Scale and Conformal Invariance

Zohar Komargodski

Weizmann Institute of Science, Israel

Collaboration with Tolya Dymarsky, Adam Schwimmer, Stefan Theisen

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General Comments on Theories without a Mass Scale

Suppose we have a theory without a mass scale. Roughly speaking, this means that all the correlation functions are power laws. The naive symmetry group:

$$ISO(d-1,1) \rtimes \mathbb{R} \ .$$

Surprisingly, we often discover that the symmetry group is actually

$$SO(d,2)$$

So we have d unexpected conserved charges.

General Comments on Theories without a Mass Scale

The idea that this symmetry enhancement is a general phenomenon in QFT has been around for many decades (Migdal, Polyakov, Wilson, and others wrote about this already in the 70s).

It has been realized fairly early (although I am not sure when and by whom) that unitarity is a key ingredient in having these d extra generators.

General Comments on Theories without a Mass Scale

The additional d generators are the special conformal transformations.

They are extremely important. The allow to fix three-point functions in terms of finitely many coefficients and they lead to many other constraints, such as inequalities among anomalous dimensions.

There is experimental, numerical, and theoretical evidence that we have the $SO(d,2)$ enhanced symmetry in unitary theories. Let us formulate the question a little more precisely:

General Comments on Theories without a Mass Scale

Suppose that

$$T^{\mu}_{\mu} = \partial^{\nu} V_{\nu}$$

for some **local** operator V_{ν}. Then the theory is scale invariant and we have the conserved current

$$S_{\mu} = x^{\nu} T_{\mu\nu} - V_{\mu} \ .$$

The theory is conformal if an only if we can find an $L_{\mu\nu}$ such that

$$V_{\nu} = \partial^{\mu} L_{\mu\nu} \ , \qquad i.e. \qquad T^{\mu}_{\mu} = \partial^{\mu}\partial^{\nu} L_{\mu\nu}$$

General Comments on Theories without a Mass Scale

If indeed $T^\mu_\mu = \partial^\mu \partial^\nu L_{\mu\nu}$ then we can define some

$$T'_{\mu\nu} = T_{\mu\nu} - \hat{\mathcal{O}}^{\mu\nu} L_{\mu\nu}$$

such that $(T')^\mu_\mu = 0$ and $\hat{\mathcal{O}}^{\mu\nu}$ is some second order differential operator that one can easily write out explicitly.

The condition $T^\mu_\mu = \partial^\mu \partial^\nu L_{\mu\nu}$ can be sharpened in unitary theories: A unitary scale invariant theory is conformal if and only if there exists L such that

$$T^\mu_\mu = \Box L$$

with scalar **local** L. This sharpening is achieved via an analysis similar to [Grinstein-Intriligator-Rothstein].

General Comments on Theories without a Mass Scale

Let us summarize: to prove that a unitary scale invariant theory is conformal, one needs to show that

$$T^{\mu}_{\mu} = \Box L$$

for some local L.

Solution for $d = 2$

There is a remarkably short and nice argument solving the problem in $d = 2$ [Polchinski, 1988].

$d = 2$ is exceptionally simple because the scaling dimension of L is zero. So we can essentially forget about L and we just need to prove that in unitary scale invariant theories

$$T^{\mu}_{\mu} = 0 \ .$$

This can be done by showing that the two point function $\langle T^{\mu}_{\mu}(x) T^{\mu}_{\mu}(0)\rangle = 0$ at $x \neq 0$.

Solution for $d = 2$

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(-q)\rangle = B(q^2)\tilde{q}_\mu \tilde{q}_\nu \tilde{q}_\rho \tilde{q}_\sigma \ ,$$

with $\tilde{q}_\mu = \epsilon_{\mu\nu} q^\nu$. This is the most general decomposition satisfying conservation and permutation symmetry. In a scale invariant theory we must take by dimensional analysis

$$B(q^2) = \frac{1}{q^2} \ .$$

Then,

$$\langle T^\mu_\mu(q) T^\rho_\rho(-q)\rangle \sim q^2 \ .$$

This is a contact term, thus, $T^\mu_\mu = 0$.

The Difficulty of the Problem for $d > 2$

There is no hope to repeat an argument of this kind in $d > 2$ because of two main reasons

- It is not true that unitarity and scale invariance imply that $T^\mu_\mu = 0$. Indeed, in many examples one finds a nontrivial L:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2$$

  leads to $T^\mu_\mu = \frac{2-d}{4}\Box(\phi^2)$, i.e. $L = \frac{2-d}{4}\phi^2$.

- **It is not true** that scale+unitarity implies conformal invariance. The only known counter-example is a free scalar with shift symmetry, i.e. the model above with $\phi \to \phi + c$ gauged for any real c. Then, L is not well defined and hence $T^\mu_\mu \neq \Box L$ for local L.

The Difficulty of the Problem for $d > 2$

These two novelties in $d > 2$ lead to two difficulties

- One really needs to show the existence of a local L such that $T^\mu_\mu = \Box L$. This L should be constructed ab initio. How can we do something like that even in principle?
- One must explain why the scalar with shift symmetry is an exception. (In 3d it is dual to a free Maxwell field, and in 4d to a free two-form.) This must require **dynamical** input, since all the kinematical constraints due to anomalies/correlation functions are satisfied by this example.

The Difficulty of the Problem for $d > 2$

Take a free scalar field, Φ. The set of operators where Φ appears only with derivatives is closed under the OPE. This theory is scale invariant. It is local because we have the EM tensor

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2}\eta_{\mu\nu}(\partial\Phi)^2 \ .$$

It is not conformal because the improvement $\sim \left(\partial_\mu\partial_\nu - \partial^2\eta_{\mu\nu}\right)\Phi^2$ is not an allowed operator.

In flat space this theory is indistinguishable from the ordinary scalar, it has consistent separated points correlation functions, OPE, consistent anomalies etc.

The Difficulty of the Problem for $d > 2$

Therefore, it is not true that scale+unitarity implies conformality!! Also it bodes ill for looking for a generalization of Polchinski's argument in some n-point function $\langle TTT...T \rangle$.

One may hope to prove that this 2-form theory is the only exception. This is what we will argue. Note that this theory has no dimension two operators, so even this assumption cannot be enough. One needs "dynamical" input.

The Difficulty of the Problem for $d > 2$

Actually, our argument does not depend on the existence of a local scale current (just the existence of a charge), neither does the recent paper by Farnsworth, Luty, Prelipina (we used the same anomaly structure under global scale transformations).

I will not comment further on their argument at this point, except to mention that our strategy is closely related to a generalization of the a-theorem sum rules (that I will describe if i have time), while the logic of FLP does not seem to be related to any monotonicity theorem. It also remains to be seen how the 2-form fits their story.

The Difficulty of the Problem for $d > 2$

The problem is conceptually simpler in perturbation theory because no operator needs to be divined. We have a clear list of candidates for L and V_μ and we "just" check if the equations are satisfied. This has been checked very explicitly in many models [Grinstein-Fortin-Stergiou] and a beautiful general argument was offered by [Luty-Polchinski-Rattazzi] as well as [Osborn] and [Grinstein-Fortin-Stergiou].

The Difficulty of the Problem for $d > 2$

The problem also simplifies when there is a weakly coupled holographic dual [Nakayama]. There is some evidence that all solutions to $10d/11d$ Einstein equations with fluxes that are scale invariant are also conformal invariant.

Some simplification also takes place in SUSY theories, see for example [Antoniadis-Buican, Zheng, Nakayama, Fortin-Grinstein-Stergiou]

Outline of the Argument

Our argument has two central parts.

- We consider any local SFT and couple it to some background metric $g_{\mu\nu}$. Derivatives with respect to $g_{\mu\nu}$ give correlation functions of the EM tensor. Using unitarity we prove that when acting on the vacuum with an arbitrary number of $T^\mu_\mu(p)$ with $p^2 = 0$, we never create nontrivial states in the Hilbert space:

$$\langle VAC|\,T^\mu_\mu(p_1)....T^\mu_\mu(p_n)|Anything\rangle_{connected} = 0\ , \qquad p_i^2 = 0.$$

- The above is a nontrivial necessary condition for conformal invariance. We explain why it is sufficient too, at least for interacting theories.

Outline of the Argument

We have therefore established that scale invariance in conjunction with unitarity leads to conformal invariance under some seemingly mild assumptions about QFT. In particular, as we will see, the 2-form theory is an exception because it is free.

Outline of the Argument

The discussion below does not do justice to various subtle issues that are nevertheless important – it is only a hand-wavy explanation of the structure of the argument.

A Proof of the Vanishing Theorem

We couple any SFT to a background metric. Then we can consider the generating functional $W[g_{\mu\nu}]$. Its UV divergences are characterized by the local counterterms

$$\int d^4x\sqrt{g}\left(\Lambda + \mathfrak{a}R + \mathfrak{b}R^2 + \mathfrak{c}W^2\right) \ ,$$

where we only neglected to write total derivative terms. If we consider metric of the type

$$g_{\mu\nu} = (1+\Psi)^2\eta_{\mu\nu}$$

with $\partial^2\Psi = 0$ then neither of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ contribute.

A Proof of the Vanishing Theorem

Whence it follows that $W[\Psi]$ is well defined up to the cosmological constant term, or more precisely, any derivative with respect to momentum of $W[\Psi]$ is a good observable in QFT.

We define

$$A_n(p_1, ..., p_{2n}) = \frac{\delta^n W[\Psi]}{\delta\Psi(p_1)\delta\Psi(p_2)...\delta\Psi(p_{2n})}$$

and we will choose all the momenta to be null, $p_i^2 = 0$.

A Proof of the Vanishing Theorem

Let us start from $n = 2$. We can prepare forward kinematics $p_3 = -p_1$ and $p_4 = -p_2$. We have the dispersion relation

$$A_4(s) = \frac{1}{\pi} \int ds' \frac{ImA_4(s')}{s - s'} + subtractions \ , \qquad s = (p_1 + p_2)^2 \ .$$

We immediately see that $ImA_4 = 0$. Had it not been zero, we would have needed a subtraction which goes like s^2. On the other hand, $ImA_4 = \kappa s^2$ by scale invariance. Hence, $\kappa = 0$.

A similar argument proceeds for all the amplitudes A_{2n}, in other words, in forward kinematics

$$ImA_{2n} = 0$$

.

A Proof of the Vanishing Theorem

Now we use unitarity, more precisely, the optical theorem.

One can use it since the structure of the correlation functions we are considering with $p_i^2 = 0$ has the same formal properties as the S-matrix of massless particles.

All the contributions to ImA_4 are positive definite since there is just one cut (s-channel and t-channel, depending on whether $s > 0$ or $s < 0$).
Hence,

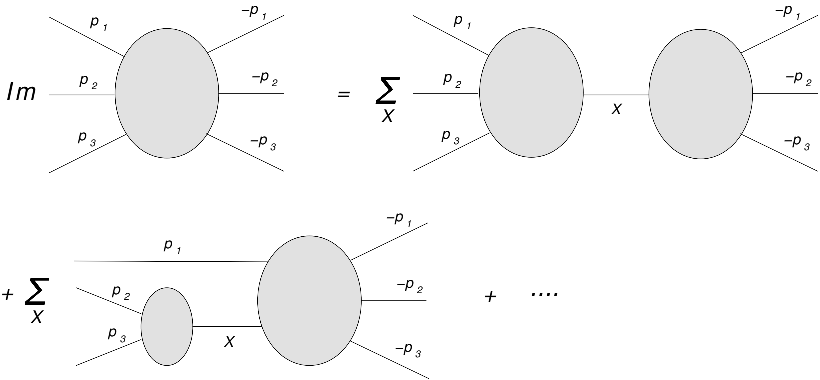
$$\langle \Psi(p_1)\Psi(p_2)|Anything\rangle = 0 \ , \qquad p_1^2 = p_2^2 = 0$$

A Proof of the Vanishing Theorem

The result $\langle\Psi(p_1)\Psi(p_2)|Anything\rangle = 0$ (with null momenta) is precisely the Luty-Polchinski-Rattazzi theorem. It can be useful in theories with marginally relevant operators, where they showed that this implies that scale+unitarity=conformal.

Upon trying to generalize their result to $n = 3$, one immediately finds hits a difficulty that various unitarity cuts of 3-3 scattering are not manifestly positive.

A Proof of the Vanishing Theorem



The cuts of the type appearing in the second line are not necessarily positive.

A Proof of the Vanishing Theorem

However, the trick is that from the $n = 2$ result is follows that the non-positive cuts are absent, and we can conclude, by induction,

$$\langle\Psi(p_1)\Psi(p_2)....\Psi(p_n)|Anything\rangle = 0 \ , \qquad p_1^2 = p_2^2 = .. = p_n^2 = 0$$

If one thinks of Ψ as a massless, weakly coupled, dilaton particle, the result above means that it is completely decoupled from the physical theory.

It is a highly nontrivial necessary condition for conformal invariance. Indeed, if the theory is conformal $T_\mu^\mu = \Box L$ the coupling of the dilaton looks like

$$\int d^4x \Psi T_\mu^\mu = \int d^4x \Box\Psi L$$

and hence vanishes with null external momenta for Ψ.

A Proof of the Vanishing Theorem

This necessary condition is nontrivial, as many (non-unitary) SFTs violate it.

A sufficient condition

As we see, it is very natural to focus on light-like kinematics in this context. Indeed, the equation $T^{\mu}_{\mu} = \Box L$ almost screams that it wants a massless particle to be coupled to it, and that one just needs to show that this particle has a trivial S-matrix.

More formally, let us assume the following property of QFT:

"decoupling property"

If the S-matrix of some particle scattering is trivial in a unitary theory, then the particle can be rendered free after some change of variables.

Of course, if the particle is free after some change of variables, then the S-matrix is trivial. Here we allude to the converse.

A sufficient condition

Since the coupling $\int d^4x \Psi T^\mu_\mu$ leads to a trivial S-matrix for Ψ particle, and, consequently, all the transition elements between Ψ and any state in the Hilbert space vanish, we conclude that we must be able to remove the interaction by a change of variables.

But the Ψ particle

$$\int d^4x \left[-\frac{f^2}{2} \Psi \Box \Psi + \Psi T^\mu_\mu + \cdots \right]$$

can be rendered free only if $T^\mu_\mu = \Box L$. Then, we can redefine $\Psi \to \Psi - \frac{1}{f^2} L$.

A sufficient condition

This shows that scale invariant unitary theories are conformal. Let us explain how the two-form fits into this. In flat space the two-form is indistinguishable from the free scalar, hence, indeed, the S-matrix for Ψ vanishes. But we cannot solve $T^\mu_\mu = \Box L$.

This can be traced to the fact that for a free scalar field theory we can project out a subset of the operators (including L) and still maintain consistency (at least as far as the OPE and other simple requirements go). Then it provides an exception to our basic assertion about the S-matrix.

In general, in QFT one cannot project out a subset of the operators and not jeopardize the consistency of the theory. This is expected to be possible only in free field theory.

A sufficient condition

Said differently, we argue that there is always an extension of the space of local operators such that a change of variables exists, i.e. $T^\mu_\mu = \Box L$ for a good L. This extension of the space of operators should not modify the original theory.

Such extensions of the space of local operators are expected to exist only in free theories, because the space of local operators in a theory is supposed to *characterize* it (apart from topological degrees of freedom).

Hence, we argued that scale invariance and unitarily lead to conformal invariance, and we explained why the 2-form is an exception.

Open Question

- The argument fails in three dimensions because the structure of counter-terms is different and the dispersion relation thus produces only very weak constraints. Can we show that scale+unitarity implies conformality (other than in free Maxwell)? should apply to boiling water...! – perhaps by using the f-theorem?
- The free Maxwell and free two-forms have various special properties that we did not need to use, such as that they don't have relevant operators, no virial current etc. Maybe there exists another argument where this is important.
- Verify our vanishing theorem in weakly coupled models (not completely trivial for seagull terms etc.).
- Are there unitary non-free CFTs above $d = 6$?
- Learn how to exploit better the power of $SO(d, 2)$.

Connection to the a-Theorem

Consider renormalization group flows between two conformal field theories, CFT_{uv} and CFT_{ir}. A simple generalization of the a-theorem sum rule

$$a_{uv} - a_{ir} = \frac{1}{4\pi} \int \frac{ds}{s^3} Im\mathcal{A}_4(s) \ ,$$

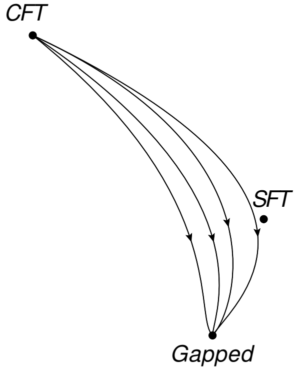
is

$$(a_{uv} - a_{ir}) \sum_{ij} s_{ij}^2 \sim \frac{1}{4\pi} \int \frac{d\lambda}{\lambda^3} Im\mathcal{A}_{2n}(\lambda s_{ij}) \ .$$

It is not very hard to prove this, but we won't have time to do it now.

Connection to the *a*-Theorem

Imagine that during the flow we can pass very close to a SFT. We imagine that there is a small parameter ϵ in the space of couplings such that we can get arbitrarily close to an SFT as we take ϵ to zero.



Connection to the a-Theorem

In th energy range between μ_{IR} and μ_{UV}, $\mu_{IR} \ll \mu_{UV}$, the theory is approximately scale invariant. Then, by dimensional analysis,

$$\mu_{IR}^2 \ll s_{ij} \ll \mu_{UV}^2, \ \mu_{IR}^2 \ll \lambda s_{ij} \ll \mu_{UV}^2 :$$

$$Im\mathcal{A}_n(\lambda s_{ij}) = \lambda^2 \mathcal{F}_n(s_{ij}) \ .$$

If the function $\mathcal{F}_n(s_{ij})$ is non-vanishing, such a behavior leads to a contradiction with the sum rules because the sum rules cease to converge as $\epsilon \to 0$.

Connection to the a-Theorem

In unitary theories it means that the dilaton $g_{\mu\nu} = (1+\Psi)^2\eta_{\mu\nu}$ is completely decoupled from the SFT for energies in the range between μ_{IR} and μ_{UV}.

This decoupling is consistent with the apparently nontrivial interaction term in the action

$$\int d^4x \Psi T^\mu_\mu$$

only if $T^\mu_\mu = \Box L$ in this energy range. Hence, the SFT in the middle of the flow is conformal.