).

Write momenta in terms of

$$x_i = \frac{2p_i \cdot q}{Q^2} \quad (i = 1, 2, 3),$$

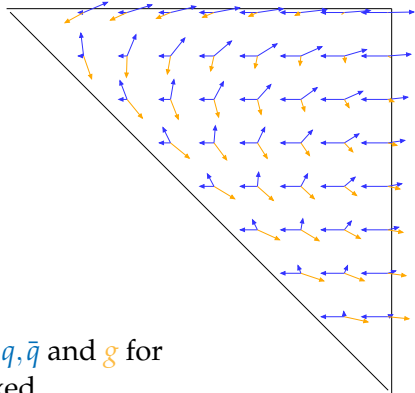
$$0 \leq x_i \leq 1, x_1 + x_2 + x_3 = 2,$$

$$q = (Q, 0, 0, 0),$$

$$Q \equiv E_{cm}.$$

Fig: momentum configuration of q, \bar{q} and g for given point (x_1, x_2) , \bar{q} direction fixed.

$(x_1, x_2) = (x_q, x_{\bar{q}})$ -plane:

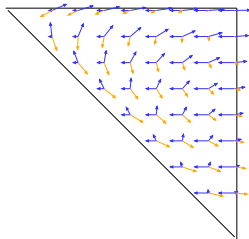
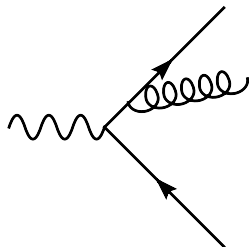


Differential cross section:

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \frac{x_1 + x_2}{(1-x_1)(1-x_2)}$$

Collinear singularities: $x_1 \rightarrow 1$ or $x_2 \rightarrow 1$.

Soft singularity: $x_1, x_2 \rightarrow 1$.



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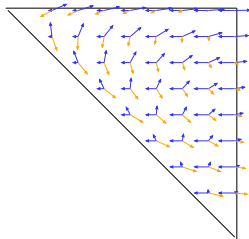
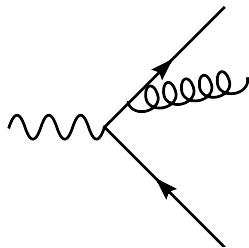
Collinear singularities: $x_1 \rightarrow 1$ or $x_2 \rightarrow 1$.

Soft singularity: $x_1, x_2 \rightarrow 1$.

Rewrite in terms of x_3 and $\theta = \angle(q, g)$:

$$\frac{d\sigma}{d\cos\theta dx_3} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \left[\frac{2}{\sin^2\theta} \frac{1 + (1-x_3)^2}{x_3} - x_3 \right]$$

Singular as $\theta \rightarrow 0$ and $x_3 \rightarrow 0$.



Can separate into two jets as

$$\begin{aligned}\frac{2d\cos\theta}{\sin^2\theta} &= \frac{d\cos\theta}{1-\cos\theta} + \frac{d\cos\theta}{1+\cos\theta} \\ &= \frac{d\cos\theta}{1-\cos\theta} + \frac{d\cos\bar{\theta}}{1-\cos\bar{\theta}} \\ &\approx \frac{d\theta^2}{\theta^2} + \frac{d\bar{\theta}^2}{\bar{\theta}^2}\end{aligned}$$

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So, we rewrite $d\sigma$ in collinear limit as

$$d\sigma = \sigma_0 \sum_{\text{jets}} \frac{d\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} C_F \frac{1+(1-z)^2}{z^2} dz$$

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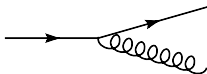
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with DGLAP splitting function $P(z)$.

Universal DGLAP splitting kernels for collinear limit:

$$d\sigma = \sigma_0 \sum_{\text{jets}} \frac{d\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} P(z) dz$$



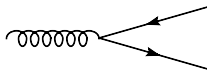
$$P_{q \to qg}(z) = C_F \frac{1+z^2}{1-z}$$



$$P_{g \to gg}(z) = C_A \frac{(1-z(1-z))^2}{z(1-z)}$$



$$P_{q \to gq}(z) = C_F \frac{1+(1-z)^2}{z}$$



$$P_{g \to qq}(z) = T_R (1 - 2z(1-z))$$

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Note: Other variables may equally well characterize the collinear limit:

$$\frac{d\theta^2}{\theta^2} \sim \frac{dQ^2}{Q^2} \sim \frac{dp_{\perp}^2}{p_{\perp}^2} \sim \frac{d\tilde{q}^2}{\tilde{q}^2} \sim \frac{dt}{t}$$

whenever $Q^2, p_{\perp}^2, t \rightarrow 0$ means “collinear”.

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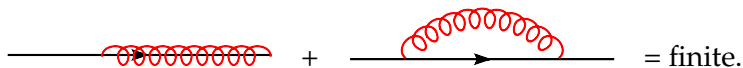
whenever $Q^2, p_{\perp}^2, t \rightarrow 0$ means “collinear”.

- ▶ θ : HERWIG
- ▶ Q^2 : PYTHIA ≤ 6.3 , SHERPA.
- ▶ p_{\perp} : PYTHIA ≥ 6.4 , ARIADNE, Catani–Seymour showers.
- ▶ \tilde{q} : Herwig++.

Need to introduce **resolution** t_0 , e.g. a cutoff in p_{\perp} . Prevent us from the singularity at $\theta \rightarrow 0$.

Emissions below t_0 are **unresolvable**.

Finite result due to virtual corrections:



The diagram shows two Feynman diagrams separated by a plus sign, followed by an equals sign and the word "finite". The first diagram is a horizontal line with a red wavy line (representing an emission) attached to it. The second diagram is a horizontal line with a red wavy line (representing a virtual correction) forming a semi-circular arc above it.

unresolvable + virtual emissions are included in Sudakov form factor via unitarity (see below!).

Starting point: factorisation in collinear limit, single emission.

$$\sigma_{2+1}(t_0) = \sigma_2(t_0) \int_{t_0}^t \frac{dt'}{t'} \int_{z_-}^{z_+} dz \frac{\alpha_S}{2\pi} \hat{P}(z) = \sigma_2(t_0) \int_{t_0}^t dt W(t) .$$

Towards multiple emissions

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Simple example:

Multiple photon emissions, strongly ordered in t .

We want

$$W_{\text{sum}} = \sum_{n=1} W_{2+n} = \frac{\int \left| \begin{array}{c} \nearrow \\ \bullet \\ \searrow \\ \text{---} \\ \nearrow \end{array} \right|^2 d\Phi_1 + \int \left| \begin{array}{c} \nearrow \\ \bullet \\ \searrow \\ \text{---} \\ \nearrow \\ \text{---} \\ \nearrow \end{array} \right|^2 d\Phi_2 + \int \left| \begin{array}{c} \nearrow \\ \bullet \\ \searrow \\ \text{---} \\ \nearrow \\ \text{---} \\ \nearrow \\ \text{---} \\ \nearrow \end{array} \right|^2 d\Phi_3 + \dots}{\left| \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \right|^2}$$

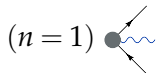
for any number of emissions.

Towards multiple emissions

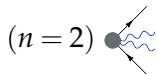
$$(n = 1) \text{ } \bullet \begin{array}{l} \nearrow \\ \text{---} \\ \searrow \end{array}$$

$$W_{2+1} = \left(\int \left| \begin{array}{l} \nearrow \\ \text{---} \\ \searrow \end{array} \right|^2 + \left| \begin{array}{l} \nearrow \\ \text{---} \\ \searrow \end{array} \right|^2 d\Phi_1 \right) / \left| \begin{array}{l} \nearrow \\ \bullet \\ \searrow \end{array} \right|^2 = \frac{2}{1!} \int_{t_0}^t dt W(t) .$$

Towards multiple emissions



$$W_{2+1} = \left(\int \left| \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} \right|^2 + \left| \begin{array}{c} \text{diagram 3} \\ \text{diagram 4} \end{array} \right|^2 d\Phi_1 \right) / \left| \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \end{array} \right|^2 = \frac{2}{1!} \int_{t_0}^t dt W(t).$$



$$W_{2+2} = \left(\int \left| \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} \right|^2 + \left| \begin{array}{c} \text{diagram 3} \\ \text{diagram 4} \end{array} \right|^2 + \left| \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \end{array} \right|^2 + \left| \begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \end{array} \right|^2 d\Phi_2 \right) / \left| \begin{array}{c} \text{diagram 9} \\ \text{diagram 10} \end{array} \right|^2$$
$$= 2^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' W(t') W(t'') = \frac{2^2}{2!} \left(\int_{t_0}^t dt W(t) \right)^2.$$

We used

$$\int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n W(t_1) \dots W(t_n) = \frac{1}{n!} \left(\int_{t_0}^t dt W(t) \right)^n.$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

$$W_{2+n} = \frac{2^n}{n!} \left(\int_{t_0}^t dt W(t) \right)^n$$

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So, in total we get

$$\sigma_{>2}(t_0) = \sigma_2(t_0) \sum_{k=1}^{\infty} \frac{2^k}{k!} \left(\int_{t_0}^t dt W(t) \right)^k = \sigma_2(t_0) \left(e^{2 \int_{t_0}^t dt W(t)} - 1 \right)$$

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Sudakov Form Factor

$$\Delta(t_0, t) = \exp \left[- \int_{t_0}^t dt W(t) \right]$$

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Sudakov Form Factor in QCD

$$\Delta(t_0, t) = \exp \left[- \int_{t_0}^t dt W(t) \right] = \exp \left[- \int_{t_0}^t \frac{dt}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z, t)}{2\pi} \hat{P}(z, t) dz \right]$$

Note that

$$\begin{aligned}\sigma_{\text{all}} &= \sigma_2 + \sigma_{>2} = \sigma_2 + \sigma_2 \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right), \\ \Rightarrow \Delta^2(t_0, t) &= \frac{\sigma_2}{\sigma_{\text{all}}}.\end{aligned}$$

Two jet rate = $\Delta^2 = P^2$ (No emission in the range $t \rightarrow t_0$).

Sudakov form factor = No emission probability .

Often $\Delta(t_0, t) \equiv \Delta(t)$.

- ▶ Hard scale t , typically CM energy or p_{\perp} of hard process.
- ▶ Resolution t_0 , two partons are resolved as two entities if inv mass or relative p_{\perp} above t_0 .
- ▶ P^2 (not P), as we have two legs that evolve independently.

Sudakov form factor from Markov property

Unitarity

$$\begin{aligned} P(\text{"some emission"}) + P(\text{"no emission"}) \\ = P(0 < t \leq T) + \bar{P}(0 < t \leq T) = 1 . \end{aligned}$$

Multiplication law (no memory)

$$\bar{P}(0 < t \leq T) = \bar{P}(0 < t \leq t_1) \bar{P}(t_1 < t \leq T)$$

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Then subdivide into n pieces: $t_i = \frac{i}{n}T, 0 \leq i \leq n$.

$$\begin{aligned} \bar{P}(0 < t \leq T) &= \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} \bar{P}(t_i < t \leq t_{i+1}) = \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} (1 - P(t_i < t \leq t_{i+1})) \\ &= \exp \left(- \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} P(t_i < t \leq t_{i+1}) \right) = \exp \left(- \int_0^T \frac{dP(t)}{dt} dt \right). \end{aligned}$$

Sudakov form factor

Again, no-emission probability!

$$\bar{P}(0 < t \leq T) = \exp\left(-\int_0^T \frac{dP(t)}{dt} dt\right)$$

So,

$$\begin{aligned} dP(\text{first emission at } T) &= dP(T)\bar{P}(0 < t \leq T) \\ &= dP(T)\exp\left(-\int_0^T \frac{dP(t)}{dt} dt\right) \end{aligned}$$

That's what we need for our parton shower! Probability density for next emission at t :

$$dP(\text{next emission at } t) = \frac{dt}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z, t)}{2\pi} \hat{P}(z, t) dz \exp\left[-\int_{t_0}^t \frac{dt}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z, t)}{2\pi} \hat{P}(z, t) dz\right]$$

Probability density:

$dP(\text{next emission at } t) =$

$$\frac{dt}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z, t)}{2\pi} \hat{P}(z, t) dz \exp \left[- \int_{t_0}^t \frac{dt}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z, t)}{2\pi} \hat{P}(z, t) dz \right]$$

Conveniently, the probability distribution is $\Delta(t)$ itself.

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Hence, parton shower very roughly from (HERWIG):

1. Choose flat random number $0 \leq \rho \leq 1$.
2. If $\rho < \Delta(t_{\max})$: no resolvable emission, stop this branch.
3. Else solve $\rho = \Delta(t_{\max})/\Delta(t)$
(= no emission between t_{\max} and t) for t .
Reset $t_{\max} = t$ and goto 1.

Determine z essentially according to integrand in front of exp.

Probability density:

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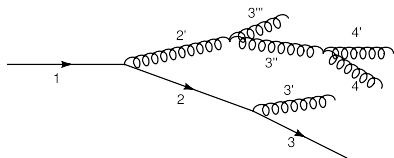
- ▶ That was old HERWIG variant. Relies on (numerical) integration/tabulation for $\Delta(t)$.
- ▶ Pythia, now also Herwig++, use the **Veto Algorithm**.
- ▶ Method to sample x from distribution of the type

$$dP = F(x) \exp \left[- \int^x dx' F(x') \right] dx .$$

Simpler, more flexible, but slightly slower.

Parton cascade

Get tree structure, ordered in evolution variable t :



- ▶ t can be θ , Q^2 , p_{\perp} , ...
- ▶ Choice of hard scale t_{\max} not fixed. "Some hard scale".
- ▶ z can be light cone momentum fraction, energy fraction, ...
- ▶ Available parton shower phase space.
- ▶ Integration limits.
- ▶ Regularisation of soft singularities.
- ▶ ...

Good choices needed here to describe wealth of data!

Soft emissions

- ▶ Only *collinear* emissions so far.
- ▶ Including *collinear+soft*.
- ▶ *Large angle+soft* also important.

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Soft emission: consider *eikonal factors*,
here for $q(p+q) \rightarrow q(p)g(q)$, soft g :

$$u(p) \not{\epsilon} \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} \longrightarrow u(p) \frac{p \cdot \epsilon}{p \cdot q}$$

soft factorisation. Universal, *i.e.* independent of emitter.
In general:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_S}{2\pi} \sum_{ij} C_{ij} W_{ij} \quad (\text{"QCD-Antenna"})$$

with

$$W_{ij} = \frac{1 - \cos \theta_{ij}}{(1 - \cos \theta_{iq})(1 - \cos \theta_{jq})} .$$

We define

$$W_{ij} = \frac{1 - \cos \theta_{ij}}{(1 - \cos \theta_{iq})(1 - \cos \theta_{qj})} \equiv W_{ij}^{(i)} + W_{ij}^{(j)}$$

with

$$W_{ij}^{(i)} = \frac{1}{2} \left(W_{ij} + \frac{1}{1 - \cos \theta_{iq}} - \frac{1}{1 - \cos \theta_{qj}} \right).$$

$W_{ij}^{(i)}$ is only collinear divergent if $q \parallel i$ etc .

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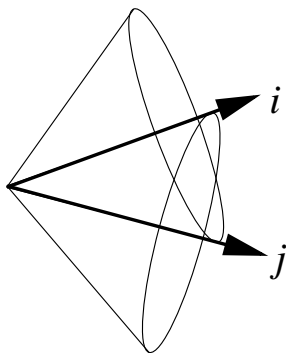
After integrating out the azimuthal angles, we find

$$\int \frac{d\phi_{iq}}{2\pi} W_{ij}^{(i)} = \begin{cases} \frac{1}{1 - \cos \theta_{iq}} & (\theta_{iq} < \theta_{ij}) \\ 0 & \text{otherwise} \end{cases}$$

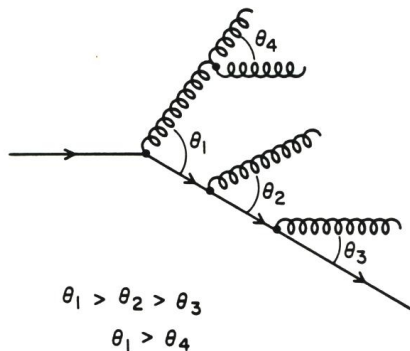
That's angular ordering.

Angular ordering

Radiation from parton i is bound to a cone, given by the colour partner parton j .



Results in angular ordered parton shower and suppresses soft gluons viz. hadrons in a jet.



Events with 2 hard (> 100 GeV) jets and a soft 3rd jet (~ 10 GeV)

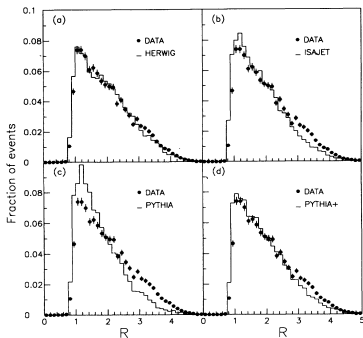


FIG. 14. Observed R distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

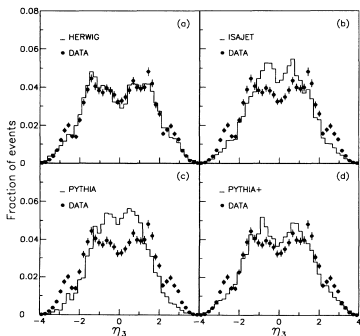


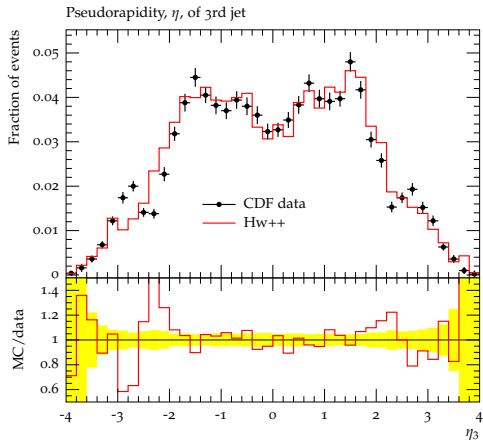
FIG. 13. Observed η_3 distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

F. Abe *et al.* [CDF Collaboration], Phys. Rev. D **50** (1994) 5562.

Best description with angular ordering.

Colour coherence from CDF

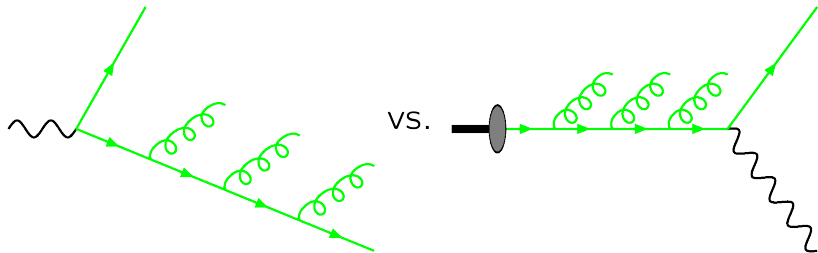
Events with 2 hard (> 100 GeV) jets and a soft 3rd jet (~ 10 GeV)



F. Abe *et al.* [CDF Collaboration], Phys. Rev. D **50** (1994) 5562.

Best description with angular ordering.

Initial state radiation



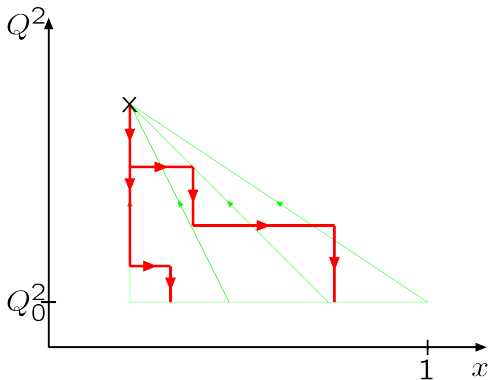
Similar to final state radiation. Sudakov form factor ($x' = x/z$)

$$\Delta(t, t_{\max}) = \exp \left[- \sum_b \int_t^{t_{\max}} \frac{dt}{t} \int_{z_-}^{z_+} dz \frac{\alpha_S(z, t)}{2\pi} \frac{x' f_b(x', t)}{x f_a(x, t)} \hat{P}_{ba}(z, t) \right]$$

Have to **divide out the pdfs**.

Initial state radiation

Evolve backwards from hard scale Q^2 down towards cutoff scale Q_0^2 . Thereby increase x .



With parton shower we *undo* the DGLAP evolution of the pdfs.

Reconstruction of Kinematics

After shower: original partons acquire virtualities q_i^2

→ boost/rescale jets:

Started with

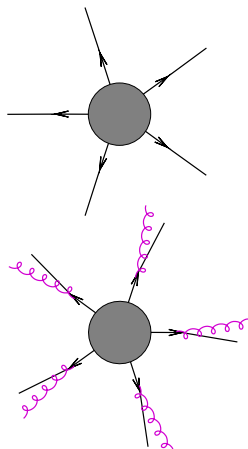
$$\sqrt{s} = \sum_{i=1}^n \sqrt{m_i^2 + \vec{p}_i^2}$$

we *rescale* momenta with common factor k ,

$$\sqrt{s} = \sum_{i=1}^n \sqrt{q_i^2 + k\vec{p}_i^2}$$

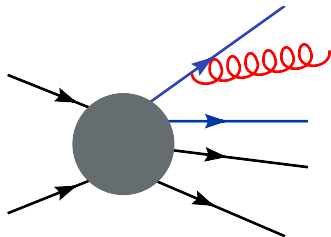
to preserve overall energy/momentum.

→ resulting jets are boosted accordingly.



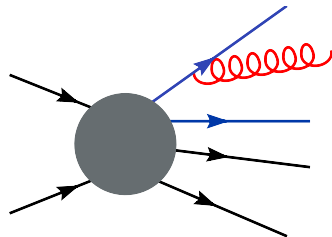
Exact kinematics when recoil is taken
by *spectator(s)*.

- ▶ Dipole showers.
- ▶ Ariadne.
- ▶ Recoils in Pythia.



Exact kinematics when recoil is taken by *spectator(s)*.

- ▶ Dipole showers.
- ▶ Ariadne.
- ▶ Recoils in Pythia.
- ▶ New dipole showers, based on
 - ▶ Catani Seymour dipoles.
 - ▶ QCD Antennae.
 - ▶ Goal: matching with NLO.
- ▶ Generalized to IS–IS, IS–FS.



- ▶ Hard scattering
 - ▶ Matrix elements and phase space
 - ▶ Mini event generator
- ▶ Parton showers
 - ▶ e^+e^- annihilation and collinear limits
 - ▶ Multiple emissions
 - ▶ Sudakov form factor
 - ▶ Parton cascades
 - ▶ Coherence/angular ordering
 - ▶ Misc aspects

- ▶ Part I — Basics
 - ▶ Introduction
 - ▶ Monte Carlo techniques
- ▶ Part II — Perturbative physics
 - ▶ Hard scattering
 - ▶ Parton showers
- ▶ Part III — Merging/Matching
 - ▶ Matrix element corrections
 - ▶ Merging multiple tree level MEs with parton showers
 - ▶ Matching NLO and parton showers
- ▶ Part IV — Non-perturbative physics
 - ▶ Hadronization
 - ▶ Hadronic decays
 - ▶ Comparison to data
- ▶ Part V — Multiple Partonic Interactions
 - ▶ Minimum Bias/Underlying Event in data
 - ▶ Modelling