Lattice $\mathcal{N} = 4$ super Yang-Mills

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Barriers to Lattice Supersymmetry

- $\{Q, \overline{Q}\} = \gamma_\mu p_\mu$. No p_μ on lattice. Equivalently: no Leibniz rule for difference ops: $\Delta(AB) \neq \Delta AB + A\Delta B$.
- Classical SUSY breaking leads to (many) SUSY violating ops via quantum corrections. Couplings must be adjusted with cut-off $(1/a)$ to achieve SUSY in continuum limit -fine tuning.
- Discretization of Dirac equation: Lattice theories contain additional fermions (doublers) which do not decouple in continuum limit. Consequence: no. fermions $\neq$ no. bosons
- Lattice gauge fields live on lattice links and take values in group. Fermions live on lattice sites and (for adjoint fields) live in algebra ....

Putting SUSY on a lattice

Goals of any successful SUSY lattice formulation:

- Reduce/eliminate fine tuning. In particular scalar masses.
- Keep exact gauge invariance. Lesson of lattice QCD (Wilson)
- Avoid fermion doubling...
- More symmetrical treatment of bosons and fermions - particularly for gauge theories.

New formulations exist with all these features

New ideas - twisting

- Rewrite continuum theory in twisted variables.
- Exposes a single scalar supersymmetry $\mathcal{Q}$ whose algebra is simple: $\mathcal{Q}^2=0$. Furthermore $S\sim\mathcal{Q}\Lambda(\Phi)$.
- Key: this SUSY can be retained on discretization: easy to build invariant lattice action.
- Fine tuning reduced (eliminated ?):

$$\textbf{Exact hypercubic symmetry} \stackrel{a\to 0}{\to} \textbf{Full Poincare invariance}$$
$$\textbf{Exact } \mathcal{Q} \textbf{ symmetry} \to \textbf{Full SUSY ?}$$

- See that all fields will live on links and take values in algebra.
- Structure of fermionic action dictated by exact SUSY - would doublers will be physical

Most interesting application: $\mathcal{N}=4$ SYM

Many lattice SUSY theories in $D<4$.
However in $D=4$ they single out a unique theory: $\mathcal{N}=4$ YM

- Fascinating QFT - finite but non-trivial. A lattice formulation gives a non-perturbative definition of theory (like lattice QCD for QCD)
- Heart of AdS/CFT correspondence.
  String theory in AdS_5 and $\mathcal{N}=4$ SYM on boundary
- Lattice formulation allows us to verify and extend holographic ideas: compute classical and quantum string corrections ... (expansion in $1/N$ and $1/\lambda$)

Twisted fields of $\mathcal{N} = 4$ YM

Basic idea:
Decompose all fields on twisted (Eucl.) Lorentz group

$$SO'(4) = \mathrm{diag}\left(SO_R(4) \times SO_{\mathrm{Lorentz}}(4)\right)$$

- Spinors: $\psi^i_\alpha \rightarrow \Psi$: 4×4 matrix: expand on products of γ matrices yields (integer spin) twisted fermions $(\eta, \psi_\mu, \chi_{\mu\nu}, \ldots)$
- Bosons: Gauge fields unchanged. Scalars become vectors. Combine with A_μ to form complex fields $\mathcal{A}_\mu$

Twisted action

Can write $\mathcal{N}=4$ continuum action in twisted form.

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{ab}\mathcal{F}_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d\right) + S_{\mathrm{closed}}$$

with

$$\mathcal{D}_a = \partial_a + \mathcal{A}_a,\ \overline{\mathcal{D}}_a = \partial_a + \overline{\mathcal{A}}_a,\ a = 1\ldots 5$$

and

$$S_{\mathrm{closed}} = -\frac{1}{4g^2}\int \epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de}$$

Most compact form for 4D action - dim. reduction of 5D theory

Scalar supersymmetry

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_a &= \psi_a \\
\mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: $\mathcal{Q}^2 = 0$ as promised...
$\mathcal{Q}S_{\mathrm{closed}} = 0$ by Bianchi

Twisted Lattice theory

- Assign all lattice fields to oriented links. Gauge transform like eg.$\psi_a \to G(x)\psi_a(x)G^\dagger(x+a)$
  ($\chi_{ab}(x)$ lives on link $x+a+b \to x$)
- Lattice is unique: 5 (complex) gauge fields $\to$ lattice with (equal) 5 basis vectors with $\sum_{a=1}^{5} \mathbf{e}^a = 0$. A_4^* lattice
- All fields take values in $U(N)$ algebra.
- Well defined prescription for gauge covariant lattice difference operators...

Derivatives

- Precise dictionary exists to translate $\mathcal{D}_\mu$ to gauge covariant difference ops. Eg

$$a\mathcal{D}_a^+\psi_b(x) = \mathcal{U}_a(x)\psi_b(x+a) - \psi_b(x)\mathcal{U}_a(x+b)$$

New field lives on link $(x \to x + a + b)$

- Naive continuum limit : $\mathcal{U}_a(x) = I + a\mathcal{A}_a(x) + \ldots$:

$$\mathcal{D}_a^+\psi_b(x) \to \frac{1}{a}\left(\psi_b(x+a) - \psi_b(x)\right) + [\mathcal{A}_a, \psi_b] + \mathcal{O}(a)$$

Gauge invariance, doublers and all that

- All terms in action local, correspond to closed loops and hence are lattice gauge invariant
- $\mathcal{U}_a$'s non compact! $\mathcal{U}_a = \sum_B T^B \mathcal{U}_a^B$ – flat measure $\int \prod D\mathcal{U}_a D\overline{\mathcal{U}}_a$. Nevertheless, still gauge invariant - Jacobians resulting from gauge transformation of $\mathcal{U}$ and $\overline{\mathcal{U}}$ cancel.
- Bosonic action (trivially) has no doublers. Exact SUSY ensures no fermion doublers ... (fermion action has Kähler-Dirac form)

Bigger question: how to generate correct naive continuum limit ?

Naive continuum limit - fixing the vev of the $U(1)$ scalar

- Need $\mathcal{U}_a = I + a\mathcal{A}_a(x) + \ldots$. Here, unlike lattice QCD, unit matrix necessary for generating kinetic terms arises from the vev of a dynamical field! $\boxed{\mathrm{Im}\mathcal{A}_a^0 = 1}$
- Ensure by adding gauge invariant potential

$$\delta S = \mu^2 \sum_{x,a} \left(\frac{1}{N} \mathrm{Tr}\,(\mathcal{U}_a^\dagger \mathcal{U}_a) - 1 \right)^2$$

Polar decomposition: $\mathcal{U}_a = (I + h_a)u_a$. Generates potential for $h_a^0 \sim \mathrm{Im}\mathcal{A}_a^0$. $U(1)$ scalar.

- Breaks $\mathcal{Q}$ SUSY softly. All breaking terms must vanish for $\mu \to 0$ (exact $\mathcal{Q}$).

Quantum corrections ...

Can show (JHEP 1104 (2011) 074, arXiv:1306.3891)

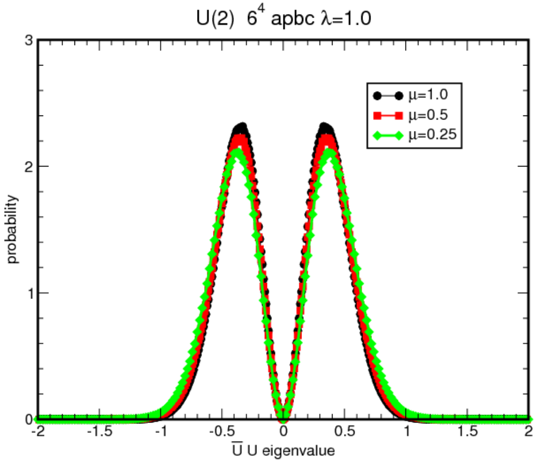
- Lattice theory renormalizable: only counterterms allowed by lattice symmetries correspond to terms in original action
- Effective potential vanishes to all orders in p. theory. No scalar mass terms!
- At one loop:
  - No fine tuning: common wavefunction renormalization
  - Vanishing beta function: Divergence structure matches continuum ....
  - Restoration of 15 additional twisted susys depends on restoration of Lorentz invariance.

Need to go beyond p. theory....

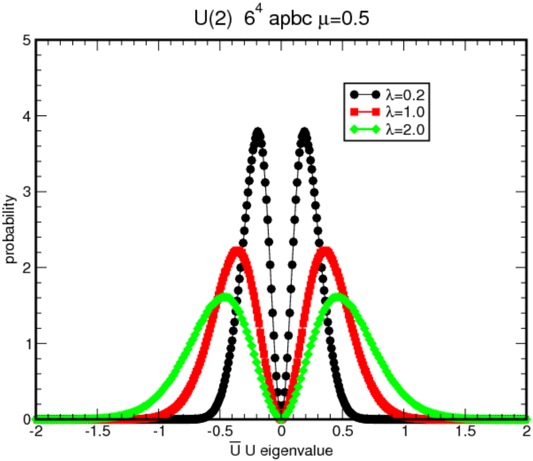
Simulations

- Integrate fermions $\to \mathrm{Pf}(M)$. Realize as $\det\left(M^\dagger M\right)^{-\frac{1}{4}}$
- Standard lattice QCD algs may be used: RHMC with Omelyan, multi time step evolution. GPU acceleration for inverter (speedup: 5-10 over single core code for $L = 8^3 \times 16$)
- Phase quenched approximation should be ok: analytical argument, numerical evidence ...
- First step: phase structure .... $U(2), L^4, apbc, \; L = 4, 6, 8$
  - Instabilities from flat directions ?
  - Supersymmetry realized ?
  - (De)confinement, (absence of) chiral symmetry breaking, phase transitions ?

Scalar eigenvalue distribution - insensitive to boson mass μ



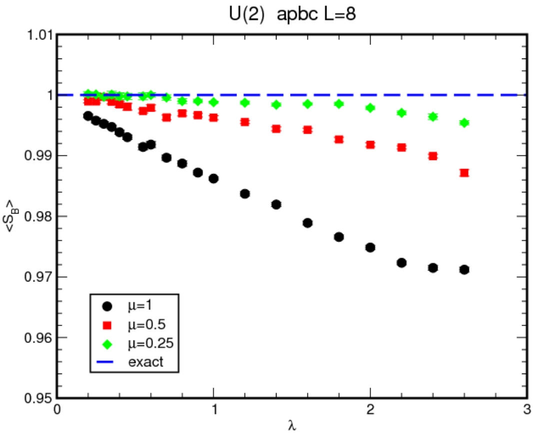
Scalar eigenvalue distribution - localized for all λ



Comments

- Common statement: "Moduli space is not lifted in $\mathcal{N} = 4$ by quantum corrections ..."
  Why is scalar distribution not flat as $\mu \to 0$?
- Pfaffian vanishes on flat directions. Formally this zero cancels against infinity from boson zero modes but latter are lifted at non-zero μ.
- Thus configurations corresponding to flat directions make no contribution to lattice path integral.
- Small fluctuations around flat directions cost increasing action as move away from origin in field space - large scalar eigenvalues suppressed.

Test of exact supersymmetry

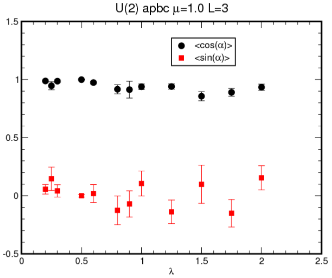


For $\mu \to 0$ S_B given by simple $\mathcal{Q}$ Ward identity.

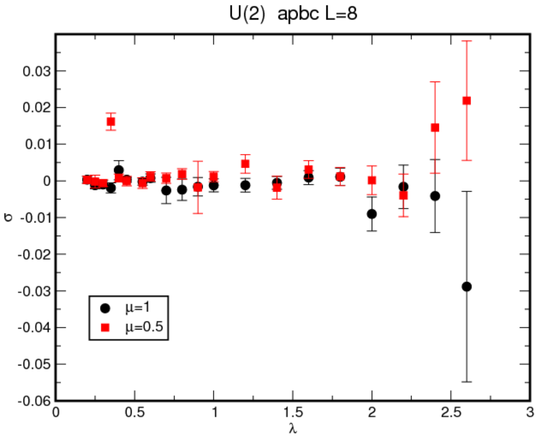
Sign problem ?

Integrate fermions: complex Pfaffian. But observed Pfaffian phase α small in phase quenched simulations..

Exact $\mathcal{Q}$ symmetry ensures $< O_{\text{inv}} >$ can be computed as $\lambda \to 0$ where $\alpha = 0$



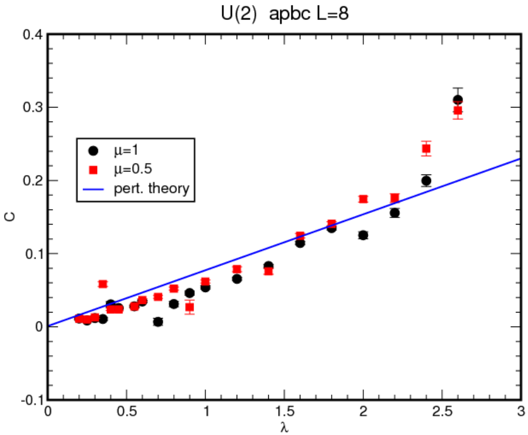
Phase structure - String tension



Extract by fitting $W(R, T)$ to $e^{-V(R)T}$ and extract σ

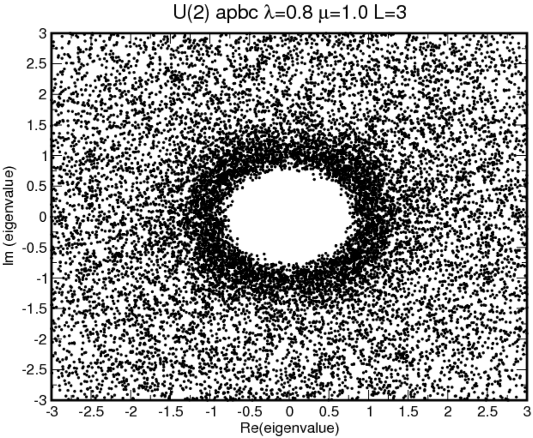
Vanishing for all λ

Coulomb fits



Fits are good and consistent with p theory .. no sign of Maldacena $\sqrt{\lambda}$ behavior

Chiral symmetry breaking - or lack of it ..



Eigenvalues excluded from origin: insensitive to μ and λ

In progress ...

- Push to larger lattices. Developed parallel code (MILC) $8^3 \times 16$ and $16^3 \times 32$ lattices.
- Push to larger λ. Care needed – lattice $U(1)$ sector can undergo transition. Suppress with coupling to $\det(U_P)$. Expanded phase diagram.
- Anomalous dimensions: Measure Konishi and SUGRA (BPS protected) ops. Agreement with e.g. conformal bootstrap bounds ?
- Restoration of additional SUSYs: general arguments show that restoration of Lorentz invariance should be enough...
- Black hole physics via holography ...

Conclusions

- Simulations of $\mathcal{N}=4$ YM look promising: gauge invariance and (some) SUSY can be preserved. No instabilities from flat directions, no sign problem.
- Prelim investigations show no sign of any phase transitions as vary λ. String tension small. Only Coulomb term needed. Evidence for single, deconfined phase. Consistent with pert theory: 1 loop calc shows $\beta_{\rm latt}(\lambda)=0$

Lots of work to do ! ... lots of interesting questions to address ..!

The end