Resurgence and Quantum Field Theory

Gerald Dunne
University of Connecticut

DESY Workshop "Non-Perturbative QFT", September 24-27, 2013

with Mithat Ünsal: arXiv:1210.2423 (JHEP), 1210.3646 (PRD), 1306.4405, ...

also: Gökçe Başar 1308.1108 (->JHEP), Robert Dabrowski, 1306.0921 (PRD),
Daniele Dorigoni, Aleksey Cherman, Ünsal, 1308.0127

related: Argyres & Ünsal: arXiv:1204.1661 (PRL), 1206.1890 (JHEP)

Physical Motivation

- Infrared renormalon puzzle in asymptotically free QFT
  - (i) IR renomalons: perturbation theory ill-defined
  - (ii) $\mathcal{I}\bar{\mathcal{I}}$ interaction: non-pert. instanton gas ill-defined

- non-perturbative physics without instantons

The Bigger Picture:

- non-perturbative definition of QCD in the continuum

- ``exact'' asymptotics in QFT and string theory

- analytic continuation of path integrals

- dynamical and non-equilibrium physics from path integrals

IR renormalon problem

SU(N) Yang-Mills on $\mathbb{R}^4$ and $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$

- asymptotically free
- instantons, theta vacua, ...

two serious long-standing problems:

- perturbative sector: infrared (IR) renormalons
  $\Rightarrow$ perturbation theory ill-defined
- non-perturbative sector: instanton scale moduli
  $\Rightarrow$ instanton gas picture ill-defined

't Hooft, 1979; Affleck, 1980; David, 1981

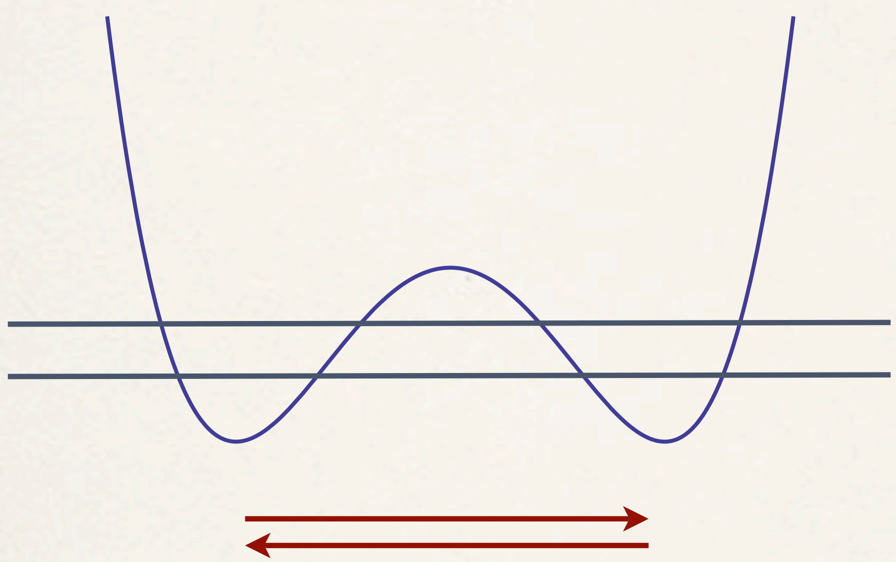
new idea: ``resurgence''

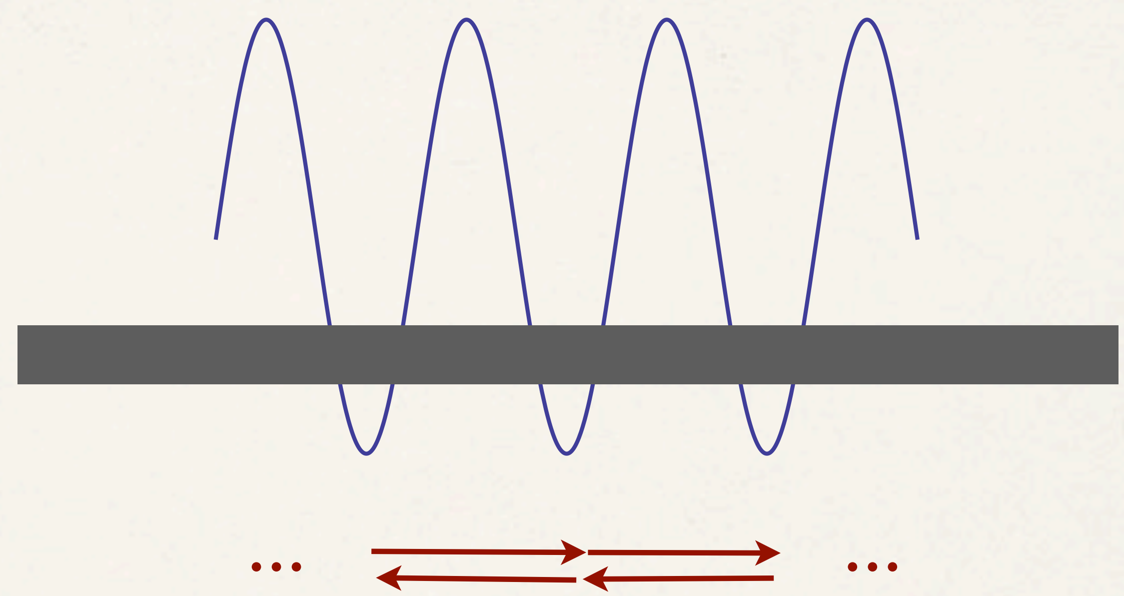
J. Écalle (1980); Stokes (1850), ...

- unify perturbation theory and non-perturbative physics
- ``trans-series'': $f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$
- mathematics: differential equations, improved asymptotics
- physics: quantum mechanics, and recent applications to QFT

analogue of IR renormalon problem in QM: Bogomolny/Zinn-Justin (BZJ)

degenerate classical vacua: double-well or Sine-Gordon





single-instanton sector: (i) level or band splitting $\sim e^{-S_{\text{instanton}}}$

(ii) real and unambiguous

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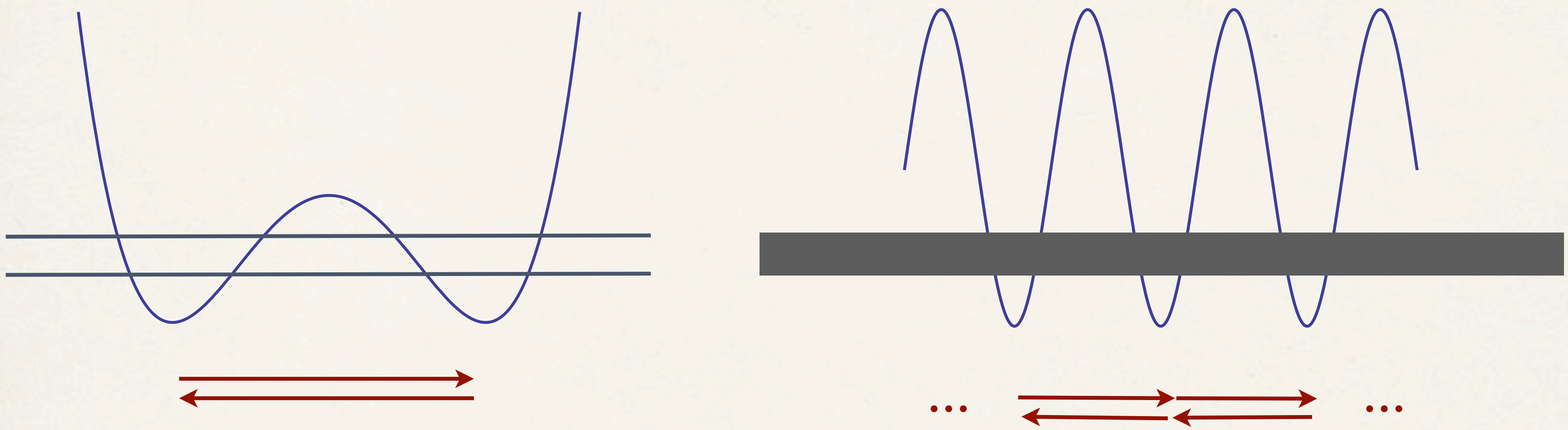

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perturbation theory is non-Borel-summable:

(i) imaginary contribution to real energy

(ii) ambiguous

(iii) $\sim \pm i\, e^{-2\, S_{\rm instanton}}$

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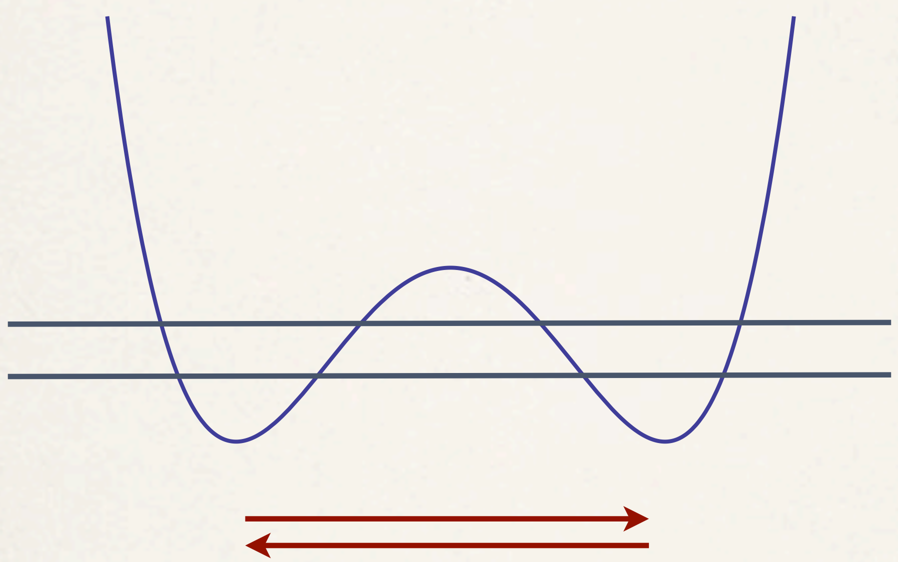

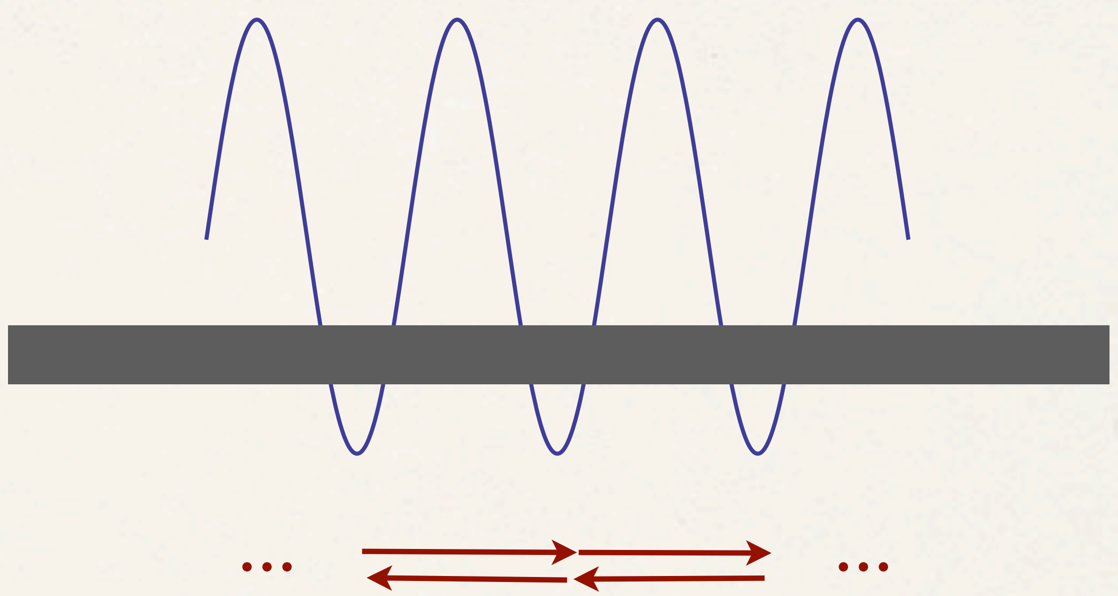

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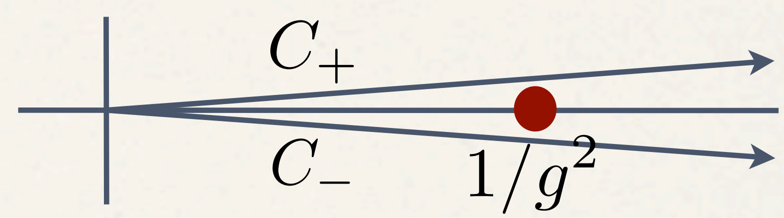
recap: basics of Borel summation

(i) divergent, alternating: $$\sum_n (-1)^n n!\,(g^2)^n = \int_0^\infty dt\,\frac{e^{-t}}{1+g^2\,t}$$

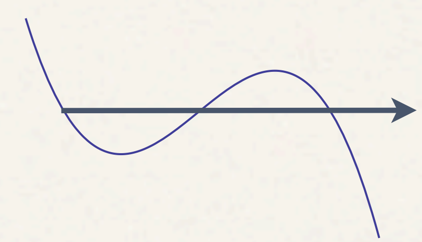
(ii) divergent, non-alternating: $$\sum_n n!\,(g^2)^n = \int_0^\infty dt\,\frac{e^{-t}}{1-g^2\,t}$$

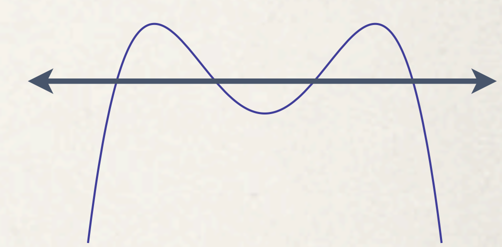
$\Rightarrow$ ambiguous imaginary non-perturbative term: $\pm\dfrac{i\pi}{g^2}\,e^{-1/g^2}$

"lateral Borel sums": $$\int_{C_\pm} dt\,\frac{e^{-t}}{1-g^2 t}$$



often identified with vacuum instability





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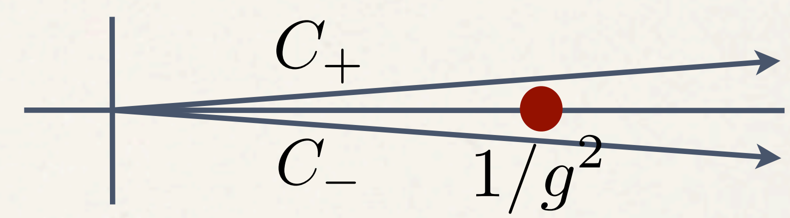

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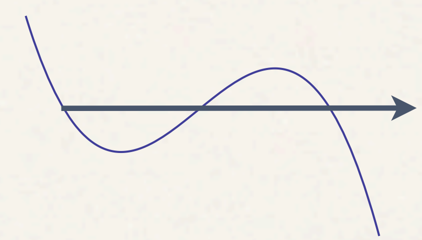
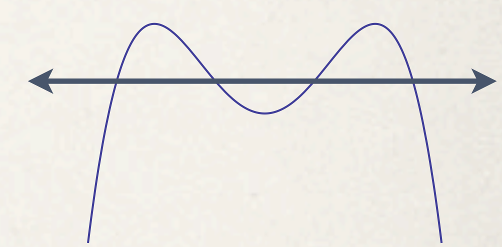


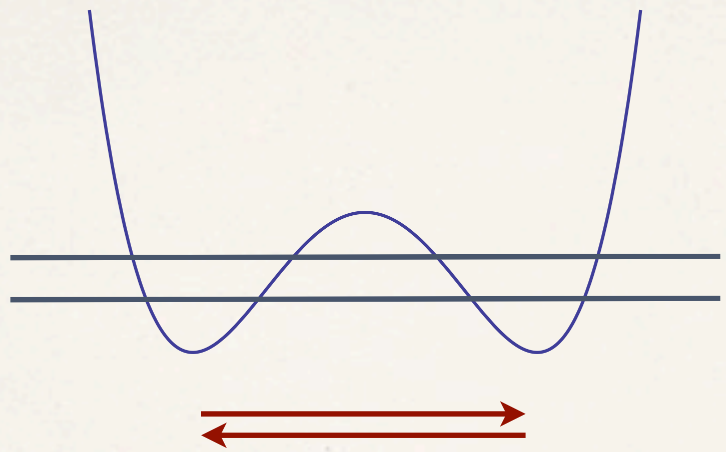


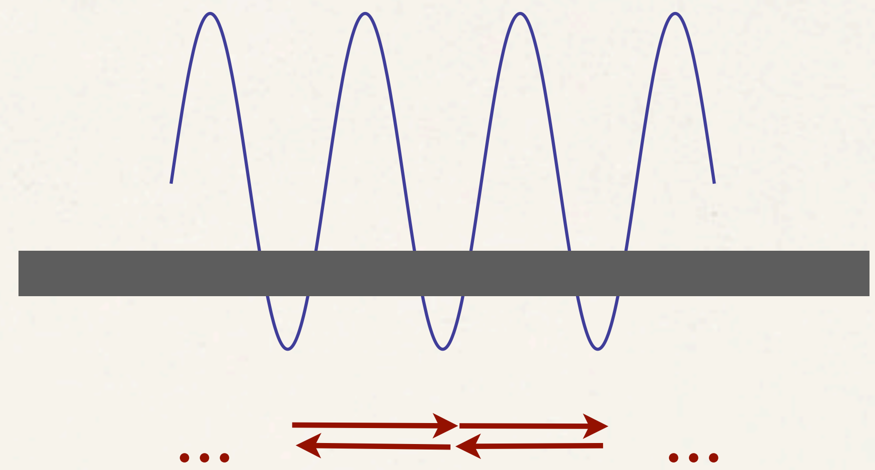
non-Borel-summable $\quad$ positive Borel pole $\quad$ ambig. imag. NP term

$$\sum_n n!\left(\frac{g^2}{c}\right)^n \quad\Leftrightarrow\quad t=\frac{c}{g^2} \quad\Leftrightarrow\quad \pm\frac{i\pi\,c}{g^2}\,e^{-c/g^2}$$

resolution in QM: Bogomolny / Zinn-Justin (BZJ) mechanism

degenerate classical vacua: double-well or Sine-Gordon





perturbation theory is non-Borel-summable:

(i) ambiguous imaginary contrib. to real energy

(ii) $\sim \pm i\, e^{-2 S_{\text{instanton}}}$

BZJ idea: non-perturbative sector: $\mathcal{I}\bar{\mathcal{I}}$ attractive

rotate $g^2 \to -g^2$; interaction repulsive; rotate back again

ambiguous imaginary non-perturbative contribution $\mp i\, e^{-2 S_{\text{instanton}}}$
which exactly cancels the term from perturbation theory

Bogomolny, 1980; Zinn-Justin, 1981; Balitsky / Yung 1986

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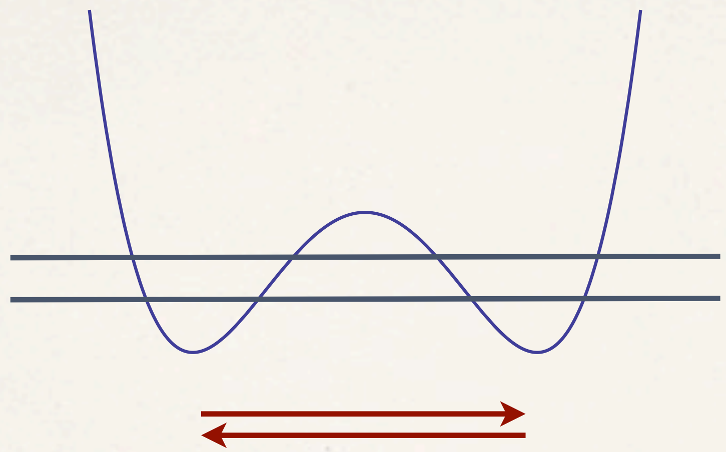

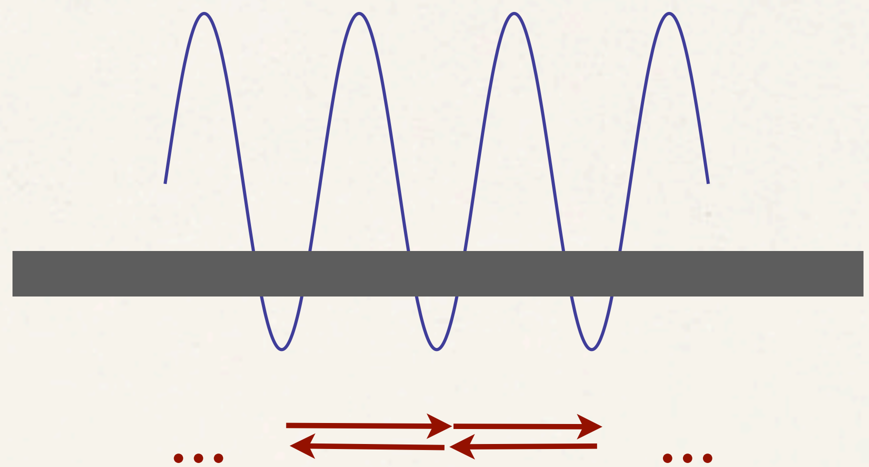

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``*resurgence*''

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analogous problem in asymptotically free QFT

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- asymptotically free, instantons, chiral symmetry breaking, ...

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$$\pm i\, e^{-2\, S_{\rm instanton}/\beta_0} \qquad\qquad \beta_0 \sim N$$

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cannot cancel!

perturbation theory remains incomplete and inconsistent

't Hooft, 1980; David, 1981

resolution: correct BZJ mechanism for SU(N) YM or $\mathbb{CP}^{N-1}$

Argyres/Ünsal, GD/Ünsal, 2012

- instanton gas picture has another problem: instanton scale moduli
- regulate with compactification: instantons fractionalize
- temporal compactification: information only about deconfined phase

$\mathbb{R}^1 \times S_\beta^1$

$\mathbb{R}^1$ $\longrightarrow$ $\mathbb{R}^2$

high T   low T

- spatial compactification: semiclassical (small L) continuously connected to large L: ``principle of continuity''

$S_L^1 \times \mathbb{R}^1$

$\mathbb{R}^1$ $\longrightarrow$ $\mathbb{R}^2$

"continuity"

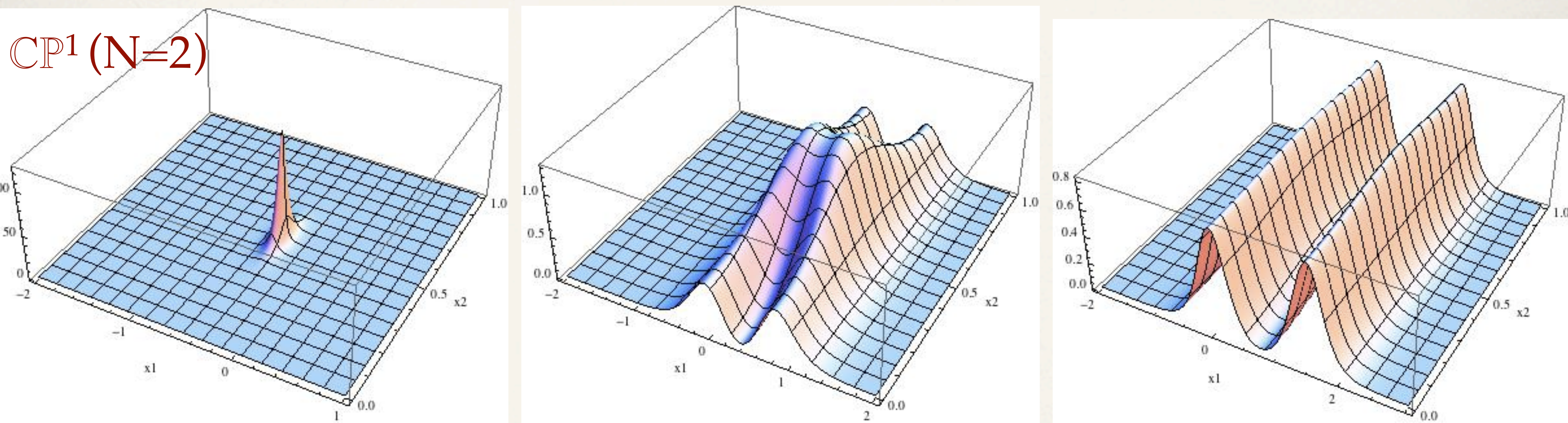
SUSY (Witten); non-SUSY (Ünsal, Yaffe, Poppitz, Shifman, Argyres, Schaefer, ...)

Fractionalized Instantons in the $\mathbb{CP}^{N-1}$ Model on $S_L^1 \times \mathbb{R}^1$

$\mathbb{Z}_N$ twisted boundary conditions: $v(x_1, x_2 + L) = \Omega_N \, v(x_1, x_2)$

$\mathbb{CP}^1$ on $S_L^1 \times \mathbb{R}^1$:
$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2 \, e^{-\frac{2\pi}{L} z}\right) \, e^{\frac{2\pi}{L} \mu_2 \, z} \end{pmatrix}$$

(twist in x_2) + (holomorphicity) $\Rightarrow$ fractionalization in x_1 direction



$\mathbb{CP}^{N-1}$: Q=1 instanton splits into N distinct Q=1/N "kink-instantons"

- technically analogous to 3d monopole constituents of 4d calorons

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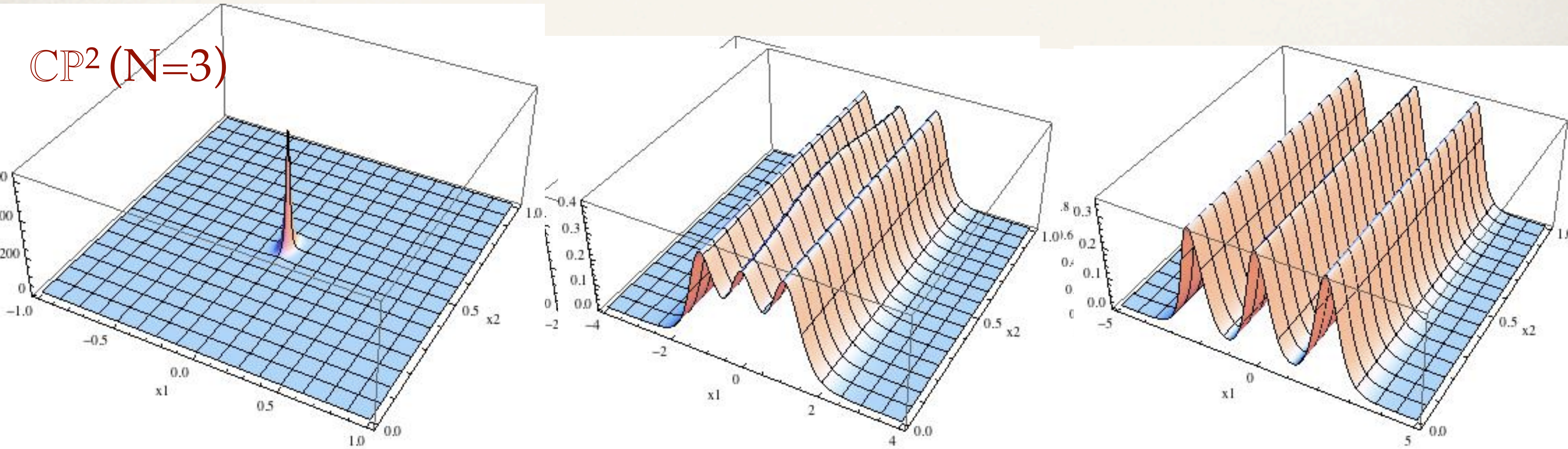
$\mathbb{CP}^2$ (N=3)



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Resurgence in 2d asymptotically free QFT: $\mathbb{CP}^{N-1}$

GD, Ünsal: 1210.2423, 1210.3646

perturbative sector: Borel-Écalle summation

$$B_\pm \, \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt \; B\mathcal{E}(t) \; e^{-t/g^2} = \mathrm{Re} B \, \mathcal{E}(g^2) \mp i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

non-perturbative sector: bion-bion amplitudes

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_\pm = \left(\ln\left(\frac{g^2 \, N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

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cancel

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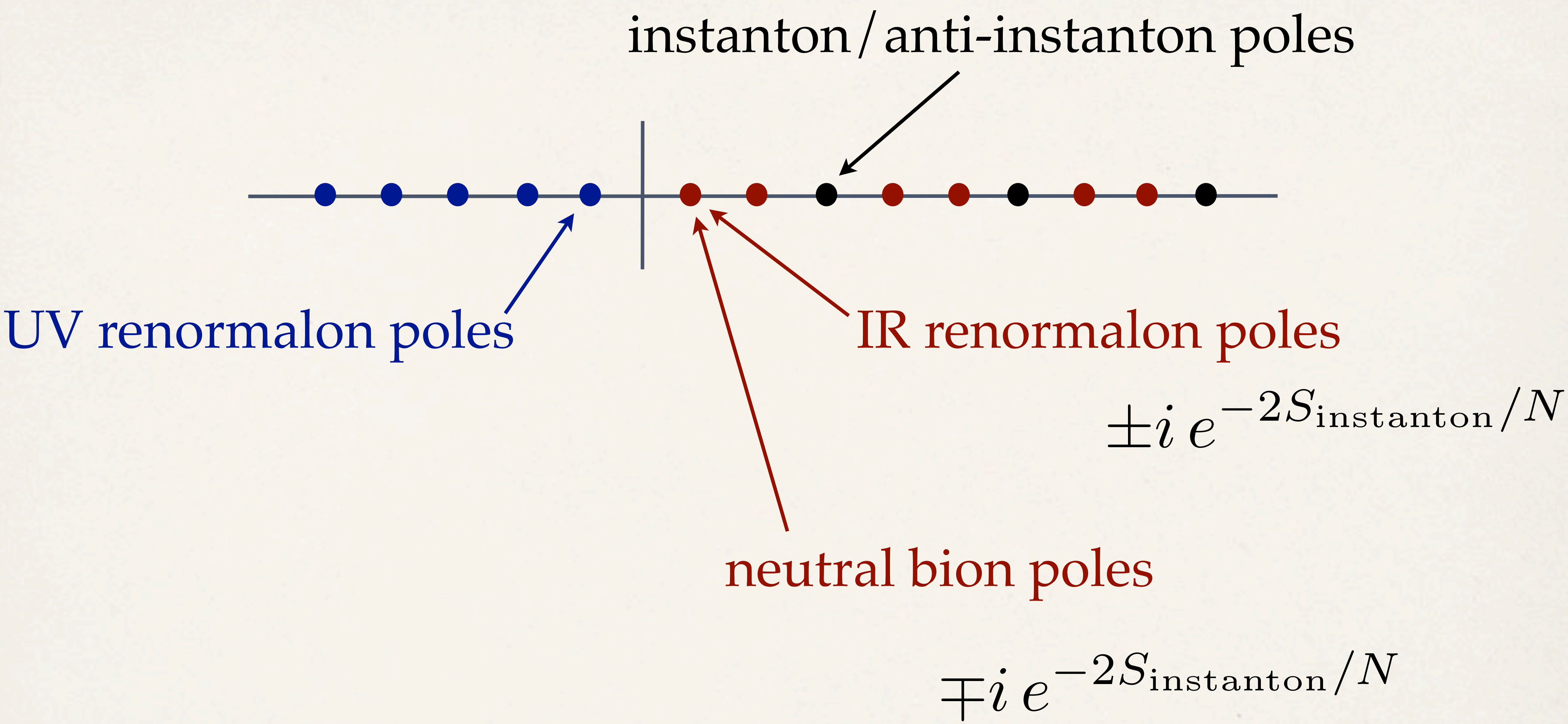
resolution: correct BZJ mechanism for $\mathbb{CP}^{N-1}$

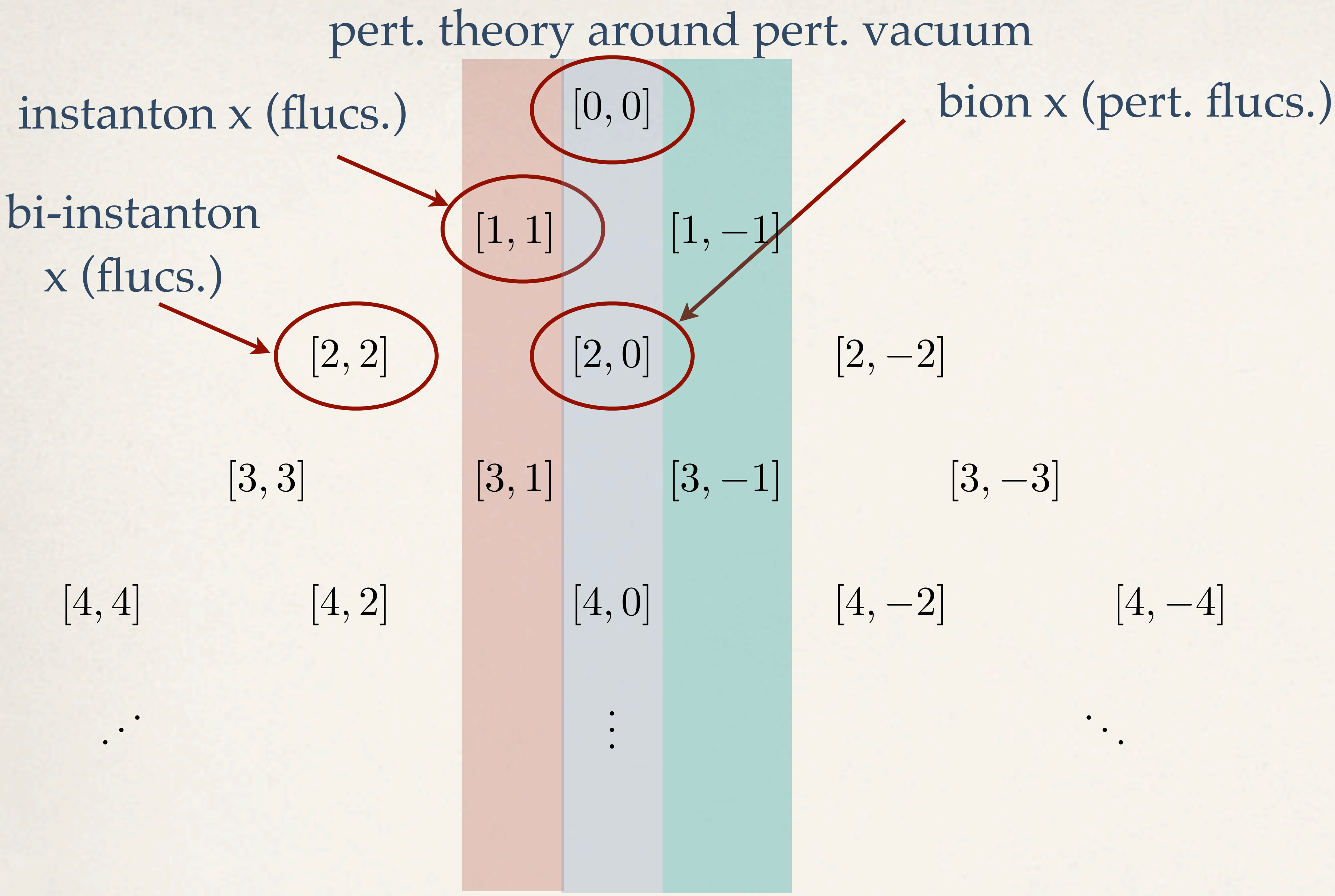


- spatially compactified $\mathbb{CP}^{N-1}$ generates fractionalized instantons and bions, cancelling perturbative IR renormalon ambiguities against non-perturbative ambiguities in instanton/bion gas picture

GD, Ünsal: 1210.2423, 1210.3646

graded resurgence triangle and extended SUSY

GD, Ünsal: 1210.2423, 1210.3646



sectors with different Θ dependence cannot mix or cancel

graded resurgence triangle and extended SUSY

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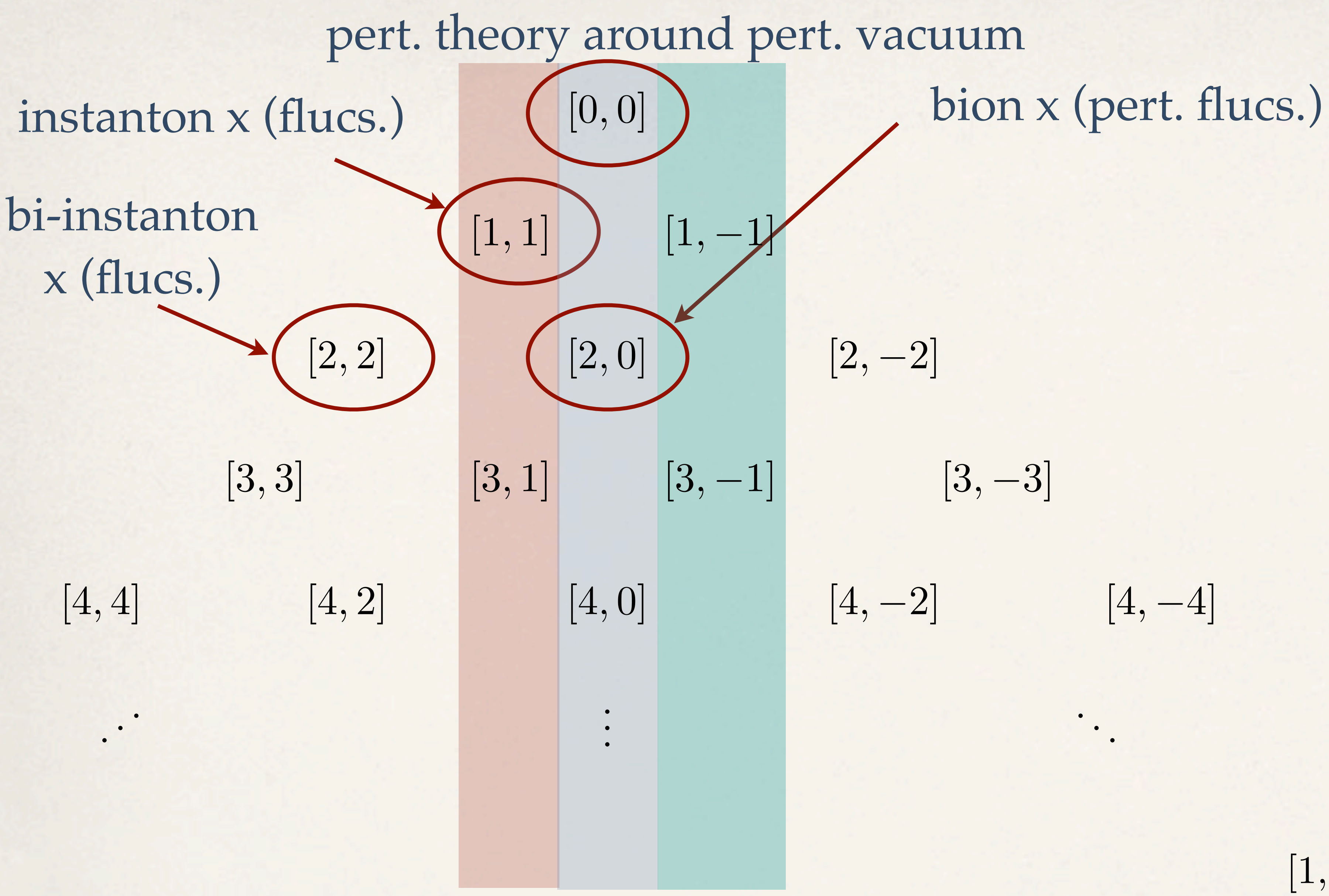

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extended SUSY: no superpotential; no bions; no condensates:

$$\begin{array}{ccccccccc} & & & & [0,0] & & & & \\ & & & [1,1] & & [1,-1] & & & \\ & & [2,2] & & \varnothing & & [2,-2] & & \\ & [3,3] & & \varnothing & & \varnothing & & [3,-3] & \\ [4,4] & & \varnothing & & \varnothing & & \varnothing & & [4,-4] \\ \iddots & & & & \vdots & & & & \ddots \end{array}$$

perturbative expansions must be Borel summable

graded resurgence triangle and extended SUSY

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pert. theory around pert. vacuum

instanton x (flucs.)

bi-instanton x (flucs.)

bion x (pert. flucs.)

$$
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perturbative expansions must be Borel summable

cf. Russo, 2012

the bigger picture: resurgence and non-perturbative QFT

"*resurgence*" unifies perturbative and non-perturbative sectors in such a way that the combination is unambiguous and well-defined under analytic continuation of the expansion parameter

Resurgence and Trans-Series

$$f(g^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$

J. Écalle (1980): set of functions with these trans-monomial elements is closed; "any reasonable function" has a trans-series expansion

(Borel transform) + (analytic continuation) + (Laplace transform)

- exponentially improved asymptotic expansions (dlmf.nist.gov)
- philosophical shift:

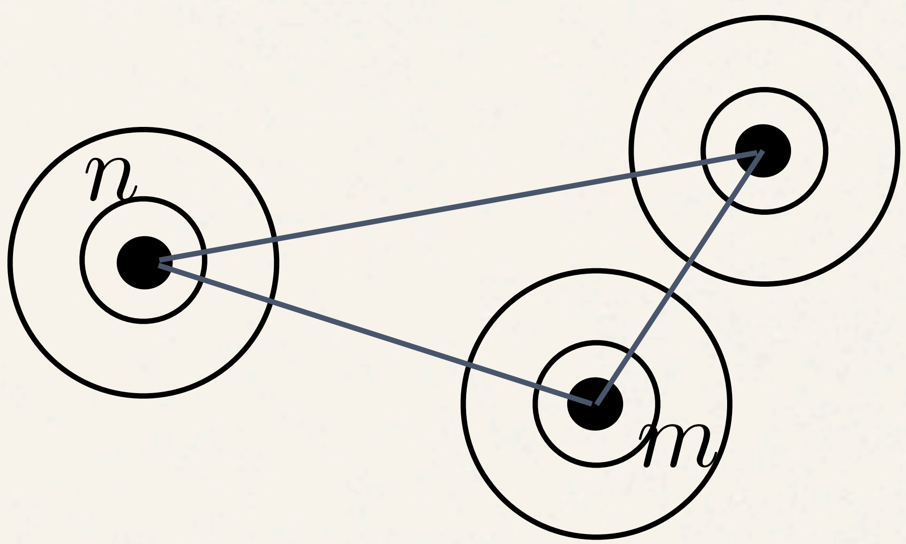
view semi-classical asymptotic expansions as `exact encoding' of the function

dramatic consequence: expansion coefficients extremely constrained (cf. BZJ cancellation mechanism)

What is Resurgence?

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the "origin". Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

Écalle, 1980



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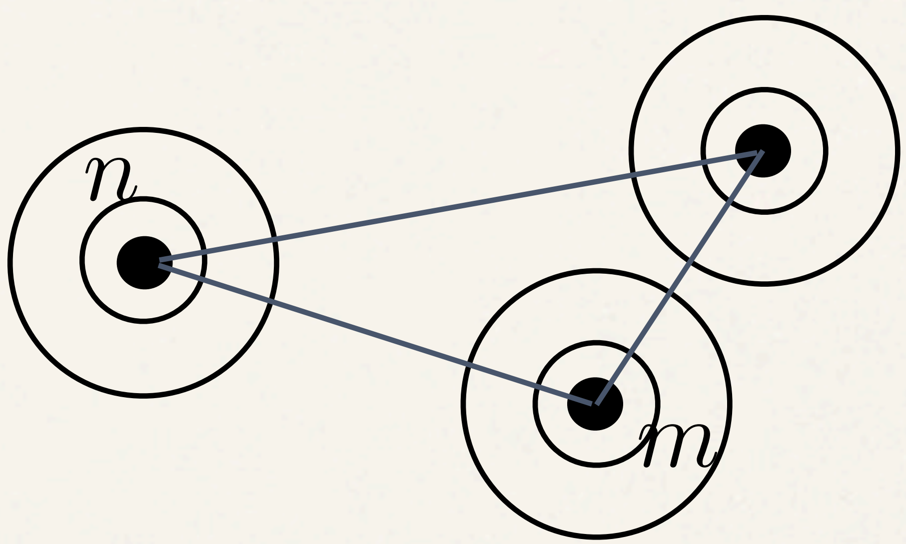

`in principle', we can reconstruct the full function from the perturbative series

cf. GD, Ünsal: 1306.4405

Full Trans-Series from Perturbation Theory

$$E^{(N)}(g^2) = E^{(N)}_{\text{pert. theory}}(g^2) + \sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\sum_{p=0}^{\infty} c_{k,l,p}\left(\frac{1}{g^{2N+1}}\,\exp\left[-\frac{c}{g^2}\right]\right)^k \left(\ln\left[-\frac{1}{g^2}\right]\right)^l g^{2p}$$

vacuum saddle: $E^{(N)}_{\text{pert. theory}}(g^2) = 1 + a_1^{(N)} g^2 + a_2^{(N)} g^4 + \dots$

1-instanton saddle: $\Delta E^{(N)}_{1\,\text{instanton}}(g^2) = \frac{1}{N!\,g^{2N}}\frac{e^{-S_{\text{inst}}/g^2}}{\sqrt{\pi\, g^2}}\left(1 + b_1^{(N)} g^2 + b_2^{(N)} g^4 + \dots\right)$

state-of-the-art: entire trans-series encoded in these two series, and a (conjectured) exact quantization condition

Zinn-Justin/Jentschura, 2004

new results: (i) equivalent to uniform WKB boundary condition
(ii) instanton fluctuation follows immediately from perturbative expansion

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the perturbative expansion contains ALL information of the trans-series

Resurgence prototype: Gamma function and Stirling's Formula

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

leading (Stirling)

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leading (Stirling) (divergent!) correction

functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot \pi z$

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reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot \pi z$

`perturbative' asymptotic expansion is incompatible with reflection formula

Resurgence prototype: Gamma function and Stirling's Formula

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

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non-perturbative terms generated from a resurgent analysis of the perturbative asymptotic expansion

unlike the ``perturbative'' asymptotic series, a resurgent trans-series expansion is fully compatible with global analyticity properties

precisely this gamma function example appears in many QFT & string computations

- Euler-Heisenberg effective actions
- de Sitter/AdS effective actions
- exact S-matrices
- Chern-Simons partition functions
- matrix models
- Painlevé
- ...

Gopakumar/Vafa, 1998, 1999;
Das/Dunne, 2006;
Mariño/Schiappa/Weiss, 2007, 2008;
Mariño 2012;
Aniceto/Schiappa/Vonk/Vaz, 2010, 2011;
Garoufalidis/Its/Kapaev/Mariño, 2012; ...

for QFT we should understand resurgence using <u>path integrals</u>

Resurgence in d=0 Path Integrals: Steepest Descents

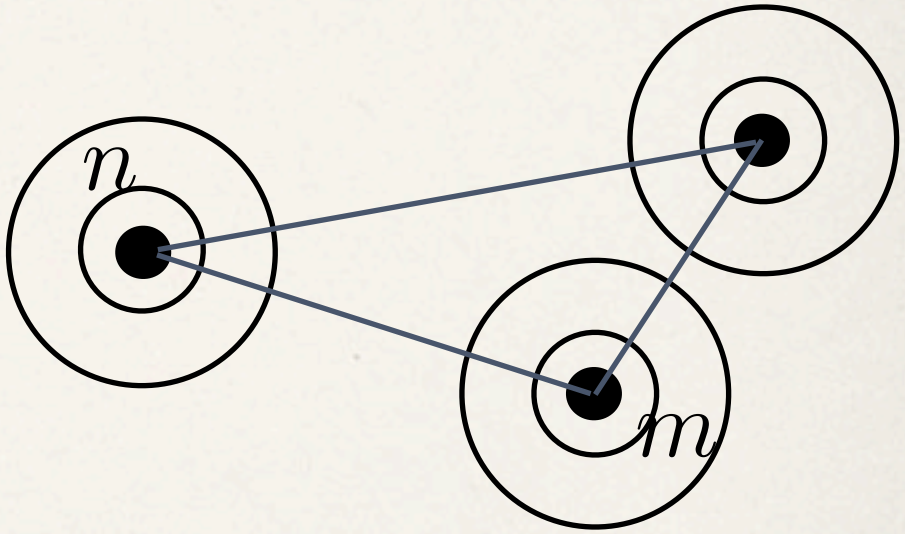
Berry/Howls, 1991 (``hyperasymptotics'')

resurgence in saddle-point integrals

$$I^{(n)}(g^2) = \int_{\Gamma_n} dz\, e^{-f(z)/g^2} \quad \Rightarrow \quad I^{(n)}(g^2) \sim \frac{1}{g}\, e^{-f_n/g^2}\, T^{(n)}(g^2)$$

$$T^{(n)}(g^2) = \sum_{r=0}^{\infty} g^{2r}\, T_r^{(n)}$$

Darboux's theorem: large orders of expansion around one critical point governed by nhd. of nearest singularity = other critical point



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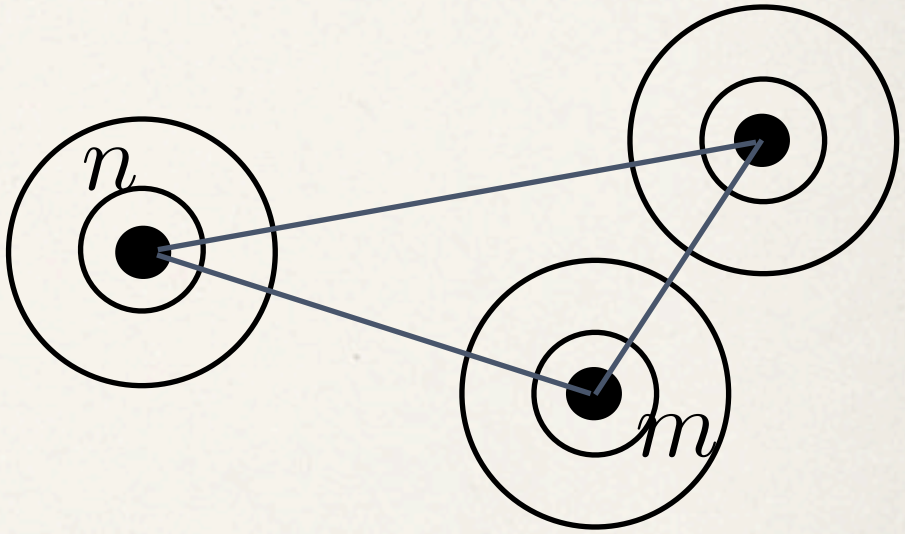


Resurgence: large orders of fluctuations around an instanton governed by low orders about "nearby" instanton(s)

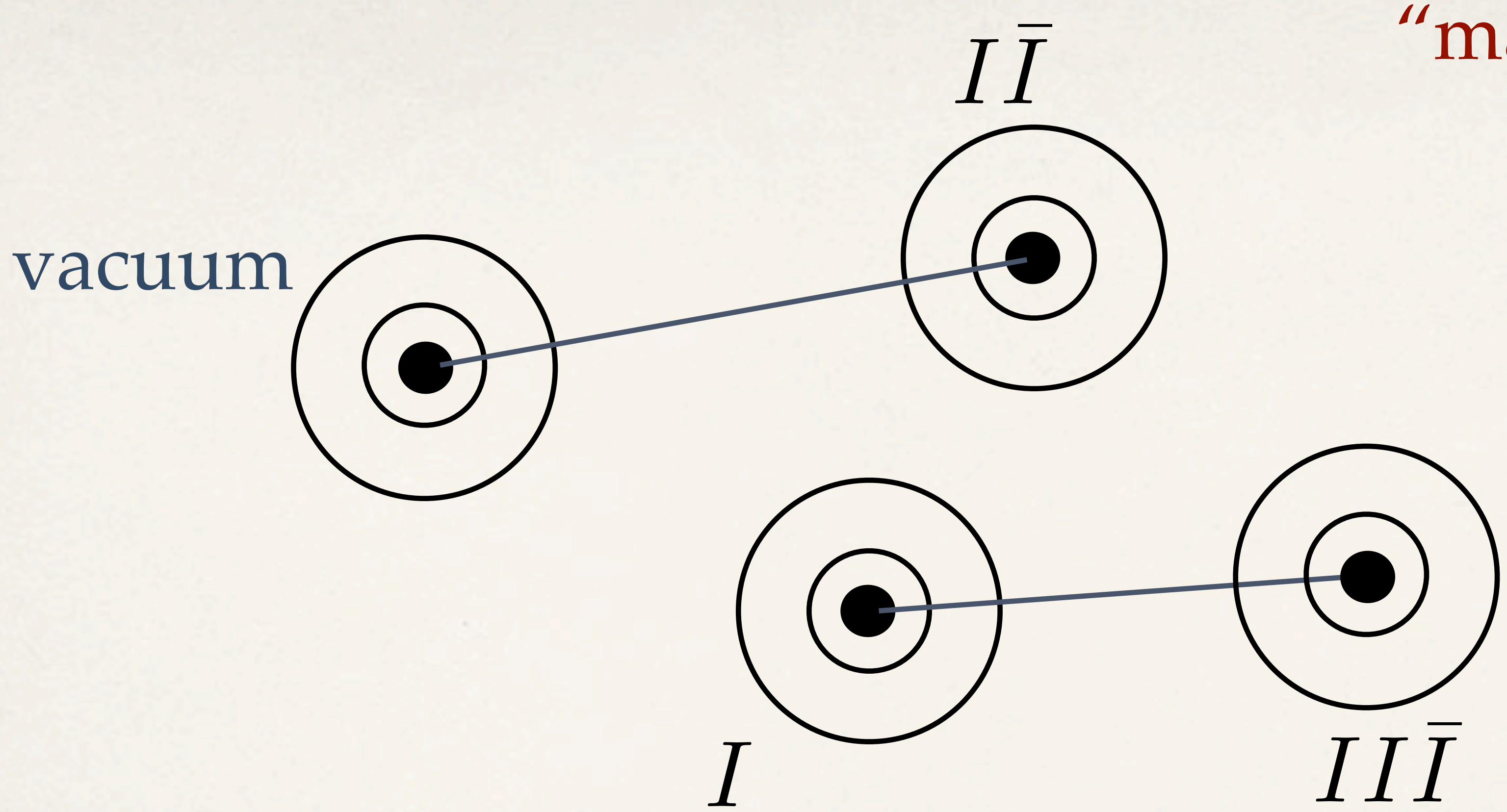
$$T_r^{(n)} \sim \frac{1}{2\pi i} \sum_m (-1)^{\gamma_{mn}} \frac{(r-1)!}{|f_n - f_m|^r} \left(T_0^{(m)} + \frac{|f_n - f_m|}{r-1} T_1^{(m)} + \dots \right)$$

resurgent trans-series structure is a basic property of all-orders saddle-point expansions of <u>ordinary integrals</u>

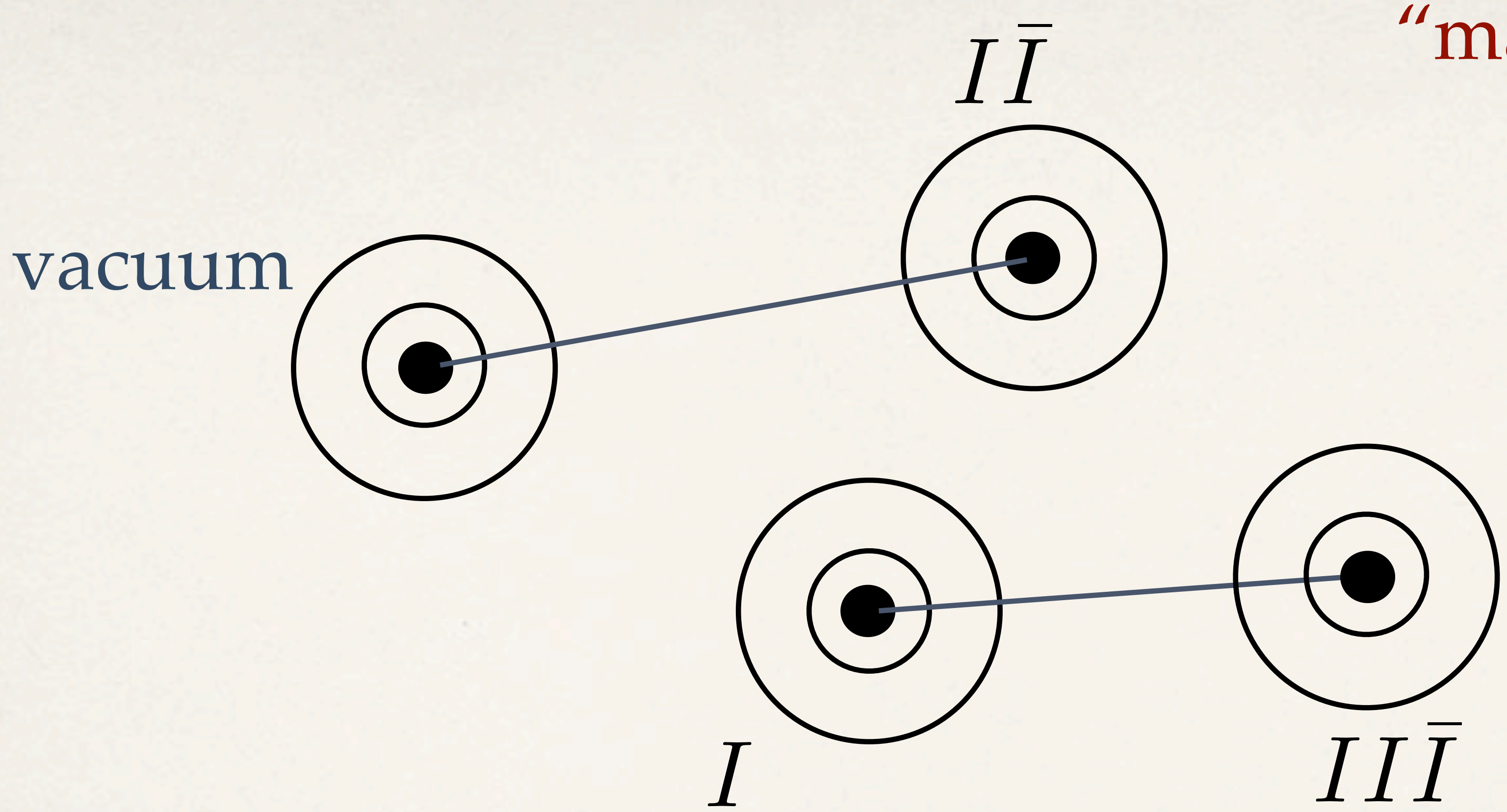
deeply embedded in perturbation theory and semi-classical analysis in QM and QFT, but its origin is (presumably) very basic

QFT: "functional Darboux theorem"

“map” of all saddle points



network of correspondences within trans-series lead to the “required” cancellations “map” of all saddle points



network of correspondences within trans-series lead to the “required” cancellations

$$(1 + a_1 g^2 + a_2 g^4 + \cdots + \ldots) \; + e^{-S/g^2} (1 + b_1 g^2 + b_2 g^4 + \cdots + \ldots)$$

$$+ e^{-2S/g^2} (1 + c_1 g^2 + c_2 g^4 + \cdots + \ldots) + e^{-3S/g^2} (1 + d_1 g^2 + d_2 g^4 + \cdots + \ldots)$$

$$+ \ldots \qquad + (\text{log terms})$$

resurgence in zero dimensional QFT

$$I^{(n)}(g^2) = \int_{\Gamma_n} dz\, e^{-f(z)/g^2} \qquad \Rightarrow \qquad I^{(n)}(g^2) \sim \frac{1}{g}\, e^{-f_n/g^2}\, T^{(n)}(g^2)$$

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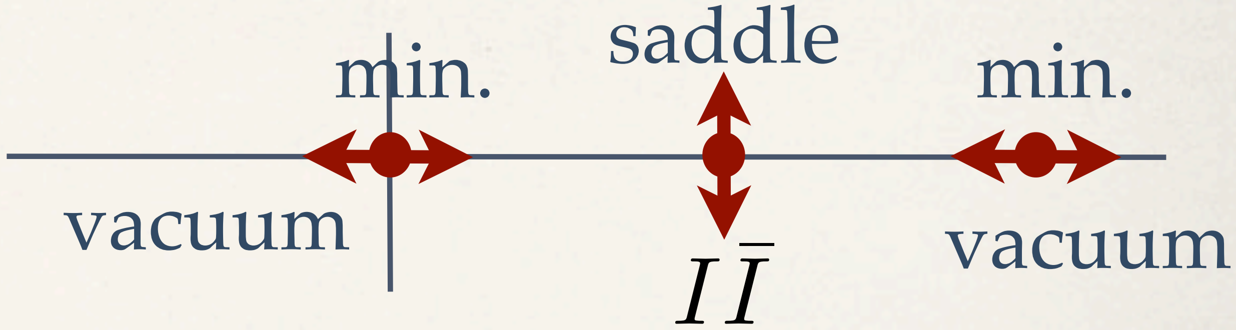

$$T^{(0)} = g \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)}\, g^{2n}$$

$$\Rightarrow \quad c_n \sim (n-1)! - \frac{1}{4}(n-2)! + \frac{9}{32}(n-3)! - \frac{75}{128}(n-4)! + \dots$$

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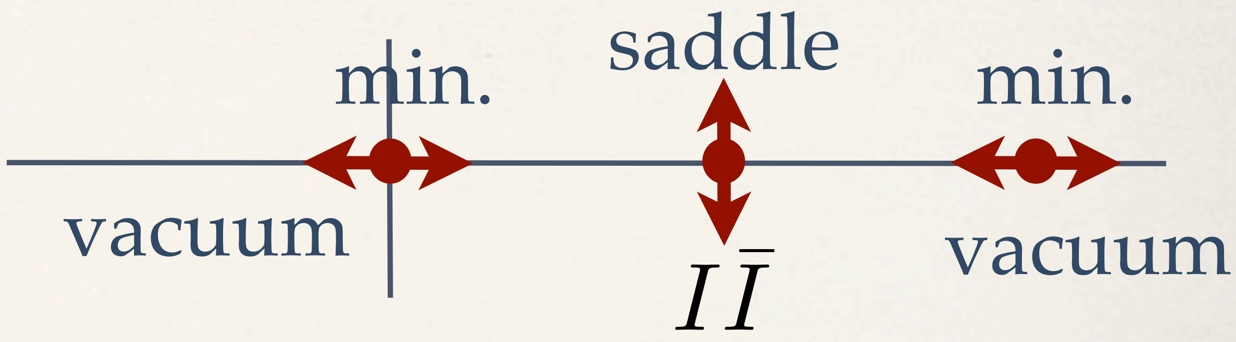

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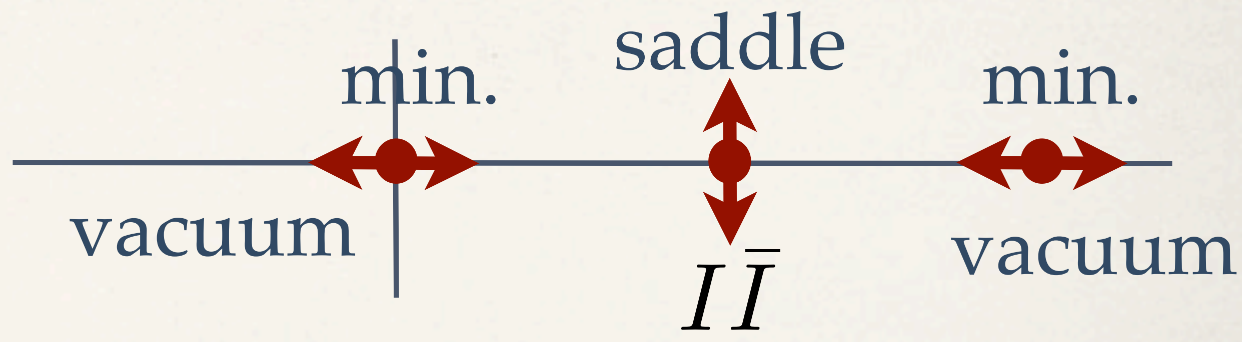

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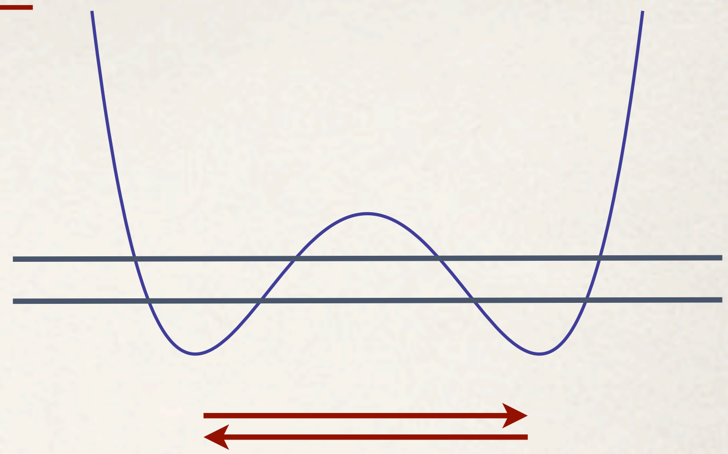
low orders in fluctuations around “$I\bar{I}$” saddle determine large-order behavior of fluctuations around the “vacuum”

cf. Darboux’s theorem

resurgence in one dimensional QFT : QM



$$V(x) = x^2(1+g\,x)^2$$

$$E = \sum_n c_n g^{2n}$$

pert. theory: $$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

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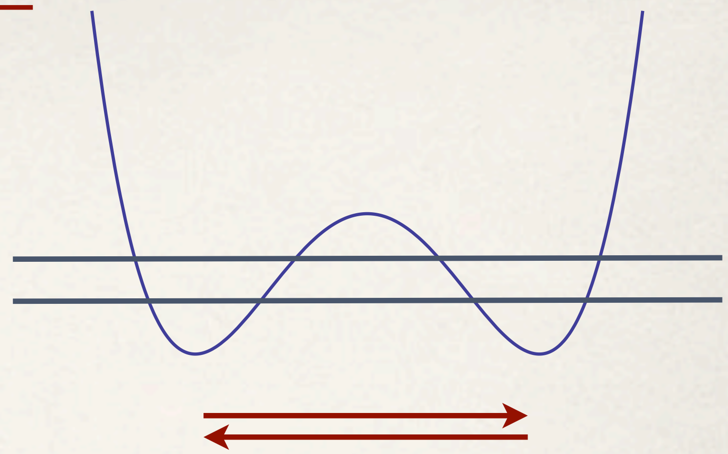

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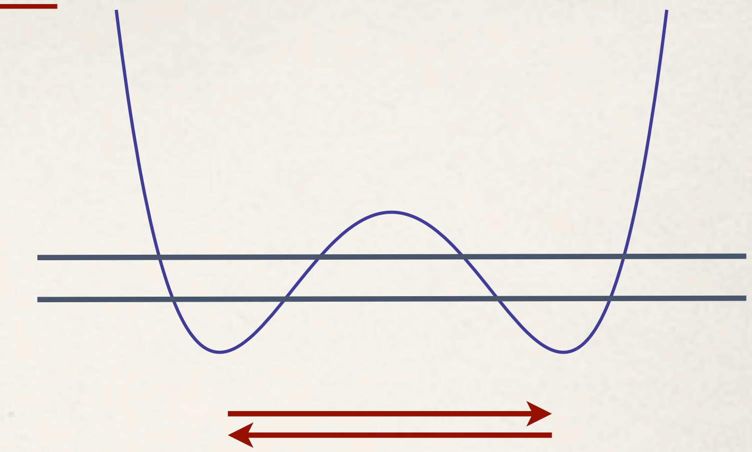

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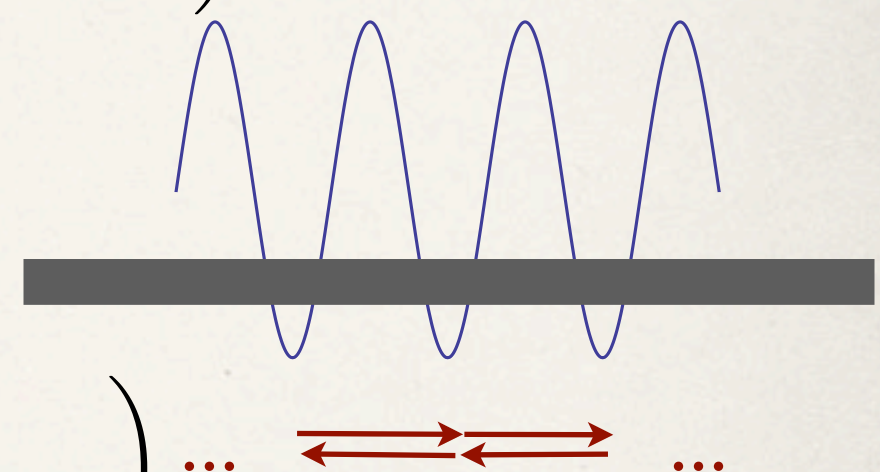

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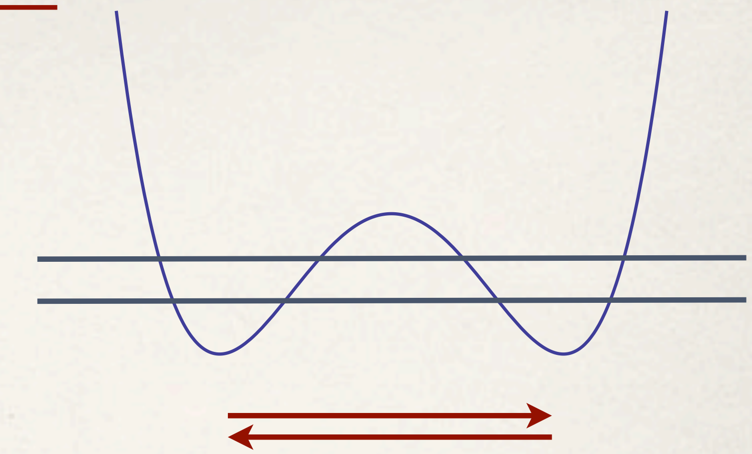

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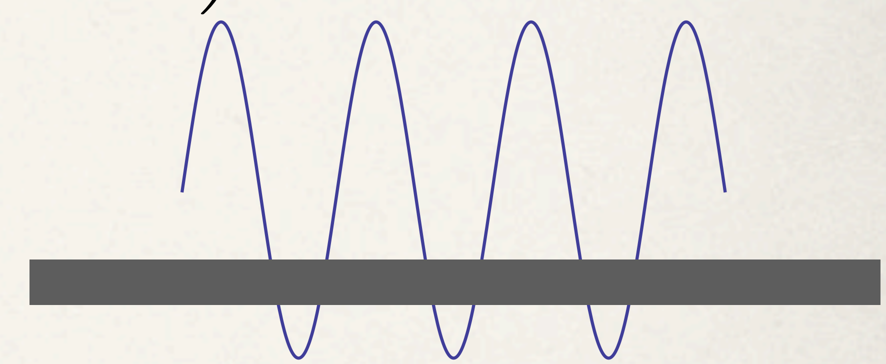

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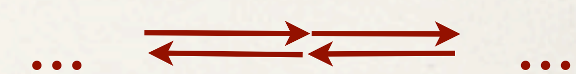

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flucs. around $I\bar{I}$ saddle determine large-order of flucs. around vacuum

Resurgence in 2d asymptotically free QFT: $\mathbb{CP}^{N-1}$

GD, Ünsal: 1210.2423, 1210.3646

perturbative sector: Borel-Écalle summation

$$B_\pm \,\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\; B\mathcal{E}(t)\; e^{-t/g^2} = \mathrm{Re} B\,\mathcal{E}(g^2) \boxed{\mp\, i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}}$$

cancel

non-perturbative sector: bion-bion amplitudes

$$[\mathcal{K}_i\bar{\mathcal{K}}_i]_\pm = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \boxed{\pm\, i\pi\frac{16}{g^2 N}e^{-\frac{8\pi}{g^2 N}}}$$

Analytic Continuation of Path Integrals: Ghost instantons

$$V(x) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,x; m)$$

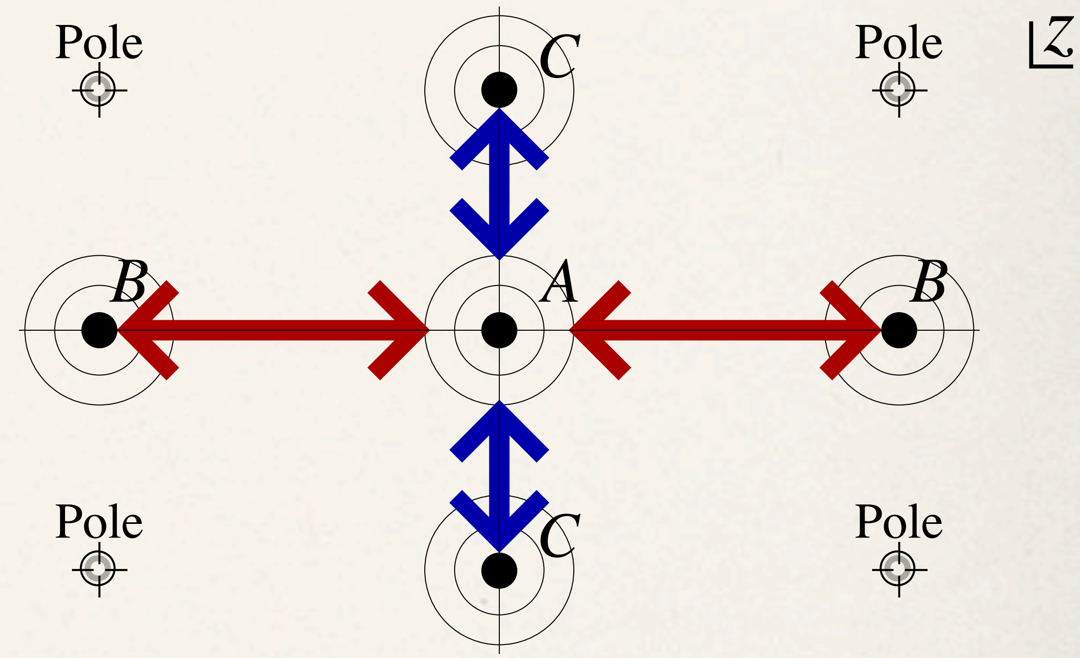
Başar, GD, Ünsal: 1308.1108

path integral : sum over real paths $\displaystyle \int \mathcal{D}x\, e^{-\frac{1}{g^2} S[x]}$

- periodic potential with both real and complex instantons

$$S_I = \frac{2\arcsin(\sqrt{m})}{\sqrt{m(1-m)}}$$

$$S_G = \frac{2\arcsin(\sqrt{1-m})}{\sqrt{m(1-m)}}$$



$$a_n(m) \sim -\frac{16}{\pi}\, n! \left(\frac{1}{(S_{I\bar{I}}(m))^{n+1}} - \frac{(-1)^{n+1}}{(S_{G\bar{G}}(m))^{n+1}} \right)$$

both real and complex instantons contribute to physical properties

Beyond instantons

Dabrowski, GD: 1306.0921;
Cherman, Dorigoni, GD & Ünsal: 1308.0127

Daniele Dorigoni:
Wed. 17:10

- YM and CPN have unstable non-self-dual classical solutions, with finite Euclidean action
- 2d O(N), PCM, ..., have no instantons, but still have IR renormalon problems in the perturbative sector
- these theories also have unstable non-BPS classical solutions, with finite Euclidean action

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

small g^2 : dominated by critical points;
i.e., finite action solutions of classical Euclidean equations

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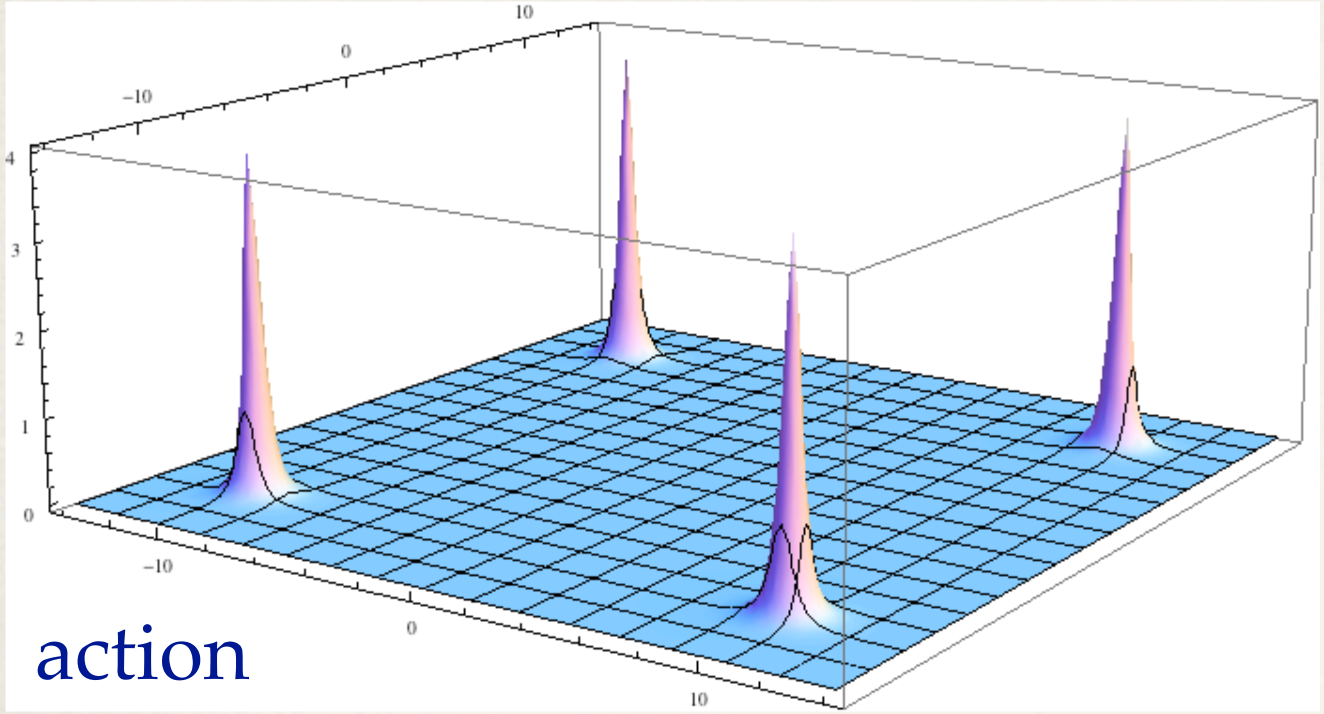
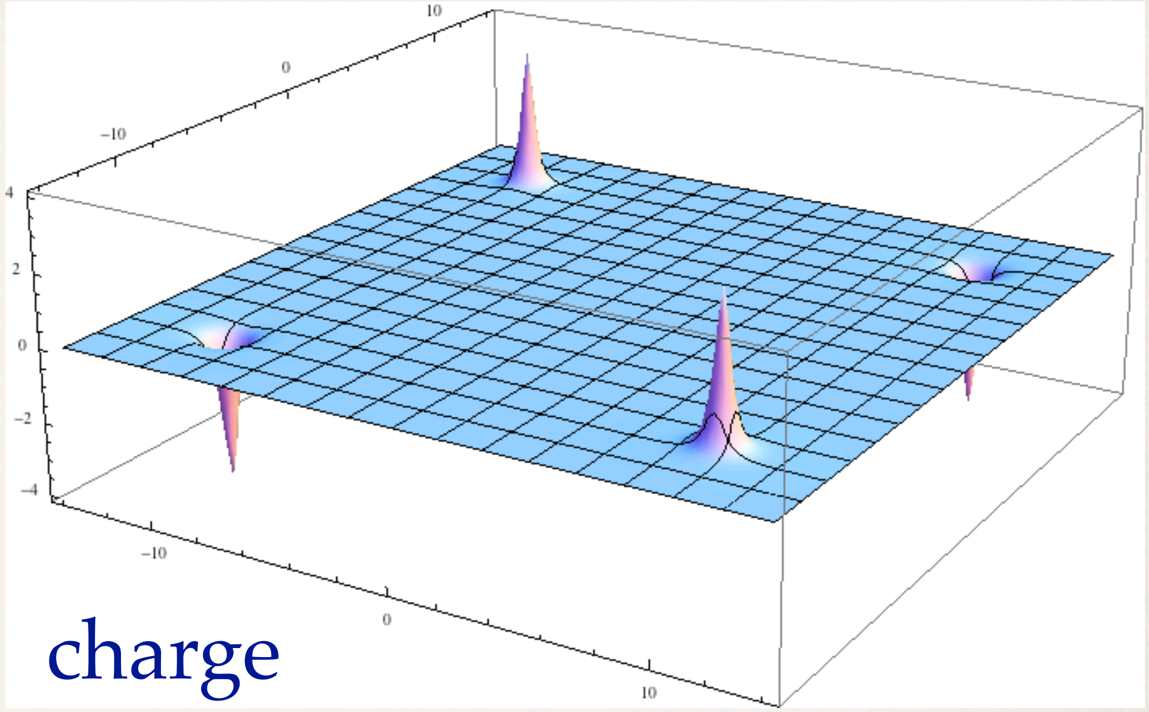
small g^2 : dominated by critical points; i.e., finite action solutions of classical Euclidean equations

- minima: BPS instantons (self-dual) $\Rightarrow \sim e^{-\frac{1}{g^2}S_{\text{inst}}}$
- saddle points: non-self-dual but finite action
  - $\mathbb{CP}^{N-1}$ (Din & Zakrzewski); YM (Sibner, Sibner & Uhlenbeck; Sadun, ...)
  - "unstable": instantons & anti-instantons
  - physics: resurgent non-perturbative contributions

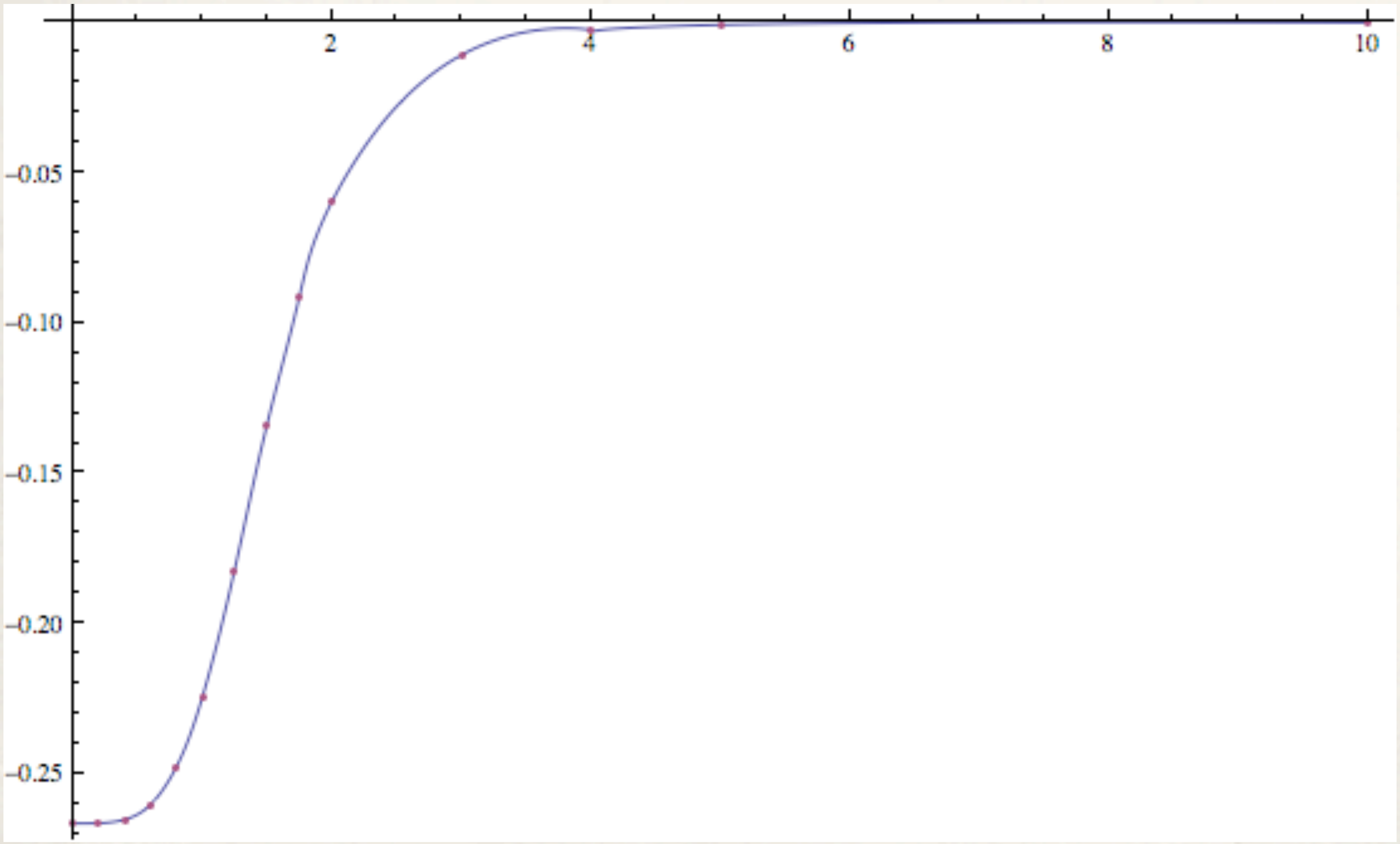
Dabrowski, GD: 1306.0921

"instability" of non-self-dual solutions





zero modes: 2 x 6 = 12, but far-separated count: 4 x 6 = 24



"instability": (some) zero modes become negative modes

contribution to semiclassical expansion is complex!

resurgence: must cancel something from perturbation theory

Conclusions

moral: perturbative series expansions are typically divergent, and incompatible with global analytic continuation properties. Resurgence fixes this.

- non-Borel-summable perturbation theory is incomplete and inconsistent
- corresponding non-perturbative instanton gas picture is similarly incomplete and inconsistent
- together, a resurgent trans-series expansion is complete & consistent

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- non-perturbative physics without instantons
- analytic continuation of ODEs and path integrals
- effect of running couplings
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- practical computational tool ?

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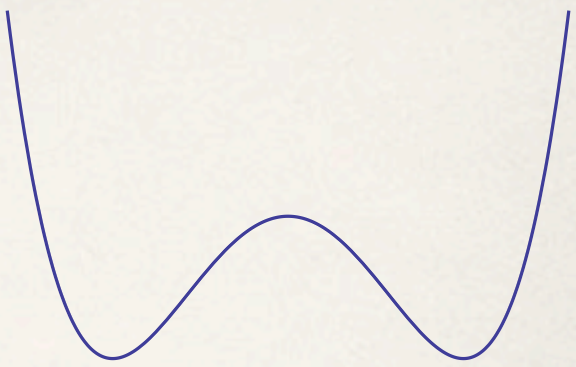
Daniele Dorigoni: Wed. 17:10

Uniform WKB, Resurgence and Trans-Series

GD, Ünsal, 2013

e.g. double-well potential: $V(x) = x^2(1 - g\,x)^2$



uniform WKB approximation: $\psi = \dfrac{D_\nu\left(\frac{1}{g}u(g\,x)\right)}{\sqrt{u'(g\,x)}}$

perturbation theory: $E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$ (non-Borel-summable)

$g = 0 \;\Rightarrow\; \nu = N$: usual perturbation theory for N^{th} oscillator level

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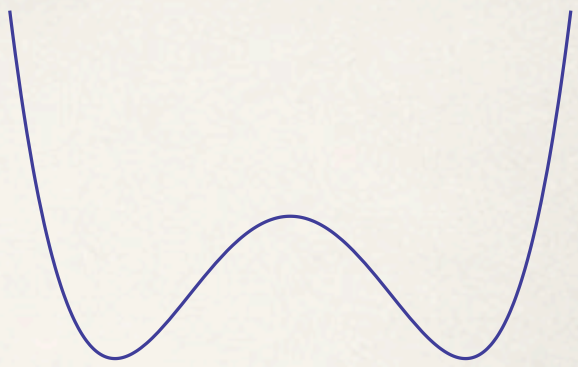

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g>0: Neumann or Dirichlet b.c. at barrier midpoint

non-Borel-summability: analytically continue $g \to g \pm i\epsilon$

need **complex** asymptotics of D_ν

Uniform WKB, Resurgence and Trans-Series

pert. theory (non-Borel-summable): $E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$

$$D_\nu(z) \sim z^\nu \, e^{-z^2/4} \, [1 + \dots] + e^{\pm i\pi\nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} \, e^{z^2/4} \, [1 + \dots] \qquad \frac{\pi}{2} < \pm \arg(z) < \pi$$

$$\boxed{\frac{1}{\Gamma(-\nu)} \left(\frac{e^{\pm i\pi} \, 2}{g^2} \right)^{-\nu} = \frac{e^{-S_{\rm instanton}/g^2}}{\sqrt{\pi g^2}} H_0(\nu, g^2)}$$

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determines ν as a function of g²: exponentially close to an integer N

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feed back into perturbative expansion ⇒ trans-series expression

$$E^{(N)}(g^2) = E^{(N)}_{\text{pert. theory}}(g^2) + \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} \sum_{p=0}^{\infty} c_{k,l,p} \left(\frac{1}{g^{2N+1}} \exp\left[-\frac{c}{g^2}\right]\right)^k \left(\ln\left[-\frac{1}{g^2}\right]\right)^l g^{2p}$$

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generic: property of parabolic cylinder functions