

Weyl Symmetry & the Structure of 4D RG Flows

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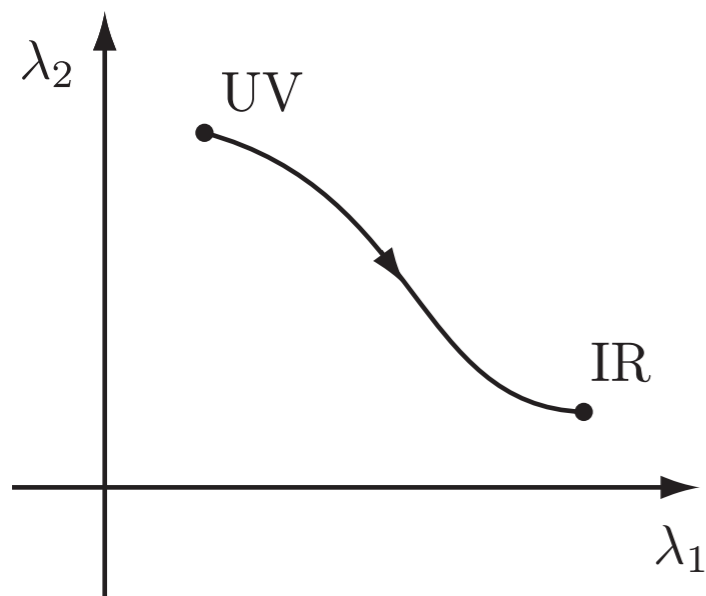
M. Luty, J. Polchinski, RR

[arXiv:1204.5221](https://arxiv.org/abs/1204.5221)

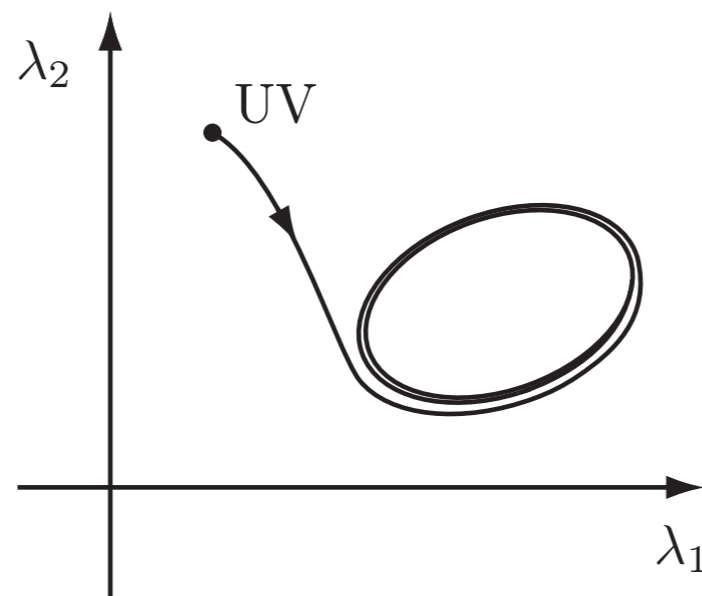
F. Baume, B. Keren-Zur, RR, L. Vitale

in preparation

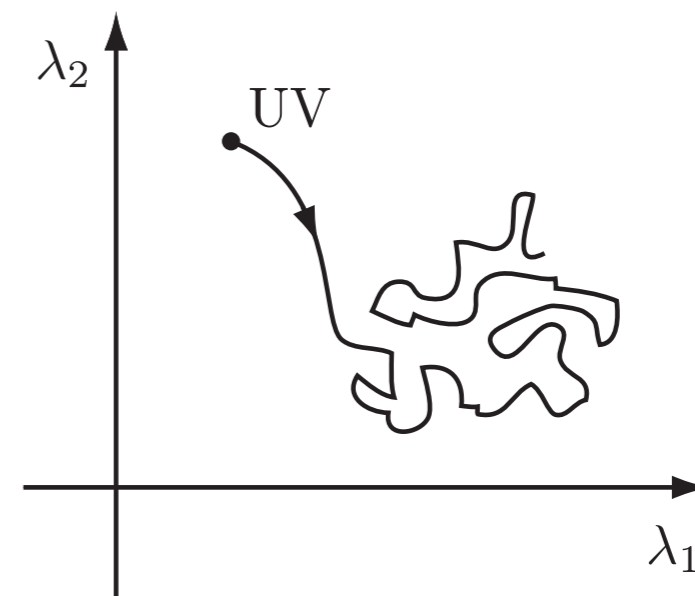
conceivable RG flows



(a)



(b)



(c)

but all known examples asymptote to a CFT fixed point

Is there a way to understand that?

Two approaches
to constrain
RG-flow structure

- Wess-Zumino consistency conditions for Weyl anomaly off-criticality

Jack, Osborn 1990

Osborn 1991

- Dispersion relations for $\langle T \dots T \rangle$
Optical theorem for scattering amplitudes of background dilaton

Komargodski and Schwimmer 2011

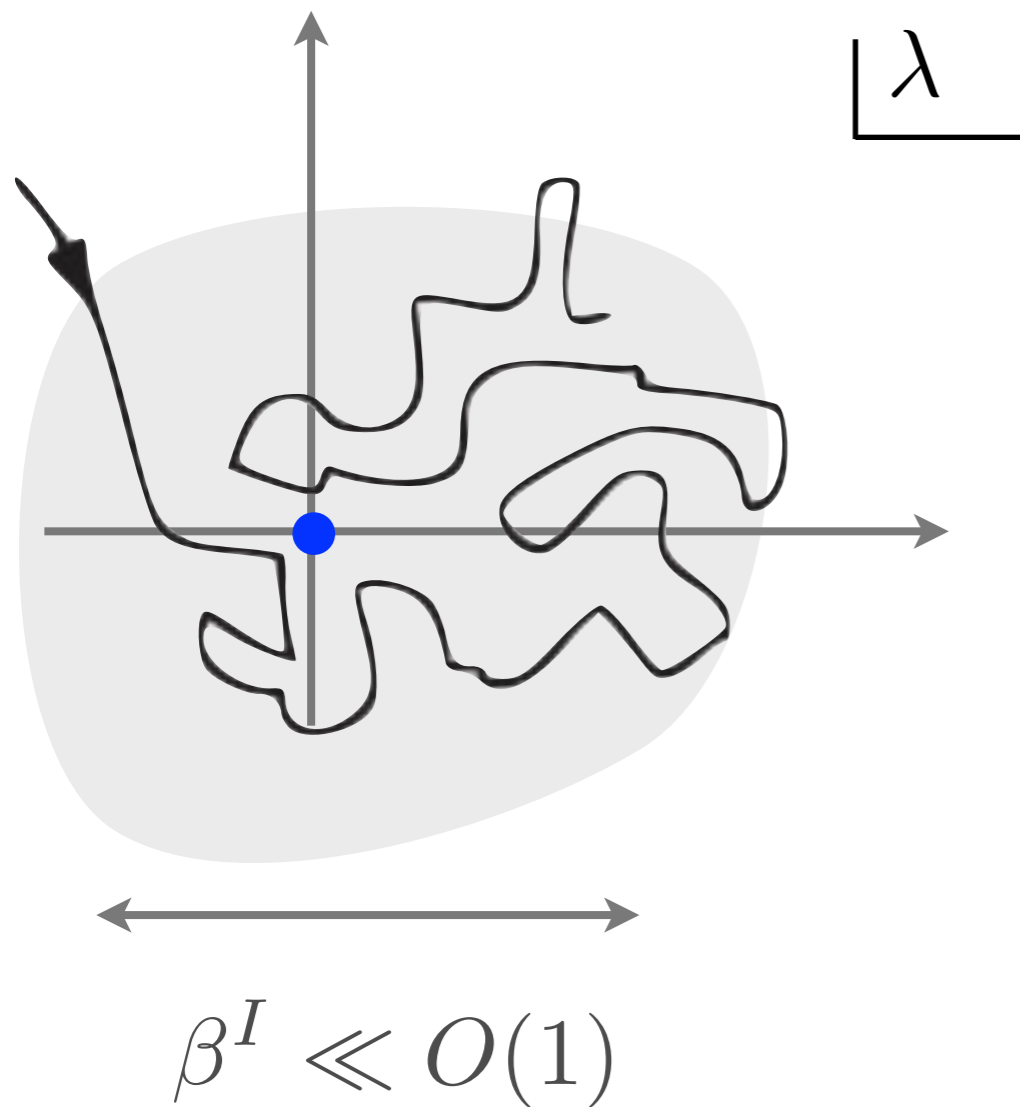
- $a_{UV} > a_{IR}$

- CFT is the only possible asymptotics in weakly coupled 4D QFT

- the occurrence of SFTs is severely constrained (ruled out...) even beyond perturbation theory

Results

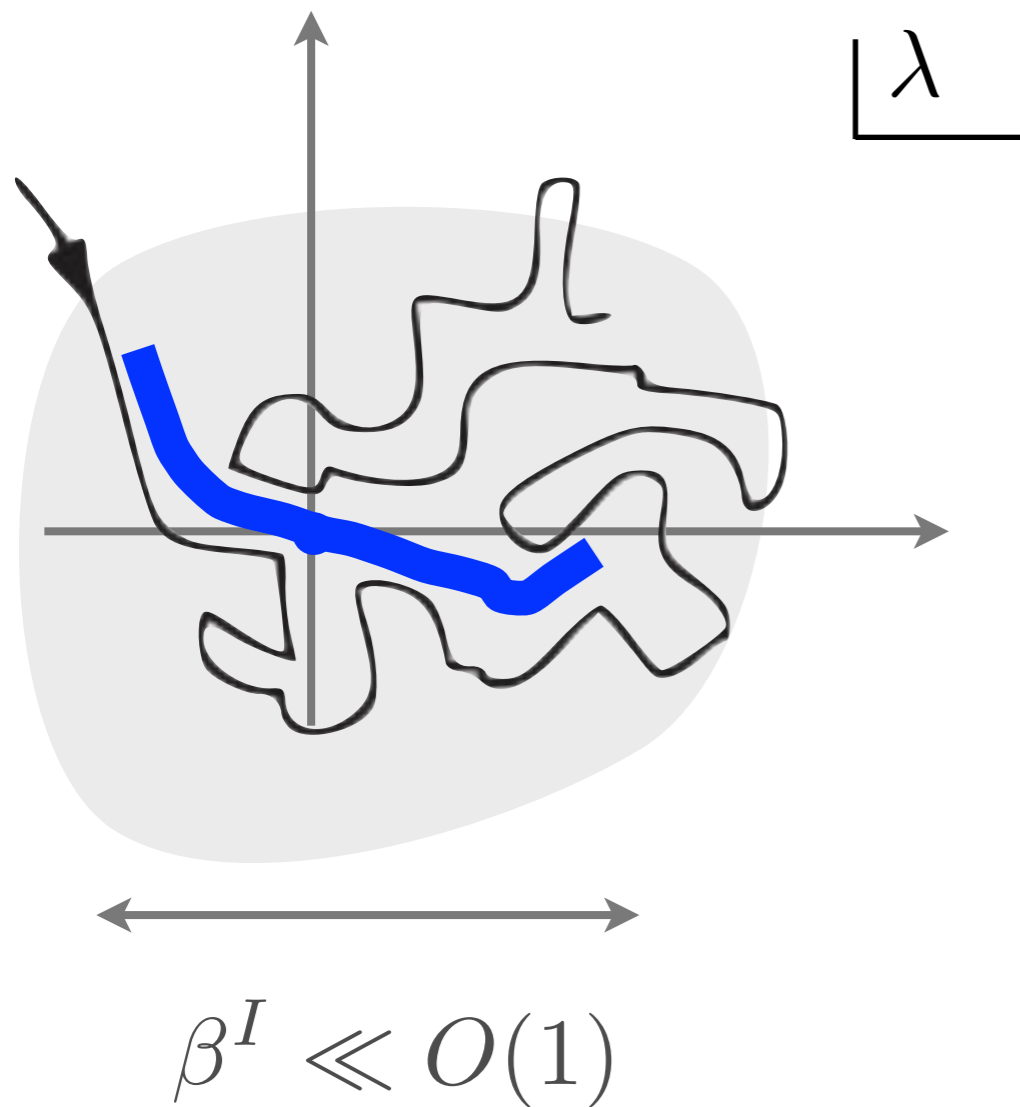
Goal: study RG flow in a domain around a fixed point



$$\mathcal{L} = \mathcal{L}_{CFT} + \sum_I \lambda^I \mathcal{O}_I$$

- CFT, not necessarily free
- $\lambda^I \ll O(1)$ not necessary
provided $\beta^I \ll O(1)$
in whole domain

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RG flow

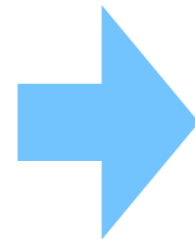


correlators of $T \equiv T_{\mu}^{\mu}$

current for flavor symm broken by $\lambda^I \neq 0$



complete 'basis' for T_{μ}^{μ}



$\mathcal{O}_I, \partial^{\mu} J_{\mu}^A, \square \mathcal{O}_a$

- Ex: free CFT with scalars and fermions

relevant object \equiv effective action for sources

$$T_{\mu\nu} \leftrightarrow g_{\mu\nu}(x)$$

$$\mathcal{O}_I \leftrightarrow \lambda_I(x)$$

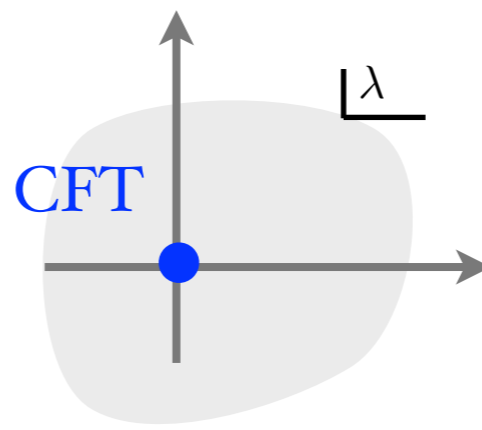
$$J_\mu^A \leftrightarrow A_\mu^A(x)$$

$$\mathcal{O}_a \leftrightarrow m_a(x)$$

$$W \equiv W[g_{\mu\nu}, \lambda^I, A_\mu^A, m_a, \dots]$$

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} W \quad \mathcal{O}_I(x) = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \lambda_I(x)} W \quad \text{etc ...}$$

At fixed point



$$W[g_{\mu\nu}, \lambda = 0, A_\mu = 0, m_a = 0]$$

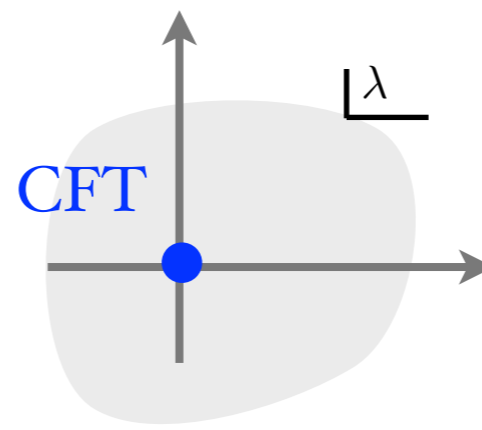
Weyl invariant up to anomaly

Capper, Duff '73

$$\frac{2g^{\mu\nu}}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} W = aE_4 - bR^2 - cW^2 - d\Box R$$

$$= aE_4 - cW^2 - \delta(F_{\text{local}})$$

At fixed point



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Weyl invariant up to anomaly

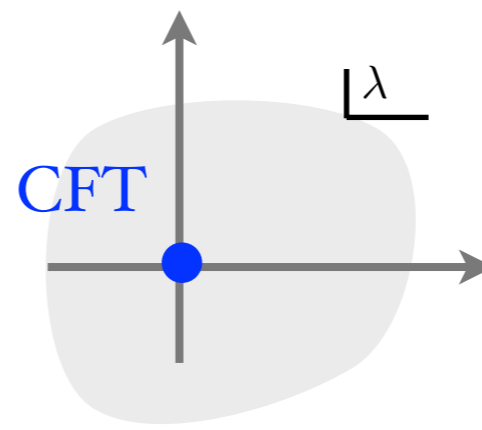
Capper, Duff '73

WZ consistency

$$\frac{2g^{\mu\nu}}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} W = aE_4 - \cancel{bR^2} - cW^2 - d\Box R$$

$$= aE_4 - cW^2 - \delta(F_{\text{local}})$$

At fixed point



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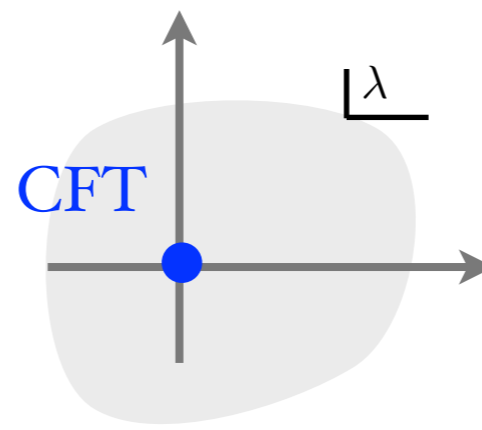
Capper, Duff '73

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Weyl invariant up to anomaly

Capper, Duff '73

WZ consistency

$$\frac{2g^{\mu\nu}}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} W = aE_4 - \cancel{bR^2} - cW^2 - d\Box R \quad \boxed{+ e\Lambda^2 R + f\Lambda^4}$$

$$= aE_4 - cW^2 - \delta(F_{\text{local}})$$

Weyl anomaly equation can be extended off criticality
by assigning transformation properties to sources



local Callan-Symanzik equation

Osborn 1991

$$\int d^4x \left\{ \sigma(x) \left[2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \tilde{m}^a \frac{\delta}{\delta m^a(x)} \right] + \right.$$

$$\left. + \nabla_\mu \sigma(x) \left[\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} - S^A \frac{\delta}{\delta A_\mu^A(x)} \right] - \square \sigma(x) t^a \frac{\delta}{\delta m^a(x)} \right\} W =$$

$$= \int d^4x \sigma(x) \mathcal{A}(x)$$

- $2\tilde{m}^a = 2m^b (\delta_b^a + \gamma_b^a) + \frac{1}{3} \eta^a R + d_I^a \square \lambda^I + \frac{1}{2} \epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J$
- $\mathcal{A}(x) =$ all possible dim 4 covariant terms

Easy to derive local CS eq. in ordinary (near free) QFT using dim reg

$$S_0 = S_1[\text{fields, sources}] + S_{CT}[\text{sources}]$$

- S_1 obviously Weyl invariant

$$\lambda_0^I \rightarrow e^{\epsilon\sigma(x)} \lambda_0^I \quad \rightarrow \quad \delta_\sigma(\lambda^I) = \sigma\beta^I$$

- S_{CT} not Weyl invariant (unless new sources added)

$$\delta_\sigma S_{CT} = \sigma\mathcal{A}$$

Redundancies in source parametrization

$$\delta_\sigma \equiv \sigma(x) \left[2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \tilde{m}^a \frac{\delta}{\delta m^a(x)} \right] +$$

$$+ \nabla_\mu \sigma(x) \left[\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} - S^A \frac{\delta}{\delta A_\mu^A(x)} \right] - \square \sigma(x) t^a \frac{\delta}{\delta m^a(x)}$$

$$m^a \rightarrow m^a + \frac{1}{6} f^a R(g) + f_I^a \square \lambda^I$$



$$T_{\mu\nu} \rightarrow T_{\mu\nu} + f^a (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \mathcal{O}_a$$

$$\mathcal{O}_I \rightarrow \mathcal{O}_I + \theta_{Ia} \square \mathcal{O}_a$$

$$t^a \rightarrow t^a + f^a$$

$$\theta_I^a \rightarrow \theta_I^a + f_I^a$$

scheme choice

$$t^a = 0$$

$$\theta_I^a = 0$$

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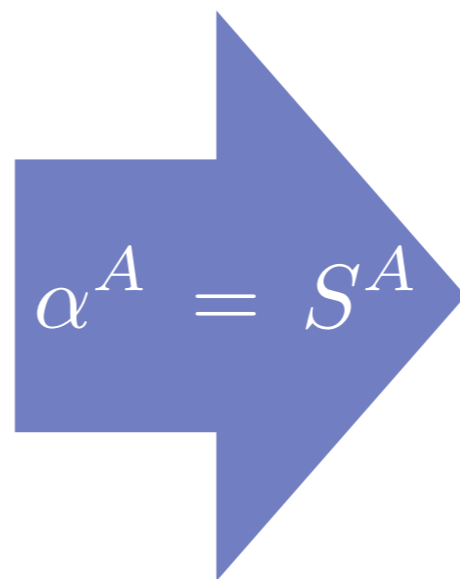
“Flavor freedom” in defining Weyl transformation

$$\delta'_\sigma = \delta_\sigma + \delta_{\sigma\alpha^A}^{\text{flavor}}$$

local flavor rotation
with Lie parameter $\sigma\alpha^A$

$\alpha^A(\lambda) =$ Flavor adjoint constructed with couplings

$$\begin{aligned}\beta^I &\rightarrow \beta^I - (\alpha^A T_A \lambda)^I \\ \tilde{m}^a &\rightarrow \tilde{m}^a - (\alpha^A T_A \tilde{m})^a \\ \rho_I^A &\rightarrow \rho_I^A + \partial_I \alpha^A \\ S^A &\rightarrow S^A - \alpha^A\end{aligned}$$


$$\alpha^A = S^A$$

$$\begin{aligned}B^I &= \beta^I - (S^A T_A \lambda)^I \\ \tilde{M}^a &= \tilde{m}^a - (S^A T_A \tilde{m})^a \\ P_I^A &= \rho_I^A + \partial_I S^A\end{aligned}$$

The Weyl transformation operator can be finally simplified as

$$\int \sigma(x) \left[2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - B^I \frac{\delta}{\delta \lambda^I(x)} - P_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \tilde{M}^a \frac{\delta}{\delta m^a(x)} \right] W = \int \sigma \mathcal{A}$$

up to contact terms:

$$T_\mu^\mu = \sum_I B^I \mathcal{O}_I$$

$$(g_{\mu\nu} = \eta_{\mu\nu}, \quad \nabla_\mu \lambda^I = A_\mu^A = m_a = 0)$$

CFT $B^I = 0$

SFT $B^I = N^A (T_A \lambda)^I \equiv$ pure flavor rotation

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] = 0$$

Two types of consistency conditions

I. On coefficients of δ_σ

- $\frac{\delta}{\delta A_\mu^A}$

$$B^I P_I^A = 0$$

$$T(x)T(y) = \dots + \delta^4(x-y) \cancel{B^I P_I^A} \partial^\mu J_{A\mu}$$

- $\frac{\delta}{\delta m^a}$

similar story

II. genuine WZ condition: $\int [\sigma_1(y)\delta_{\sigma_2(x)}\mathcal{A}(y) - \sigma_2(x)\delta_{\sigma_1(y)}\mathcal{A}(x)] = 0$

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$$\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A} = & \sigma\left(\beta_a W^2 + \beta_b E_4 + \frac{1}{9}\beta_c R^2\right) - \nabla^2\sigma\left(\frac{1}{3}dR\right) \\
& + \sigma\left(\frac{1}{3}\chi_I^e \nabla_\mu\lambda^I \nabla^\mu R + \frac{1}{6}\chi_{IJ}^f \nabla_\mu\lambda^I \nabla^\mu\lambda^J R + \frac{1}{2}\chi_{IJ}^g G^{\mu\nu} \nabla_\mu\lambda^I \nabla_\nu\lambda^J\right. \\
& \quad \left. + \frac{1}{2}\chi_{IJ}^a \nabla^2\lambda^I \nabla^2\lambda^J + \frac{1}{2}\chi_{IJK}^b \nabla_\mu\lambda^I \nabla^\mu\lambda^J \nabla^2\lambda^K + \frac{1}{4}\chi_{IJKL}^c \nabla_\mu\lambda^I \nabla^\mu\lambda^J \nabla_\nu\lambda^K \nabla^\nu\lambda^L\right) \\
& + \nabla^\mu\sigma\left(G_{\mu\nu}w_I \nabla^\nu\lambda^I + \frac{1}{3}RY_I \nabla_\mu\lambda^I + S_{IJ} \nabla_\mu\lambda^I \nabla^2\lambda^J + \frac{1}{2}T_{IJK} \nabla_\nu\lambda^I \nabla^\nu\lambda^J \nabla_\mu\lambda^K\right) \\
& - \nabla^2\sigma\left(U_I \nabla^2\lambda^I + \frac{1}{2}V_{IJ} \nabla_\nu\lambda^I \nabla^\nu\lambda^J\right) \\
& + \sigma\left(\frac{1}{2}p_{ab}\hat{m}^a\hat{m}^b + \hat{m}^a\left(\frac{1}{3}q_a R + r_{aI} \nabla^2\lambda^I + \frac{1}{2}s_{aIJ} \nabla_\mu\lambda^I \nabla^\mu\lambda^J\right)\right) \\
& + \nabla_\mu\sigma\left(\hat{m}^a j_{aI} \nabla^\mu\lambda^I\right) - \nabla^2\sigma\left(\hat{m}^a k_a\right) \\
& + \sigma\left(\frac{1}{4}\kappa_{AB}F_{\mu\nu}^A F^{B\mu\nu} + \frac{1}{2}\zeta_{AIJ}F_{\mu\nu}^A \nabla^\mu\lambda^I \nabla^\nu\lambda^J\right) + \nabla^\mu\sigma\left(\eta_{AI}F_{\mu\nu}^A \nabla^\nu\lambda^I\right)
\end{aligned} \tag{2.49}$$

10 differential constraints involving 25 tensorial coefficients

all but a few constraints can be “solved”

$$A = \underbrace{A_{R^2} + A_{W^2}}_{\text{manifestly consistent}} + \underbrace{A_{E_4} + A_{F^2}}_{\text{non-trivial}} + \underbrace{\delta_{Weyl} F_{local}}_{\text{trivial (scheme dep)}}$$

$$\frac{1}{\sqrt{g}}\sigma\mathcal{A}_{R^2} = \sigma\left(\frac{1}{2}b_{ab}\Pi^a\Pi^b + \frac{1}{2}b_{aIJ}\Pi^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

$$\Pi^{IJ} = \nabla_\mu\lambda^I\nabla^\mu\lambda^J - B^{(I}\Lambda^{J)} \quad \longrightarrow \quad \Lambda^J \propto \left(\square\lambda^J + \frac{1}{6}B^J R(g)\right)$$

$$\Pi^a = m^a - \frac{1}{6}t^a R(g) - \theta_I^a \Lambda^I$$

$$\delta_\sigma\Pi^{IJ} = \sigma(\dots) + \nabla_\mu\sigma(\dots) + \nabla^2\sigma(\dots)$$

absence of derivative terms: consistency is manifest

$$\frac{1}{\sqrt{g}} \sigma \mathcal{A}_{R^2} = \sigma \left(\frac{1}{2} b_{ab} \Pi^a \Pi^b + \frac{1}{2} b_{aIJ} \Pi^a \Pi^{IJ} + \frac{1}{4} b_{IJKL} \Pi^{IJ} \Pi^{KL} \right)$$

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$$\Pi^a = m^a - \frac{1}{6} t^a R(g) - \theta_I^a \Lambda^I$$

$$\delta_\sigma \Pi^{IJ} = \sigma (\dots) + \cancel{\nabla_\mu \sigma (\dots)} + \cancel{\nabla^2 \sigma (\dots)}$$

absence of derivative terms: consistency is manifest

Non-trivial anomalies

$$\frac{1}{\sqrt{g}} \mathcal{A}_{E_4} = \sigma a E_4 + \sigma \frac{1}{2} \chi_{IJ} G_{\mu\nu} \nabla^\mu \lambda^I \nabla^\nu \lambda^J + \nabla^\mu \sigma w_I G_{\mu\nu} \nabla^\nu \lambda^I + \dots \dots$$

$$\frac{1}{\sqrt{g}} \mathcal{A}_{F^2} = \sigma \frac{1}{4} \kappa_{AB} F_{\mu\nu}^A F^{B\mu\nu} + \sigma \frac{1}{2} \zeta_{AIJ} F_{\mu\nu}^A \nabla^\mu \lambda^I \nabla^\nu \lambda^J + \nabla^\mu \sigma \eta_{AI} F_{\mu\nu}^A \nabla^\nu \lambda^I +$$

$$\mathcal{L}[w_I] = -8\partial_I a + \chi_{IJ} B^J$$

$$\mathcal{L}[\eta_{AI}] = \kappa_{AB} P_I^B + \zeta_{AIJ} B^J - \chi_{IJ}^g (T_A \lambda)^J$$

$$0 = \eta_{AI} B^I + w_I (T_A \lambda)^I$$

Gradient flow equation

$$\tilde{a} \equiv a + \frac{1}{8} w_I B^I$$

$$\delta \partial_I \tilde{a} = (\chi_{IJ} + \partial_I w_J - \partial_J w_I + P_I^A \eta_{AJ}) B^J$$

- non-trivial constraint on perturbative expansion of B^I
- at fixed points $\tilde{a}(\lambda)$ is stationary
- along line of fixed points $\tilde{a} = a = \text{const}$

$$\delta \mu \frac{d\tilde{a}}{d\mu} \equiv \delta B^I \partial_I \tilde{a} = \chi_{IJ} B^I B^J$$

$$\langle \mathcal{O}_I(x) \mathcal{O}_J(0) \rangle = \frac{\chi_{IJ}}{x^8} + O(\partial B, B) \quad \text{by unitarity} \quad \chi_{IJ} > 0$$

$$\delta\mu \frac{d\tilde{a}}{d\mu} = \chi_{IJ} B^I B^J \geq 0$$

$$\tilde{a}(\lambda(\mu_1)) - \tilde{a}(\lambda(\mu_2)) = \frac{1}{8} \int_{\mu_1}^{\mu_2} \chi_{IJ} B^I B^J d \ln \mu$$

since \tilde{a} is finite the only possible asymptotics must satisfy $B^I = 0$

CFT, free or interacting, is the only possible asymptotics

The story with dilatons and dispersion relations

$$\langle T \dots T \rangle$$

K.S.



scattering amplitudes
of background dilaton

$$g_{\mu\nu} \equiv \Omega(x)^2 \eta_{\mu\nu}$$

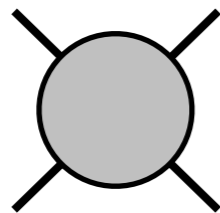
analyze
counterterms at $\square\Omega = 0$



$W[\Omega^2 \eta_{\mu\nu}]$ is finite up to CC term

Luty, Polchinski, RR 2012

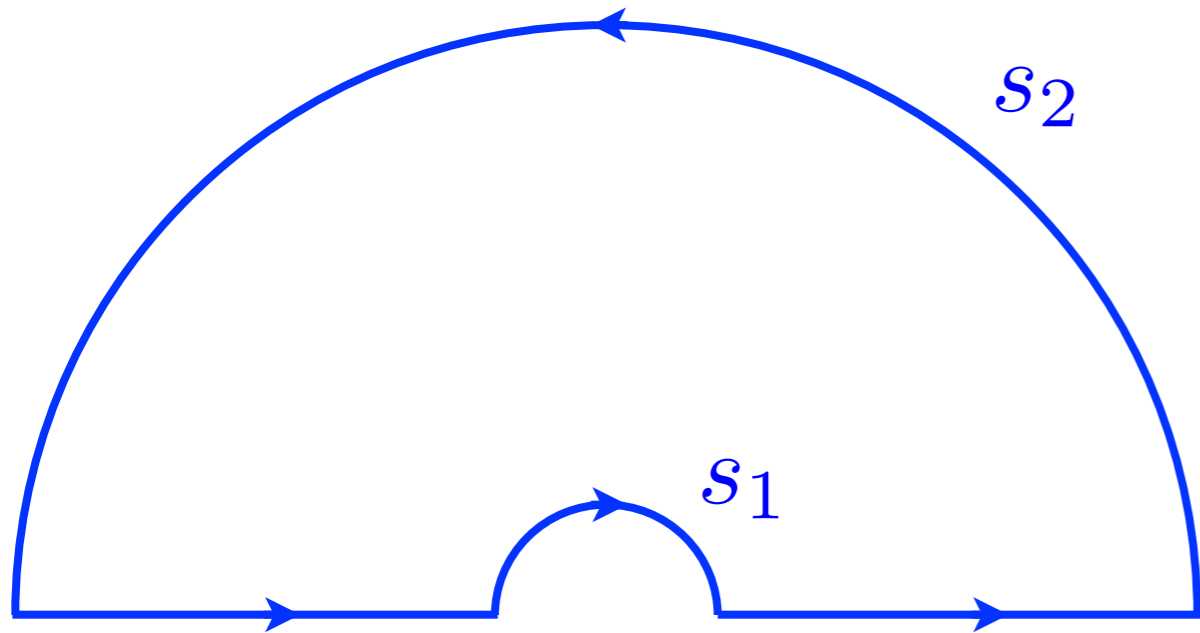
forward
amplitude



$$A(s) = -\alpha(\lambda(\sqrt{s})) s^2 + \Lambda$$

CFT limit

$$A(s) = -8a s^2$$



$$\bar{\alpha}(s) \equiv \frac{1}{\pi} \int_0^\pi d\theta \alpha(se^{i\theta})$$

$$\bar{\alpha}(s_2) - \bar{\alpha}(s_1) = \frac{2}{\pi} \int_{s_1}^{s_2} \frac{ds}{s} \text{Im } \alpha(s) \geq 0 \quad \text{by unitarity}$$

$\bar{\alpha}(s)$ finite



$$\lim_{s \rightarrow \pm\infty} \text{Im } \alpha(s) = 0$$

Local Callan-Symanzik elucidates both sides of dispersion relation

▲ $\bar{\alpha}(s) = 8 \tilde{a}(s) + O(B^2)$

this ensure a scheme choice exists where $\bar{\alpha}(s) = 8 \tilde{a}(s)$

▲
$$\begin{aligned} \text{Im } \alpha(s) &= \frac{1}{s^2} \sum_{\Psi} |\langle \Psi | B^I (\delta_I^J + \partial_I B^J) \mathcal{O}_J(p_1 + p_2) + B^I B^J \mathcal{O}_I(p_1) \mathcal{O}_J(p_2) | 0 \rangle|^2 \\ &= B^I B^J G_{IJ} \end{aligned}$$

$$G_{IJ} = \frac{1}{s^2} \sum_{\Psi} \langle 0 | \mathcal{O}_I + \partial_I B^L \mathcal{O}_L + B^L \mathcal{O}_I \mathcal{O}_L | \Psi \rangle \langle \Psi | \mathcal{O}_J + \partial_J B^K \mathcal{O}_K + B^K \mathcal{O}_J \mathcal{O}_K | 0 \rangle \geq 0$$

$$s \frac{d\bar{\alpha}(s)}{ds} = \frac{2}{\pi} G_{IJ} B^I B^J$$

There thus exists a scheme where

$$\bar{\alpha} = \tilde{\alpha} \quad \chi_{IJ} = \frac{4}{\pi} G_{IJ} + \Delta_{IJ} \quad \begin{aligned} G_{IJ} &\geq 0 \\ \Delta_{IJ} B^I B^J &= 0 \end{aligned}$$

G_{IJ} is the 4D analogue of Zamolodchikov metric in 2D

but 2D case simpler
(just 2-point functions)

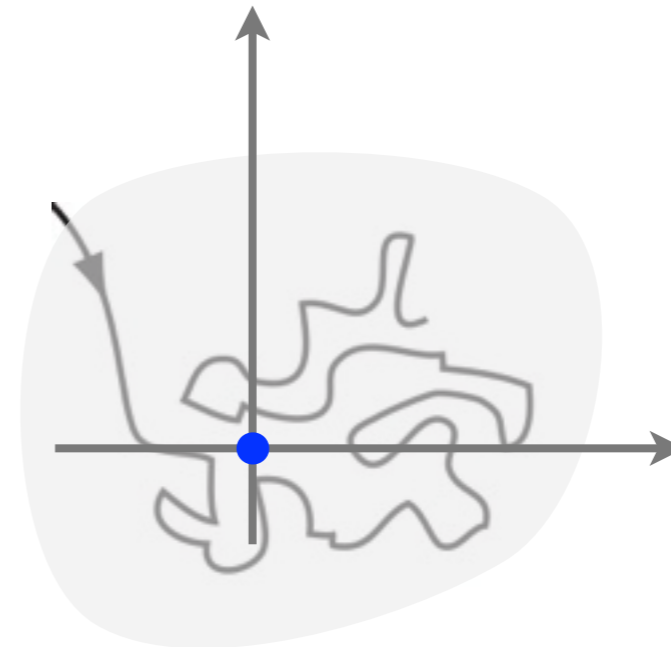
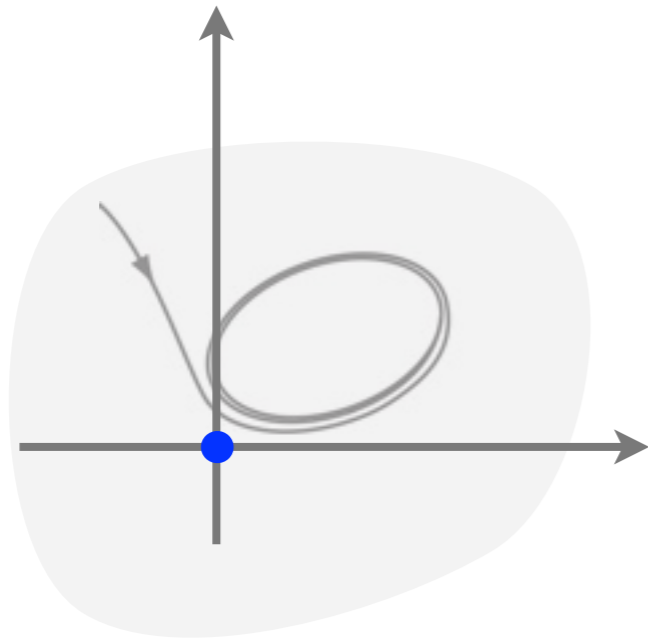
$$G_{IJ} = \frac{1}{p^2} \sum_{\Psi} \langle 0 | \mathcal{O}_I(p) | \Psi \rangle \langle \Psi | \mathcal{O}_J(p) | 0 \rangle$$

without dilaton as guideline harder to figure things out in 4D

Near CFT fixed point, irreversibility of RG flow concretely expressed by

$$\delta\mu \frac{d\tilde{a}}{d\mu} \equiv \delta B^I \partial_I \tilde{a} = \chi_{IJ} B^I B^J$$

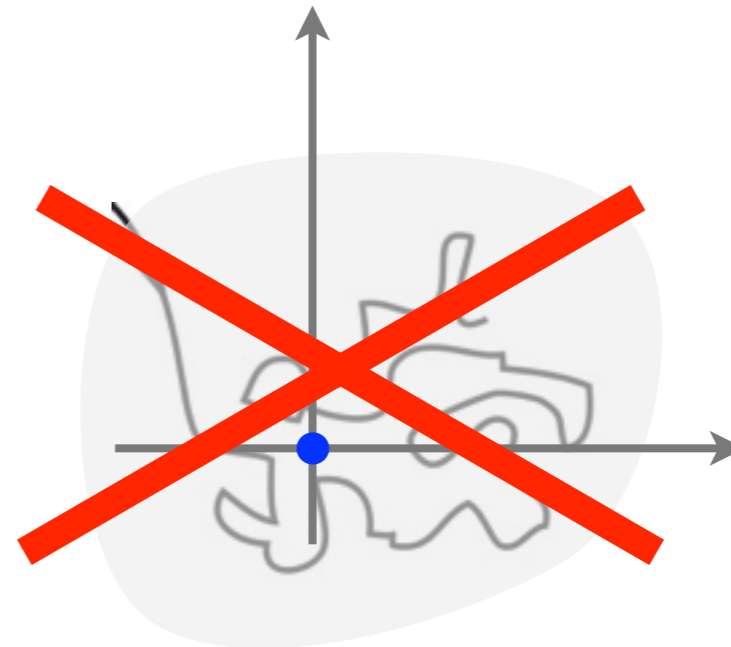
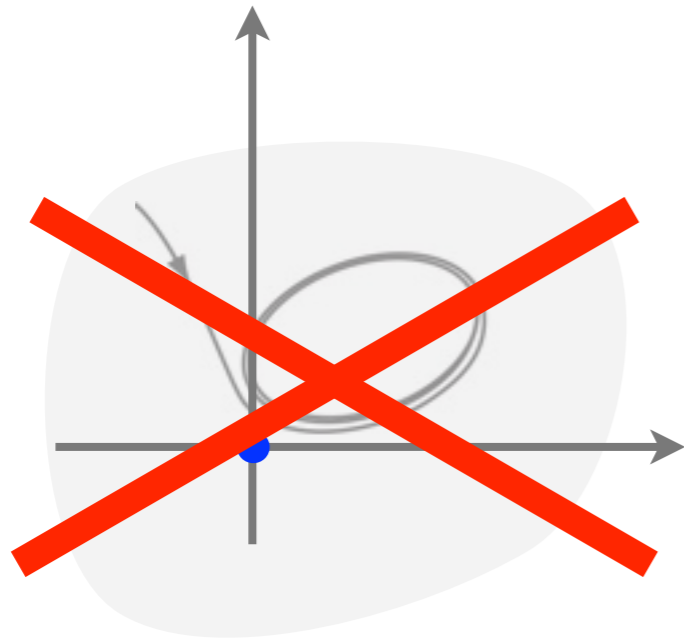
$$\chi_{IJ} > 0$$



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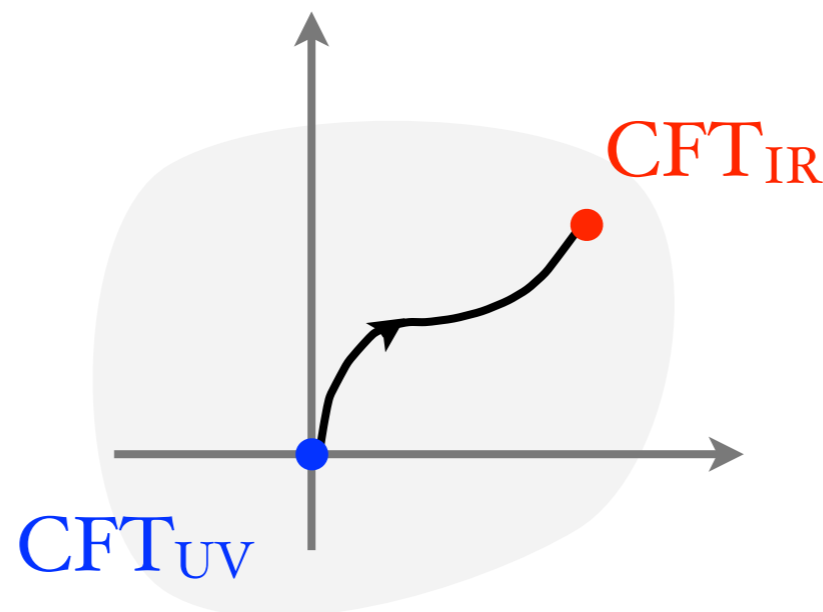
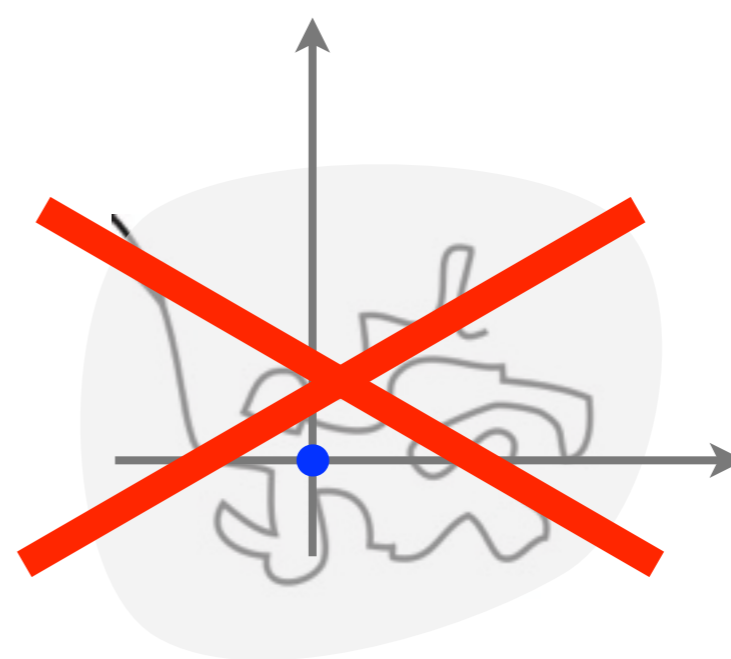
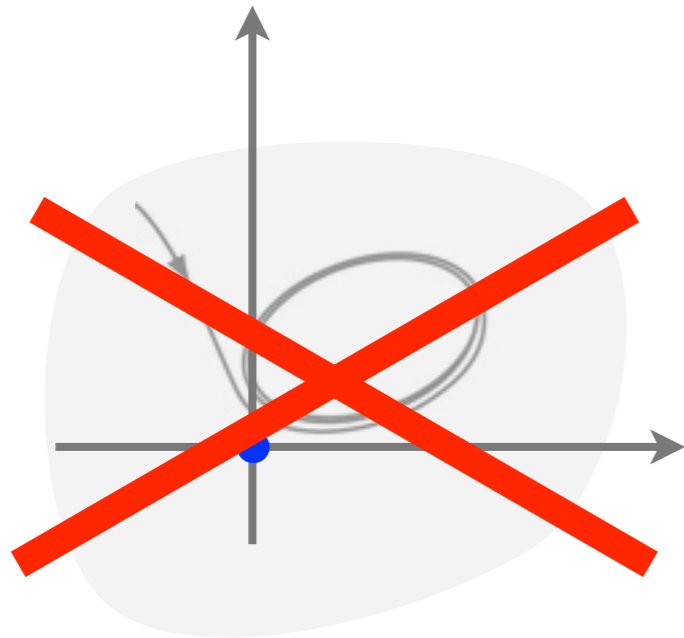
$$\delta\mu \frac{d\tilde{a}}{d\mu} \equiv \delta B^I \partial_I \tilde{a} = \chi_{IJ} B^I B^J$$

$$\chi_{IJ} > 0$$



Near CFT fixed point, irreversibility of RG flow concretely expressed by

$$\delta\mu \frac{d\tilde{a}}{d\mu} \equiv \delta B^I \partial_I \tilde{a} = \chi_{IJ} B^I B^J \quad \chi_{IJ} > 0$$



Only option

Luty, Polchinski, RR 2012
Fortin, Grinstein, Stergiou 2012

More on the local Callan-Symanzik equation:

- Any lessons hidden in the remaining consistency condition?
- What about the special case of supersymmetry?
- What about flows around CFT that break parity?