Non-perturbative effects in large N theory and string theory

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[Hatsuda-M.M.-Moriyama-Okuyama, 1306. 1734
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Non-perturbative effects in large N theories

The large N or 't Hooft expansion provides an asymptotic expansion of observables in *(S)U(N)* gauge theories:

free energy
$$F(\lambda, N) = \sum_{g \geq 0} N^{2-2g} F_g(\lambda)$$
$$\lambda = g^2 N$$
't Hooft coupling

There might be exponentially small corrections which are *invisible* in the 't Hooft expansion

$$\sim \exp\left(-N S(\lambda)\right)$$

One possible source for these corrections are *instantons*

$$S(\lambda) = \frac{A}{\lambda} + \cdots$$

If the large *N* theory has a string dual, non-perturbative effects in the *1/N* expansion correspond to non-perturbative effects in the string coupling constant

$$N \sim \frac{1}{g_{\rm st}}$$

These are typically due to D-brane or membrane instantons and are hard to calculate in string theory. Therefore, one should be able to use large *N* duals to understand and compute these stringy non-perturbative effects

Goal of this talk: address these issues in detail by focusing on an exactly solvable model: the partition function on the three-sphere of ABJM theory

In this model one can compute the full series of non-perturbative effects in the free energy. They display a beautiful mathematical structure related to topological string theory and integrable models. They also provide a novel realization of the perturbative/non-perturbative connection.

The answer provides many interesting lessons, in particular it gives *quantitative evidence* that perturbative strings are *radically insufficient*, and they need to be supplemented by membranes to obtain a consistent theory

A B J M theory

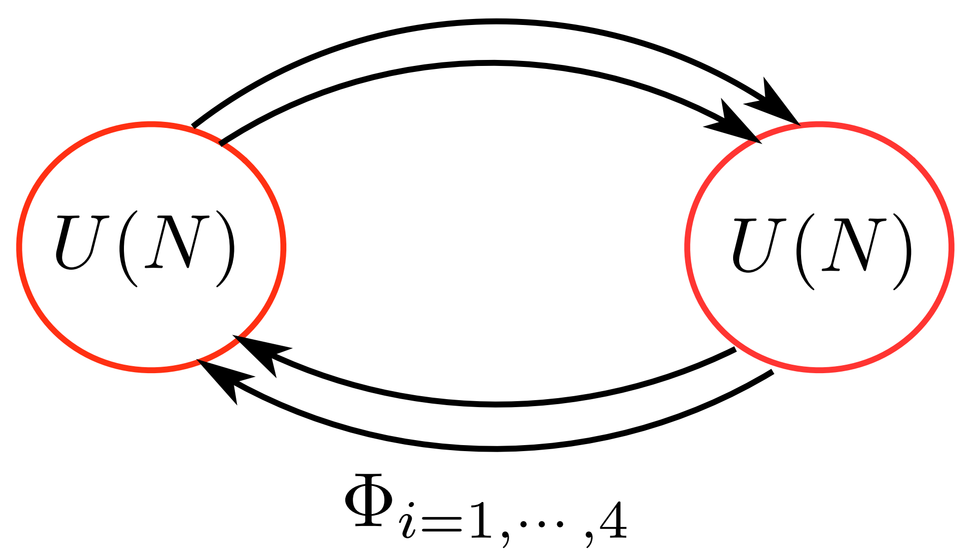
Basic building block: *Chern-Simons theory* (a TQFT in 3d) and its supersymmetric extensions

CS level (must be an integer)

$$S_{\mathrm{CS}} = -\frac{k}{4\pi}\int_M \mathrm{tr}\Big(A\wedge \mathrm{d}A + \frac{2\mathrm{i}}{3}A\wedge A\wedge A\Big)$$

ABJM =2 (super)CS theories +
4 *N*=2 hypers in the bifundamental
N=6 SUSY

$$U(N)_k \times U(N)_{-k}$$



This is a SCFT in 3d and we can consider, diagramatically, its ‘t Hooft large *N* limit:

$$k \sim \frac{1}{g^2} \Rightarrow \quad \lambda = \frac{N}{k}$$ ‘t Hooft parameter

We will be interested in a particular observable, namely the partition function on the three-sphere $Z(N,k)$. This is a sort of supersymmetric version of the thermal free energy and can be computed *exactly* in terms of a matrix integral by using supersymmtric localization [Kapustin-Willett-Yaakov]

$$Z(N,k) = \frac{1}{N!^2} \int \frac{\mathrm{d}^N \mu}{(2\pi)^N} \frac{\mathrm{d}^N \nu}{(2\pi)^N} \prod_{i<j} \left[2\sinh\left(\frac{\mu_i - \mu_j}{2}\right)\right]^2$$

$$\prod_{i<j} \left[2\sinh\left(\frac{\nu_i - \nu_j}{2}\right)\right]^2 \prod_{i,j} \left[2\cosh\left(\frac{\mu_i - \nu_j}{2}\right)\right]^{-2} \exp\left[\frac{\mathrm{i}k}{4\pi} \sum_{i=1}^N (\mu_i^2 - \nu_i^2)\right]$$

This simplifies the problem drastically, from a field theory path integral to a matrix integral. In particular, the *1/N* expansion of the free energy

$$F(N,k) = \log Z(N,k) = \sum_{g \geq 0} N^{2-2g} F_g(\lambda)$$

is now reduced to the *1/N* expansion of matrix integral -a much studied problem since the pioneering work of [Brezin-Parisi-Itzykson-Zuber]

String theory dual

ABJM theory is dual to a type IIA string on the 10d target

$$\mathrm{AdS}_4 \times \mathbb{CP}^3$$

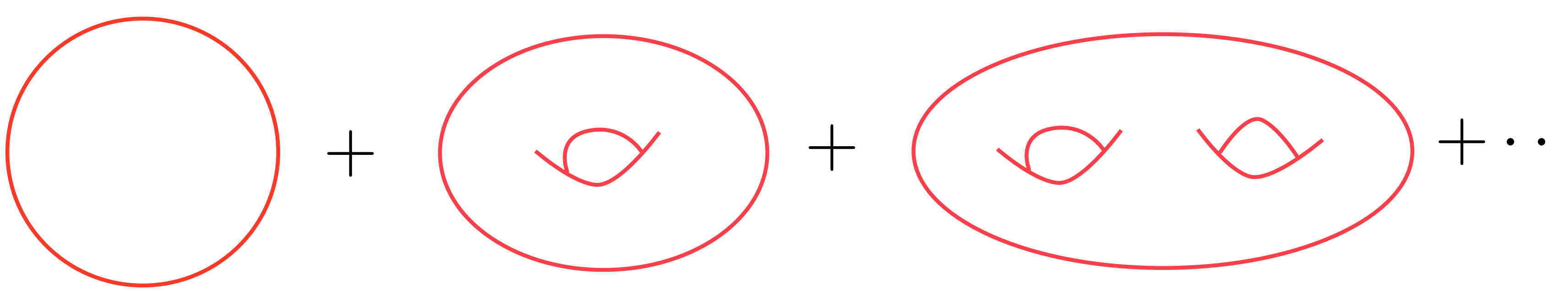
This means that the genus expansion of this string theory computes the *1/N* expansion of the gauge theory, with the dictionary

$$\lambda \sim \left(\frac{L}{\ell_s}\right)^4$$

L : AdS radius

$$g_{\mathrm{st}} \sim \frac{1}{k} \sim N$$

In particular, the free energy on the three-sphere corresponds to the free energy of the string, summed over all genera



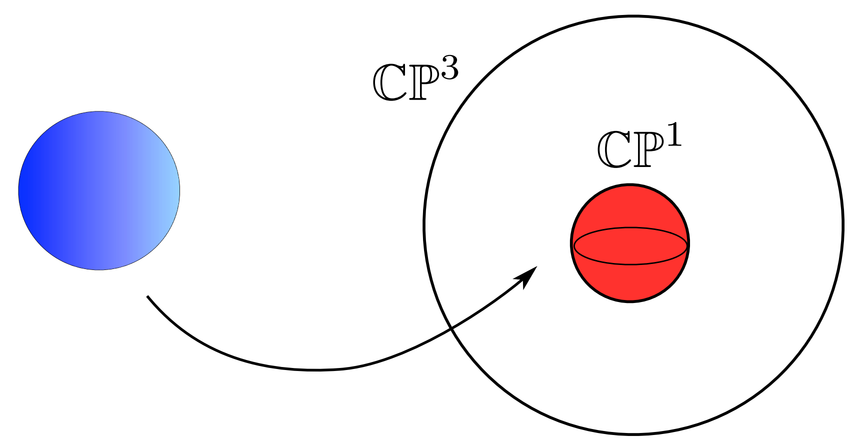
$F_0(\lambda)$ $F_1(\lambda)$ $F_2(\lambda)$

The *1/N* expansion of this quantity at *all genera* was obtained in [Drukker-M.M.-Putrov] for all values of the 't Hooft coupling. In particular, at *strong 't Hooft coupling* one makes contact with the weakly coupled non-linear sigma model at fixed genus:

worldsheet instantons!

$$F_0(\lambda) = -\frac{\pi\sqrt{2}}{3\sqrt{\lambda}} + \mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\lambda}}\right) \longleftarrow \sim \mathcal{O}\left(\mathrm{e}^{-L^2/\ell_s^2}\right)$$

$N^{3/2}$ scaling: tested with classical gravity

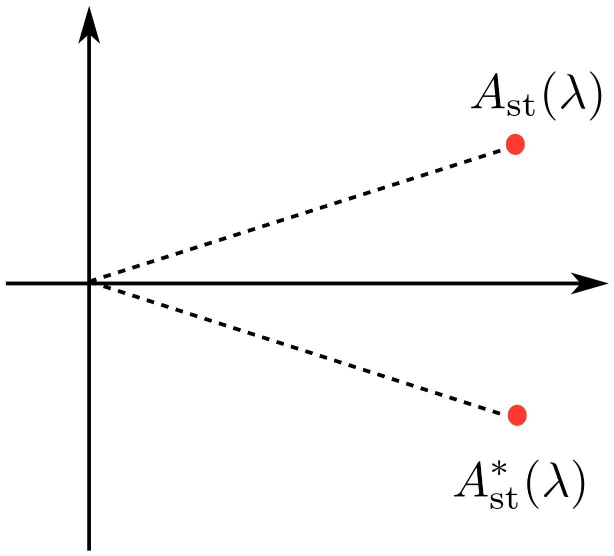


At higher genus there is a similar structure: a finite perturbative piece in λ , plus an infinite series of worldsheet instanton corrections.

One can check that the genus expansion is *asymptotic* (at fixed 't Hooft coupling) with large order behavior

$$F_g(\lambda) \sim (2g)!\,(A_{\rm st}(\lambda))^{-2g}$$

[Shenker]



The series seems to be Borel summable at finite λ. However, as we go to strong coupling, the singularities in the Borel plane move to the real axis. We expect non-perturbative corrections in this regime

NOTE: here we have a *two-parameter* problem, so the issue of asymptotics of perturbation theory is much richer than in QM or QFT

The M-theory expansion

ABJM theory can be studied in a different expansion, sometimes called the *M-theory expansion*: a *1/N expansion* at *fixed k*. This is in the strong coupling regime of the 't Hooft parameter and corresponds to type IIA strings at *finite string coupling*, i.e. to M-theory.

To study the partition function $Z(N,k)$ in this regime it is more convenient to consider the *grand potential*, i.e. go to the grand canonical ensemble

$$J(\mu,k)=\log\left(1+\sum_{N=1}^{\infty}Z(N,k)\mathrm{e}^{N\mu}\right)$$

$$\mu\sim\sqrt{Nk}$$

It turns out that one can resum the ‘t Hooft expansion to obtain the M-theory expansion. This involves resumming the worldsheet instantons at *fixed* size but *all genera* [Gopakumar-Vafa]

$$J^{\mathrm{WS}}(\mu,k)=\sum_{m=1}^{\infty} d_m(k)\mathrm{e}^{-\frac{4m\mu}{k}}$$

$d_m(k)$ ← computed from *topological string theory*

However, the coefficients have poles at *integer values of k* and cannot be the final answer: the matrix model is finite for all real *k*. The poles rather indicate a *breakdown of the genus expansion: perturbative strings are not enough and we have to go beyond the ‘t Hooft expansion*

To do this, we need a different approach to the matrix integral

The Fermi gas approach [M.M.-Putrov]

In this approach, the matrix integral $Z(N,k)$ is interpreted as the canonical partition function of an one-dimensional *ideal Fermi gas* of *N* particles, with energy levels determined by the spectral problem

$$\int_{-\infty}^{\infty} \mathrm{d}x_2 \rho(x_1,x_2) f_n(x_2) = \mathrm{e}^{-E_n} f_n(x_1)$$

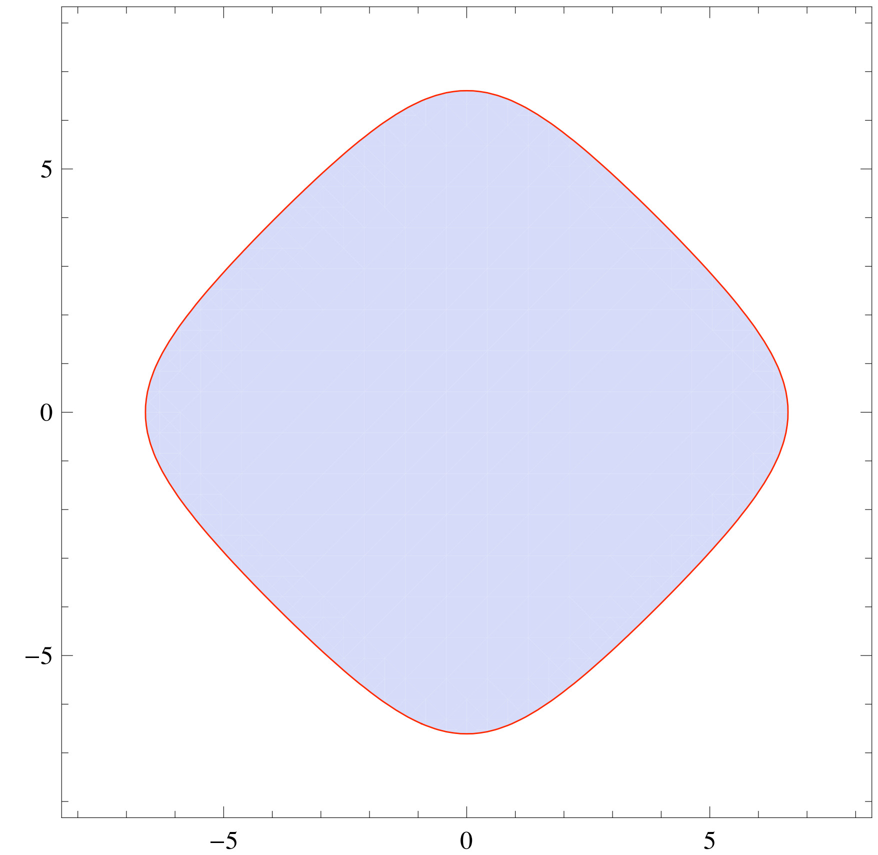
$$\rho(x_1,x_2) = \frac{1}{2\pi k} \frac{1}{\left(2\cosh \frac{x_1}{2}\right)^{1/2}} \frac{1}{\left(2\cosh \frac{x_2}{2}\right)^{1/2}} \frac{1}{2\cosh\left(\frac{x_1-x_2}{2k}\right)}$$

$J(\mu,k)$ is the grand potential of the Fermi gas. It can be seen that *k* plays the role of *Planck's constant*. The energy levels cannot be obtained in closed form, but we can calculate them with WKB

Semiclassically, the Hamiltonian is

$$H = \log\left(2\cosh\frac{p}{2}\right) + \log\left(2\cosh\frac{q}{2}\right)$$

which has periodic orbits *B* of energy *E*. Bohr-Sommerfeld-Dunham quantization gives



$$N(E)_{\mathrm{p}} = \oint_B p(E,q,k)\mathrm{d}q = 8E^2 + c + 2\sum_{\ell=1}^{\infty}\left(-4E a_\ell(k) + b_\ell(k)\right)\mathrm{e}^{-2\ell E}$$

computed from *refined topological string theory in the NS limit*

$$J^{\mathrm{M2}}(\mu,k) = C\mu^3 + D\mu + \sum_{\ell=1}^{\infty}\left(\mu^2 a_\ell(k) + \mu b_\ell(k) + c_\ell(\mu)\right)\mathrm{e}^{-2\ell\mu}$$

The exponential corrections are non-perturbative effects in the 1/N expansion and in string theory. They correspond to *membrane instantons* in M-theory

$$e^{-2\mu} \sim e^{-A_{\rm st}(\lambda)/g_{\rm st}} \sim e^{-N/\sqrt{\lambda}} \sim e^{-(L/\ell_p)^3}$$

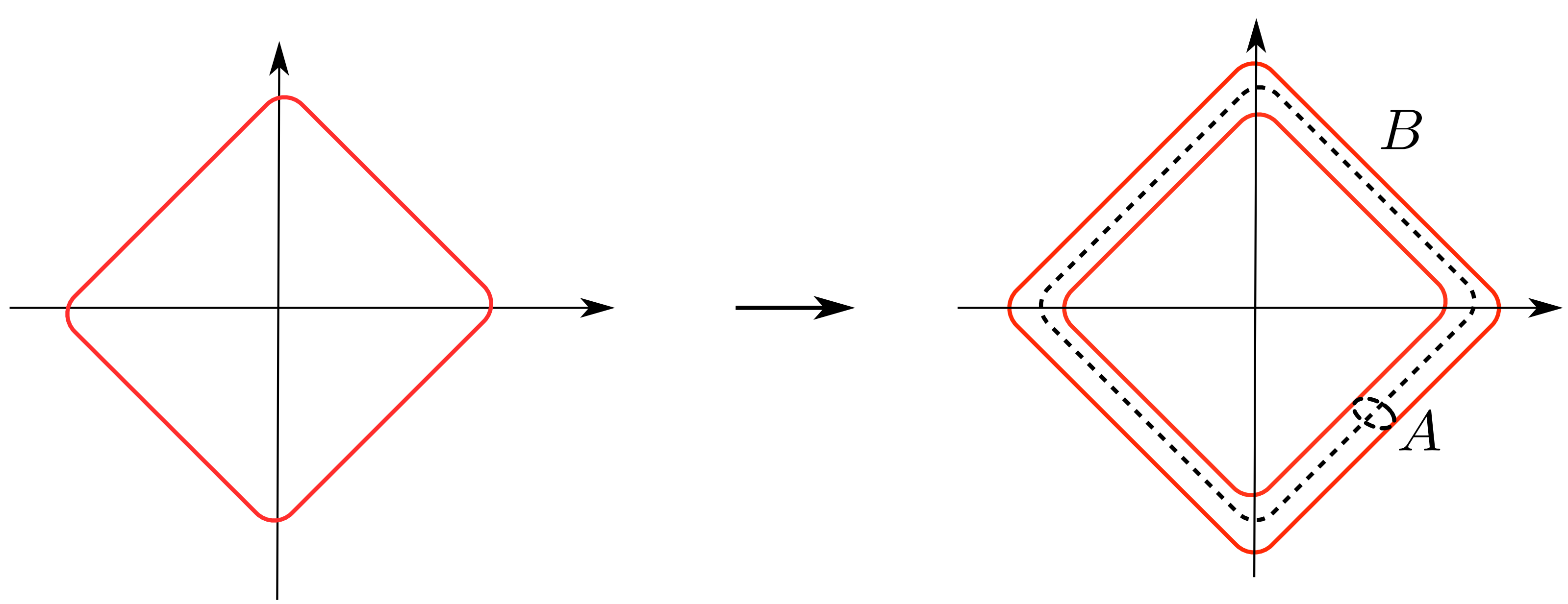
Moreover, the coefficients $b_\ell(k),\ c_\ell(k)$ have poles at integer *k*, but the sum

$$J^{\rm WS}(\mu,k) + J^{\rm M2}(\mu,k)$$

is *finite*! Therefore, *membrane instantons cure the divergence of perturbative strings*. This HMO cancellation mechanism was first conjectured by [Hatsuda-Moriyama-Okuyama] and solves the problem of the divergences in the resummed 1/N expansion

How can we see the worldsheet instantons of string theory in the WKB calculation of the spectrum? It turns out that there are *quantum-mechanical instantons* due to *complex trajectories* [Balian-Parisi-Voros]. They are *needed* to cancel the divergences due to the poles of $b_\ell(k)$ in the perturbative WKB result:

$$N(E)_{\rm np} \propto \sum_{m=1}^{\infty} \sin\left(\frac{4\pi m}{k}\right) d_m(k)(-1)^m \exp\left(\frac{{\rm i}m}{k}\oint_A p(E,q,k){\rm d}q\right)$$



However the perturbative WKB series is not asymptotic, but actually *convergent* for almost all *k*.

matrix model	gauge theory	type IIA/ M-theory	QM	topological string
$1/N$ corrections	't Hooft expansion	worldsheet instantons	QM instantons	(un-refined) topological string
$exp(-N)$ corrections	?	membrane instantons	perturbative WKB	NS topological string
coupling $1/N$	gauge coupling $1/k$	string coupling $1/k$	Planck constant k	$1/k$ (un-refined) k (NS)
eigenvalue density	master field	?	Hamiltonian	topological string target

Lessons

- The *1/N* 't Hooft expansion has non-perturbative corrections which might be crucial to make sense of the theory in some regimes. In the string dual, these effects correspond to D-brane/membrane instantons.

- In the ABJM example, one can determine these corrections in complete detail. They are *required* to cure divergences at strong coupling, and they show that the perturbative genus expansion is radically insufficient.

- Mathematically, the structures emerging here have a beautiful relationship to topological string theory, integrable systems and the complex WKB method

- There is ample room for extension of these ideas to other 3d SCFTs and topological string theories