Refined BPS invariants of del Pezzo and half K3 manifolds

Based on: M. X. Huang, A. Klemm, M. P.: arXiv:1308.0619[hep-th]

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Contents and Outline

The additional $U(1)_{\mathcal{R}}$-symmetry on local Calabi-Yau manifolds allows for the definition of an index

$$I = Tr_{BPS}(-1)^{2(J_L+J_R)} e^{-2\epsilon_L J_L} e^{-2\epsilon_R J_{\mathcal{R}} e^{\beta H}}$$

that counts the refined BPS numbers $N^{\beta}_{j_L,j_R}$.

Aim of this talk

Explain the computation of the refined BPS invariants on local Calabi-Yau manifolds over a del Pezzo and half K3 surfaces.

Outline:

- Introduction / Definition of Refined BPS invariants
- Presentation of the considered geometries
- Computation
- Applications

BPS numbers count wrapped branes

$$
\begin{tikzcd}
\begin{array}{l}\text{Type II A on}\\ \mathbb{R}^{(3,1)} \times X\end{array} \arrow[rr, "\text{Lift}"] \arrow[dr, "\text{compactify}"] & & \begin{array}{l}\text{M-theory on}\\ \mathbb{R}^{(3,1)} \times X \times S^1\end{array} \arrow[dl, "\text{compactify}"'] \\
& \displaystyle\int F_g(t_i)\lambda^{2g-2}R_+^2 & \\
\begin{array}{l}\text{D2-D0 on}\\ \mathcal{C} \in H_2(X,\mathbb{Z})\end{array} \arrow[ur, "\text{int. out}"] \arrow[rr, "\text{Lift}"] & & \begin{array}{l}\text{M2 on}\\ \mathcal{C} \in H_2(X,\mathbb{Z})\\ \& \text{ mom. along } S^1\end{array} \arrow[ul, "\text{int. out}"']
\end{tikzcd}
$$

$$\lambda = \left\langle \epsilon_1 dx_1 \wedge dx_2 - \epsilon_2 dx_3 \wedge dx_4 \right\rangle + \delta\lambda$$

Graviphoton-background field strength

$\epsilon_1 = -\epsilon_2$ unrefined topological string

$\epsilon_1 \neq -\epsilon_2$ refined topological string

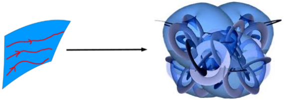
Antoniadis, Gava, Narain, Taylor '94,95
Gopakumar, Vafa '98; Nekrasov '02
Antoniadis, Hohenegger,Narain, Taylor (et al.) '10-13

Different interpretations for $SU(2)_L \times SU(2)_R$

The BPS particles are partly classified by the little group $SU(2)_L \times SU(2)_R$.

Geometrical origin of spin content:

Moduli space of curves with flat bundle



$$\underbrace{T^{2g} \longrightarrow \mathcal{M}}_{\hat{\mathcal{M}}}$$

- $SU(2)_L$ acts on the fibre via Lefshetz action
- $SU(2)_R$ acts on the base via Lefshetz action

Katz, Klemm, Vafa '99

Refined Invariants

$$N^{\beta}_{j_L, j_R}$$

not invariant under c.s. deformations

positive

Unrefined Invariants

$$n_g = \sum_{J_R} (-1)^{J_R} (2J_R + 1) N^{\beta}_{g, J_R}$$

invariant under c.s. deformations

can be negative

Mathematical approach

D2-D0 bound state defines stable pair, i.e. a sheaf $\mathcal{F}$ together with a section $s \in H^0(\mathcal{F})$, s.t.

1) $\mathcal{F}$ is pure of dimension 1 with $ch_2(\mathcal{F}) = \beta$, $\chi(\mathcal{F}) = n$
2) s generates $\mathcal{F}$ outside a finite set of points

- Toric Calabi-Yau: $(\mathbb{C}^*)^n$-action descends to the moduli space of stable pairs $P_n(X, \beta)$
- Birula-Bialynicki decomposition can be used to calculate the virtual motive

$$[P_n(X,\beta)]^{vir} = \sum_{Z \in P_n(X,\beta)^{C^*}} \left(-\mathbb{L}^{-1/2}\right)^{d_Z^+ - d_Z^-}$$

which gives rise to the refined PT-invariants.

Choi, Katz, Klemm '12

The geometries

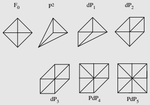
Definition

A smooth two-dimensional Fano variety is called del Pezzo surface. They enjoy a finite classification.

1) $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$
2) $Bl_n\mathbb{P}^2, \quad 0 \leq n \leq 8$ — Elliptic pencil with $9-n$ base points
3) $n = 9, \quad \frac{1}{2}K3$ — Elliptic fibration

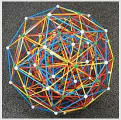
$9-n$	9	8	7	6	5	4	3	2	1	0
G	$-$	$-$	A_1	$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8	$\hat{E}_8$

Toric geometry



Versus

Modular Symmetry



Toric geometries - The algorithm

Toric Diagram

Mirror curve & map

$F^{(0,0)}$, $F^{(1,0)}$, $F^{(0,1)}$

$F^{(n,g)}$ + ambiguity

$F^{(n,g)}$

Geometrical Invariants

Refined holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} F^{(n,g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \Big(D_i D_k F^{(n,g-1)} + {\sum_{m,h}}' D_j F^{(m,h)} D_k F^{n-m,g-h} \Big)$$

Gap condition

$F^{(n,g)} = t^{-(n+g)} +$ regular

Krefl, Walcher, ’10; Huang, Kashani-Poor, Klemm, ’11; Huang, Klemm, M.P. ’13

Modular symmetry

Set-up and Goal

The $\frac{1}{2}K3$ can be considered as a elliptic fibration over a Hirzebruch surface. The aim is to determine the expansion

$$F = \sum_{n,g,n_b=0}^{\infty} (\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1} e^{2\pi i t_b n_b} F^{(n,g,n_b)}(q)$$

$$\begin{aligned}\partial_{E_2} F^{(n,g,n_b)} \quad &= \quad \frac{1}{24}\sum_{n_1=0}^{n}\sum_{g_1=0}^{g}\sum_{s=1}^{n_b-1} s(n_b-s) F^{(n_1,g_1,s)} F^{(n-n_1,g-g_1,n_b-s)} \\ &\quad + \frac{n_b(n_b+1)}{24} F^{(n,g-1,n_b)} - \frac{n_b}{24} F^{(n-1,g,n_b)}\end{aligned}$$

- use the modular symmetry to fix the free energies
- use blow-downs in order to determine the free energies for del Pezzo surfaces

Hosono, Saito, Takahashi '99; Huang, Klemm, '10; Huang, Klemm, M.P. '13

Explicit results for the dP_8-geometry

$2j_L \backslash 2j_R$	0	1
0	248	
1		1

$d = 1$

$2j_L \backslash 2j_R$	0	1	2	3
0		3876		
1			248	
2				1

$d = 2$

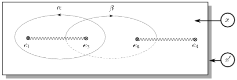
$2j_L \backslash 2j_R$	0	1	2	3	4	5	6
0	30628		151374		248		
1		4124		34504		1	
2	1		248		4124		
3				1		248	
4							1

$d = 3$

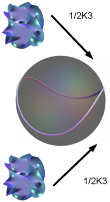
The BPS invariants $N^d_{j_L,j_R}$ for $d = 1, 2, 3$ for the local E_8 del Pezzo surface.

Huang, Klemm, M.P. '13

Results are linked to various applications

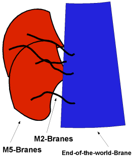


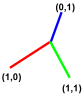
- Seiberg Witten theory with matter



- Geometries appear for stable degeneration limit of F-theory
- Counting small sized instantons in heterotic string theory

Witten '95





- Connection to E-, (p,q)- and eventually M- Strings

Vafa et al.; Gaberdiel, Zwiebach '97; Mikhailov, Nekrasov, Sethi, '98

Conclusions

We have presented a framework to compute refined BPS numbers that

- works everywhere in the moduli space
- works for toric del Pezzo surfaces
- works for non-toric del Pezzo surfaces

Outlook:

- Elaborate on the connection to E- and M-Strings
- Improve the development of the corresponding mathematical theory