Supersymmetric dualities and partition functions

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September 25, 2013 – DESY

Dualities

- Dualities in QFT: different looking descriptions of the same physics.

- IR dualities: different UV descriptions flowing in the IR to the same fixed point.
  - 4d IR (Seiberg) dualities:

$$\begin{array}{ccc} SU(N_c)_{N_f} & \longleftrightarrow & SU(N_f - N_c)_{N_f} + W_1 \\ USp(2N_c)_{N_f} & \longleftrightarrow & USp(2(N_f - N_c - 2))_{N_f} + W_3 \end{array}$$

  - 3d IR (Aharony,Giveon-Kutasov,...) dualities:

$$\begin{array}{ccc} U(N_c)_{N_f} & \longleftrightarrow & U(N_f - N_c)_{N_f} + W_1' \\ USp(2N_c)_{N_f} & \longleftrightarrow & USp(2(N_f - N_c - 1))_{N_f} + W_3' \end{array}$$

- Conformal dualities: Equivalence of different-looking conformal field theories.
  - $4d$ $\mathcal{N} = 2$ Gaiotto dualities: theories obtained by compactifying $6d$ $(2,0)$ theory down to four dimensions. ($\mathcal{N} = 4$ SYM, $\tau \leftrightarrow -\frac{1}{\tau}$)

Partition functions

- In recent years there has been a lot of progress in computing exactly certain partition functions for supersymmetric field theories in various dimensions.
  - 2d: S^2, T^2
  - 3d: $S^2 \times S^1$, S^3/Z_r
  - 4d: S^4, $S^3/Z_r \times S^1$
  - ...
- The supersymmetric partition functions usually are used for two main purposes:
  - Exact checks of dualities – the computation is quite different in the different descriptions but the result should be the same.
  - Learn something new about the properties of the theories.

Outline

- Our goal today will be:
  - Checking $4d$ $\mathcal{N}=1$ Seiberg dualities with $S^3/Z_r \times S^1$ partition functions.
  - Deducing properties of the $S^3/Z_r \times S^1$ partition functions for $4d$ $\mathcal{N}=2$ theories with no known Lagrangian.
  - Relations between physics and partition functions in $4d$ and $3d$.

Basics of the $S^3/Z_r \times S^1$ partition function

- The $S^3/Z_r \times S^1$ partition function of an $\mathcal{N}=1$ theory is given by the trace formula (Benini-Nishioka-Yamazaki 11)

$$\mathcal{I}(p,q,\cdots) = \mathrm{Tr}_{S^3/Z_r}(-1)^F\, p^{j_1+j_2-\frac{R}{2}}\, q^{j_1-j_2-\frac{R}{2}}\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\;\cdots\;.$$

- Here S^3/Z_r is given by

$$S^3/Z_r = \left(z_1, z_2;\; |z_1|^2+|z_2|^2=1\right), \qquad (z_1,\, z_2) \sim (e^{\frac{2\pi i}{r}}\, z_1,\, e^{-\frac{2\pi i}{r}}\, z_2)\,.$$

- $j_1 \pm j_2$ are rotations of $z_{1,2}$, and R is the $U(1)_R$ R-symmetry charge. The partition function is independent of β.

- Useful special cases:
  - $r=1$: the supersymmetric index, $S^3\times S^1$.
  - Sending $r\to\infty$ one obtains the $S^2\times S^1$ of the dimensionally reduced theory.*
  - Sending the radius of S^1 to zero one obtains an S^3/Z_r partition function of a dimensionally reduced theory.*

Partition function of a gauge theory

- The space $S^3/Z_r \times S^1$ contains two non-contractable cycles for $r > 1$.
- The partition function of a gauge theory with gauge group $\mathcal{G}$ localizes to flat connections and is given by a formal sum over two gauge holonomies around the two cycles

$$\mathcal{I} = \sum_{g,\,h} \mathcal{I}(g,h)\,.$$

- The holonomies have to satisfy

$$g^r = 1 \in \mathcal{G}, \qquad g\,h\,g^{-1}\,h^{-1} = 1 \in \mathcal{G}$$

- Up to simultaneous conjugation by an element of $\mathcal{G}$ this gives a discrete sum over g and, in general, an integral in h.

$$\mathcal{I} = \sum_g \frac{1}{|W_g|} \oint [dh] \Delta_g(h)\, \mathcal{I}_V^{(\mathcal{G})}(h,g) \prod_{\ell=1}^{N_\chi} \mathcal{I}_\chi^{(R_\ell,(\mathcal{R}_\ell,\hat{\mathcal{R}}_\ell))}(h,g,u)\,.$$

Partition function of a gauge theory (cont.)

- If $\mathcal{G}$ is not simply-connected let $\widetilde{g}$ and $\widetilde{h}$ be (a choice) of the lifts to the covering group $\widetilde{\mathcal{G}}$. ($\mathcal{G} = \widetilde{\mathcal{G}}/\Gamma$)

- Then we have

$$\widetilde{g}^r = \mu \in \Gamma, \qquad \widetilde{g}\,\widetilde{h}\,\widetilde{g}^{-1}\,\widetilde{h}^{-1} = \nu \in \Gamma\,.$$

- The computation of the partition function can be thus organized as a sum over different bundles labeled by $(\mu,\,\nu)$,

$$\mathcal{I}_{\mathcal{G}_c} = \frac{1}{|\Gamma|}\sum_{\mu,\nu} e^{i\,c(\mu,\nu)}\,\mathcal{Z}_{\mu,\nu}\,.$$

- Here we leave open the possibility that different bundles are weighed differently.

- In general the partition function on $S^3/Z_r \times S^1$ for $r > 1$ might be sensitive to the global structure of the gauge group.

Example of a check: $so(N)$ Seiberg dualities

- Seiberg duality for theories with $so(N)$ Lie algebras:

  $so(N)$ theory with N_f vector flavors Q and $W = 0$ is dual to

  $so(N_f - N + 4)$ theory with N_f vectors $\widetilde{Q}$, $\frac{N_f(N_f+1)}{2}$ singlets M, and $W = \widetilde{Q} M \widetilde{Q}$.

- Numerous checks of these dualities exist.
- In particular the $S^3 \times S^1$ partition function match quite non-trivially (Romelsberger 07, Dolan-Osborn 08, see also Spiridonov-Vartanov)
- However these $r = 1$ checks do not distinguish between *Spin* and *SO*.

Example of a check II: $r = 2$ (SR-Willett 13)

- The different $so(7)$ $N_f = 8$ (no mesons) sectors ($\mathcal{Z}_{\mu,\nu}$) contribute as follows ($p = q = x$)

ν/μ	1	-1
1	$2 + 72x^{\frac{3}{4}} + 1332x^{\frac{3}{2}} - 127x^2 + +16872x^{\frac{9}{4}} + 16x^{\frac{21}{8}} - 4300x^{\frac{11}{4}} + \ldots$	$1 + 36x^{\frac{3}{4}} + 666x^{\frac{3}{2}} + 56x^{\frac{15}{8}} - 62x^2 + +8436x^{\frac{9}{4}} + 1800x^{\frac{21}{8}} - 2096x^{\frac{11}{4}} + \ldots$
-1	$x^2 + 36x^{\frac{11}{4}} + \cdots$	$1 + 36x^{\frac{3}{4}} + 666x^{\frac{3}{2}} - 56x^{\frac{15}{8}} - 64x^2 + +8436x^{\frac{9}{4}} - 1800x^{\frac{21}{8}} - 2168x^{\frac{11}{4}} + \ldots$

- The different $so(5)$ $N_f = 8$ (with mesons) sectors ($\hat{\mathcal{Z}}_{\mu,\nu}$) contribute as follows

ν/μ	1	-1
1	$2 + 72x^{\frac{3}{4}} + 1332x^{\frac{3}{2}} - 127x^2 + +16872x^{\frac{9}{4}} + 8x^{\frac{21}{8}} - 4300x^{\frac{11}{4}} + \ldots$	$1 + 36x^{\frac{3}{4}} + 666x^{\frac{3}{2}} + 56x^{\frac{15}{8}} - 62x^2 + +8436x^{\frac{9}{4}} + 1808x^{\frac{21}{8}} - 2096x^{\frac{11}{4}} + \ldots$
-1	$x^2 - 8x^{\frac{21}{8}} + 36x^{\frac{11}{4}} + \cdots$	$1 + 36x^{\frac{3}{4}} + 666x^{\frac{3}{2}} - 56x^{\frac{15}{8}} - 64x^2 + +8436x^{\frac{9}{4}} - 1792x^{\frac{21}{8}} - 2168x^{\frac{11}{4}} + \ldots$

- The different dualities are:

$$
\begin{aligned}
Spin(7)_8 \leftrightarrow SO_-(5)_8 \quad &: \quad 2\mathcal{Z}_{1,1} = \hat{\mathcal{Z}}_{1,1} + \hat{\mathcal{Z}}_{-1,1} + \hat{\mathcal{Z}}_{-1,-1} - \hat{\mathcal{Z}}_{1,-1}\,, \\
Spin(5)_8 \leftrightarrow SO_-(7)_8 \quad &: \quad 2\hat{\mathcal{Z}}_{1,1} = \mathcal{Z}_{1,1} + \mathcal{Z}_{-1,1} + \mathcal{Z}_{-1,-1} - \mathcal{Z}_{1,-1}\,, \\
SO_+(7)_8 \leftrightarrow SO_+(5)_8 \quad &: \quad \mathcal{Z}_{1,1} + \mathcal{Z}_{-1,1} + \mathcal{Z}_{-1,-1} + \mathcal{Z}_{1,-1} = \\
&\qquad \hat{\mathcal{Z}}_{1,1} + \hat{\mathcal{Z}}_{-1,1} + \hat{\mathcal{Z}}_{-1,-1} + \hat{\mathcal{Z}}_{1,-1}\,.
\end{aligned}
$$

Exploring spaces of theories: $\mathcal{N} = 2$ theories of class $\mathcal{S}$

(Gaiotto, Gaiotto-Moore-Neitzke)

- $\mathcal{N} = 2$ $4d$ theories $T[\mathcal{C}]$ labeled by punctured Riemann surface $\mathcal{C}$.
- $T[\mathcal{C}]$ defined as the IR limit of the A_{N-1} $(2,0)$ theory on $\mathbb{R}^4 \times \mathcal{C}$, where $\mathcal{C}$ is a Riemann surface with punctures.
-

$4d$ **theory** $T[\mathcal{C}]$	$2d$ **Riemann surface** $\mathcal{C}$
Marginal gauge couplings	Complex moduli of $\mathcal{C}$
Flavor-symmetry factor $G \subset SU(N)$	Puncture
Gauging	Gluing
$SU(N)$ gauge group	Cylinder
Weakly-coupled frame	Pair-of-pant decomposition of $\mathcal{C}$
Different S-duality frames	Different decompositions of $\mathcal{C}$

Analytic properties and difference operators

- The $S^3/Z_r \times S^1$ partition function of theories of class $\mathcal{S}$ is a function depending on three superconformal fugacities (p, q, t) and on the holonomies for the global symmetries associated with the punctures.

$$\mathcal{I}^{(\mathcal{C})}_{p,q,t;\,r}(\{h_\ell, g_\ell\}_{\ell=1}^{s})\,.$$

- Most of the theories of class $\mathcal{S}$ are strongly-coupled and these partition functions can not be directly computed.*

- However, exploring analytical properties of these functions utilizing the S-dualities interconnecting the underlying theories one can deduce that there exist commuting difference operators $\mathcal{D}^{(\mathfrak{m})}_{p,q,t;\,N;\,r}(h,g)$ which have the property

$$\mathcal{D}^{(\mathfrak{m})}_{p,q,t;\,N;\,r}(h_i, g_i)\;\cdot\;\mathcal{I}^{(\mathcal{C})}_{p,q,t;\,r}(\{h_\ell, g_\ell\}_{\ell=1}^{s}) = \mathcal{D}^{(\mathfrak{m})}_{p,q,t;\,N;\,r}(h_j, g_j)\;\cdot\;\mathcal{I}^{(\mathcal{C})}_{p,q,t;\,r}(\{h_\ell, g_\ell\}_{\ell=1}^{s})\,.$$

- These difference operators can be explicitly computed and are interpreted as introducing surface defects into the computation of the partition function.

(Gaiotto-Rastelli-SR 12, SR-Yamazaki 13, see also Gadde-Gukov 13)

Eigenfunctions of the difference operators

- Using this property, given a set of orthonormal (under the integral measure obtained by gauging global symmetries) eigenfunctions

$$\mathcal{D}^{(\mathfrak{m})}_{p,q,t;\,N;\,r}(h,g)\cdot\psi_\lambda(h,g)=\mathcal{E}^{(\mathfrak{m})}(p,q,t;N;r)\,\psi_\lambda(h,g)\,,$$

one can write the $S^3/Z_r\times S^1$ partition function of theories of class $\mathcal{S}$ with maximal punctures as

$$\mathcal{I}^{(\mathcal{C})}_{p,q,t;\,r}(\{h_\ell,g_\ell\}_{\ell=1}^s)=\sum_\lambda C_\lambda^{2g-2+s}\prod_{\ell=1}^s\psi_\lambda(h_\ell,g_\ell)$$

- This can be made very explicit in certain cases. E.g.,
  - $r=1$ the operators are elliptic Ruijsenaars-Schneider Hamiltonians.*
  - Taking $(r=1)$ $p=0$ the eigenfunctions are proportional to (symmetric) Macdonald polynomials.
    ((Gadde-Rastelli-SR-Yan 11, Gaiotto-Rastelli-SR 12), Alday-Bullimore-Fluder 13, SR-Yamazaki 13)

- * In fact, the dualities interrelating the theories can be further exploited to solve for the spectrum of the difference operators. (SR 13)

What does this mean?

- Assuming certain dualities one derives an exact formula for the partition functions of a large class of theories: some admitting weakly-coupled descriptions but most not.
- The Lagrangian, whether known or not, is not directly visible in the above expression of the partition function.
- Is there a physical computation which directly gives the expression of the partition function of the previous slide?
- What is the physical, $4d$, meaning of the eigenfunctions?

Example of a difference operator

- Let us write an explicit example of a difference operator.
- The following is a particular operator in the A_1 case with $r > 1$,

$$
\begin{aligned}
&\mathcal{D}_{p,q,t;\,2;\,r}(h, g = e^{\frac{2\pi i m}{r}})\, f(h,g) = \\
&\frac{\theta(q^{2m}\frac{t}{pq}h^{-2};q^r)\theta(p^{2m}\frac{pq}{t}h^2;p^r)}{\theta(q^{2m}h^{-2};q^r)\theta(p^{2m}h^2;p^r)}\, f((pq)^{-\frac{1}{2}}h, g) + \\
&\frac{\theta(q^{2m}\frac{pq}{t}h^{-2};q^r)\theta(p^{2m}\frac{t}{pq}h^2;p^r)}{\theta(q^{2m}h^{-2};q^r)\theta(p^{2m}h^2;p^r)}\, f((pq)^{\frac{1}{2}}h, g) + \\
&\left(\frac{pq}{t}\right)^{\frac{2+4m-r}{r}} \frac{\theta(q^{2m}\frac{pq}{t}h^{-2};q^r)\theta(p^{2m}\frac{pq}{t}h^2;p^r)}{\theta(q^{2m}h^{-2};q^r)\theta(p^{2m}h^2;p^r)}\, f((p/q)^{\frac{1}{2}}h, e^{\frac{2\pi i}{r}}g) + \\
&\left(\frac{pq}{t}\right)^{\frac{2-4m+r}{r}} \frac{\theta(q^{2m}\frac{t}{pq}h^{-2};q^r)\theta(p^{2m}\frac{t}{pq}h^2;p^r)}{\theta(q^{2m}h^{-2};q^r)\theta(p^{2m}h^2;p^r)}\, f((q/p)^{\frac{1}{2}}h, e^{-\frac{2\pi i}{r}}g)\,.
\end{aligned}
$$

From $4d$ to $3d$: generalities

- Given a duality in $4d$ can we deduce an analogous duality in $3d$?

- For IR dualities such a deduction is not completely straightforward: the two limits, small compactification radius and focusing on small energies, do not in general commute.

- What happens in general* is that a *careful* dimensional reduction of a $4d$ duality produces a $3d$ duality between theories with same matter content and gauge interactions as in $4d$ but with additional superpotentials.

- Given such a duality in $3d$ one can can try and get rid of the superpotentials at least on one side of the duality by playing $3d$ games: e.g. turning on real masses, gauging topological symmetries.

- Following these steps many of the known $3d$ dualities can be explicitly derived from $4d$, and moreover new dualities can be deduced. (Aharony-SR-Seiberg-Willett 13×2)

From $4d$ to $3d$: partition functions

- Reducing dualities from $4d$ to $3d$ can be mimicked at the level of the partition functions.

- For example, the partition function on $S^3 \times S^1$ when the radius of S^1 is taken to zero reduces to the S^3 partition function of a theory with same matter content and gauge interactions. (Unless the reduction diverges) (Dolan-Spiridonov-Vartanov, Gadde-Yan, Imamura 11)

- In $4d$ some of the symmetries are anomalous and we are not allowed to refine the partition functions with parameters corresponding to these symmetries: thus these parameters are also absent in the reduction.

- In $3d$ what breaks these symmetries are not anomalies but the additional superpotentials one generates in the dimensional reduction. (For the partition functions at hand the only effect of a superpotential is to restrict the allowed symmetries.)

- Turning on real masses and gauging topological symmetries can be also directly implemented at the level of $3d$ partition functions (Dolan-Spiridonov-Vartanov 11, Niarchos 12, Aharony-SR-Seiberg-Willett 13, Park-Park 13, see also Benini-Closset-Cremonesi 11)

From 4d index to 3d partition functions on S^3

- Let us consider the $S^3 \times S^1$ partition function of a free chiral field.
- Taking into account the twists coming from the fugacities this partition function can be thought of as a partition function on $S^3_b \times \widetilde{S}^1$. (Imamura and Yokoyama)
- For a chiral field reducing on $\widetilde{S}^1$ one can write thus the $4d$ partition function as a product over S^3_b partition functions of the KK modes,

$$\mathcal{I}_{(4d)}(p,\, q;\, u) \propto \prod_{n=-\infty}^{\infty} \mathcal{Z}_{(3d)}(\omega_1,\, \omega_2;\, m + \frac{n}{\tilde{r}})$$

- This product should be properly regularized and the 4d partition function appropriately normalized so that the above becomes an exact equality,

$$e^{\mathcal{I}_0}\, \Gamma_e(e^{2\pi i m\widetilde{r}}; e^{2\pi i\omega_1\widetilde{r}},\, e^{2\pi i\omega_2\widetilde{r}}) = \\ e^{-\Delta} \prod_{n=-\infty}^{\infty} e^{-\mathrm{sign}(n)\, \frac{\pi i}{2\omega_1\omega_2}\left((m+\frac{n}{\tilde{r}}-\omega)^2 - \frac{\omega_1^2+\omega_2^2}{12}\right)} \Gamma_h(m + \frac{n}{\tilde{r}}; \omega_1, \omega_2)\,.$$

(This equality is mathematically precisely the $SL(3,Z)$ property of elliptic Gamma functions.)

- Sending the radius $\widetilde{r}$ to zero only the zero mass KK mode survives and we get that the 4d $S^3 \times S^1$ partition function of a chiral reduces to the 3d S^3_b partition function.

Example I: reducing *USp* Seiberg duality to 3*d*

- Consider 4*d* Seiberg duality $USp(2N_c)_{N_f} \longleftrightarrow USp(2(N_f - N_c - 2))_{N_f}$.

- Start with $N_f + 1$ flavors and dimensionally reduce to obtain a duality in 3*d* with an additional superpotential.

- Turn on a real mass for the extra flavor. Following carefully the duality in presence of the real mass the origins of the moduli space on the two sides of the duality map to each other and the superpotential on one side of the duality is removed. The duality we obtain is thus $USp(2N_c)_{N_f} \longleftrightarrow USp(2(N_f - N_c - 1))_{N_f}$.

Example II: reducing *SU* Seiberg dualities to 3*d*

- Consider now Seiberg duality $SU(N_c)_{N_f} \longleftrightarrow SU(N_f - N_c)_{N_f}$.

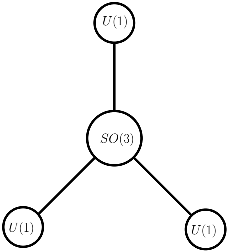
- Start with $N_f + 1$ flavors and dimensionally reduce to obtain a duality in 3*d* with an additional superpotential.

- Turn on a real mass for the extra flavor. Following carefully the duality in presence of the real mass the origin of the moduli spaces on one side of the duality maps to a non trivial vacuum on the other side: the gauge group $SU(N_f - N_c + 1)$ is broken to $SU(N_f - N_c) \times U(1)$. The duality we obtain is thus $SU(N_c)_{N_f} \longleftrightarrow SU(N_f - N_c)_{N_f} \times U(1)_{N_f+1}$.

- The subtleties with picking up the correct vacua when turning on real masses translate into subtleties with orders of limits in the partition functions: taking the real masses to infinity vs performing non-compact integrals.

Reducing difference operators to 3d and mirror symmetry

- The theories of class $\mathcal{S}$ can be dimensionally reduced to $3d$.
- In $3d$ these theories admit a mirror description a Lagrangian for which is known.
- This description takes the form of a star-shaped quiver.
- For example the mirror dual of a hypermultiplet in bifundamental representation of $SU(2)$ is



$4d$ shards of $3d$ mirrors?

- The $S^2 \times S^1$ partition function of the star-shaped quiver is schematically given by ($q = \mathfrak{q}^{\frac{1}{2}}$ and $t = \mathfrak{t}\mathfrak{q}^{\frac{1}{2}}$)

$$\mathcal{I}_{\mathfrak{q},\mathfrak{t};\, g}(\{a_\ell, m_\ell\}_{\ell=1}^{s}) = \sum_{n=-\infty}^{\infty} \oint [db]_{\mathfrak{q},\mathfrak{t}^{-1};\, g} \prod_{\ell=1}^{s} \psi_{\mathfrak{q},\mathfrak{t}^{-1}}(a_\ell, m_\ell |\, b, n)\,,$$

  where $\psi_{\mathfrak{q},\mathfrak{t}}(a, m|\, b, n)$ is the $S^2 \times S^1$ partition function of a single "leg" of the quiver. The sum over n is over magnetic fluxes through S^2.

- This has the same schematic form as the general expression for the $4d$ partition functions we obtained before. (Nishioka-Tachikawa-Yamazaki 11)

- Moreover, $\psi_{\mathfrak{q},\mathfrak{t}^{-1}}(a, m|\, b, n)$ are eigenfunctions of the dimensionally reduced difference operators,

$$\left[\lim_{r\to\infty} \mathcal{D}_{p,q,t;2;r}(b, e^{\frac{2\pi i\, n}{r}})\right] \cdot \psi_{\mathfrak{q},\mathfrak{t}}(b, n|a, m) =$$
$$(a\mathfrak{q}^{\frac{m}{2}} + a^{-1}\mathfrak{q}^{-\frac{m}{2}})(a\mathfrak{q}^{-\frac{m}{2}} + a^{-1}\mathfrak{q}^{\frac{m}{2}})\, \psi_{\mathfrak{q},\mathfrak{t}}(b, n|a, m)\,.$$

  This difference operator introduces a pair of line operators into the $S^2 \times S^1$ partition function. (Drukker-Okuda-Passerini 12)

$4d$ shards of $3d$ mirrors? (cont.)

- Thus the $4d$ eigenfunctions have a physical meaning in $3d$: they are partition functions of $T[SU(N)]$ theories, the "legs" of the star-shaped quivers.

- To obtain an interpretation of the eigenfunctions in $3d$ we have to perform a (double) mirror symmetry.

- Can this picture be lifted to $4d$?

- At least in one limit of the $S^3 \times S^1$ partition function ($p = q = 0$) the relevant eigenfunctions, which are proportional to Hall-Littlewood polynomials, are mathematically precisely what one would naively call the $S^3 \times S^1$ partition function of the $4d$ version of $T[SU(N)]$. (SR-Willett work in progress)

Summary

- Partition functions can provide refined checks of the supersymmetric dualities.
- Alternatively, *dualities* themselves give a tool for computing exactly certain partition functions.
- There are deep interrelations between properties of supersymmetric theories in different spacetime dimensions. Implications of these relations for partition functions are often tractable and can be easily studied.

Thank You!!

Summary

- Partition functions can provide refined checks of the supersymmetric dualities.

- Alternatively, *dualities* themselves give a tool for computing exactly certain partition functions.

- There are deep interrelations between properties of supersymmetric theories in different spacetime dimensions. Implications of these relations for partition functions are often tractable and can be easily studied.

Thank You!!