Energy flow at colliders: from QCD to $\mathcal{N}=4$ SYM and back

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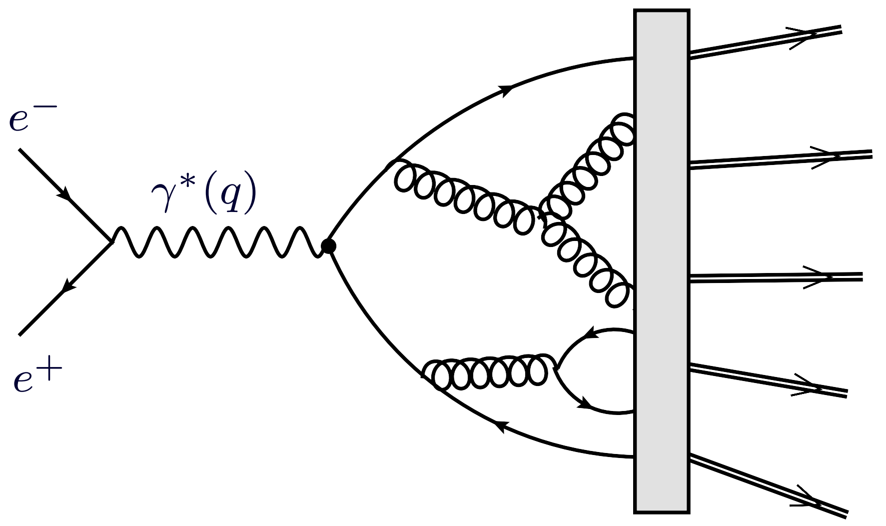
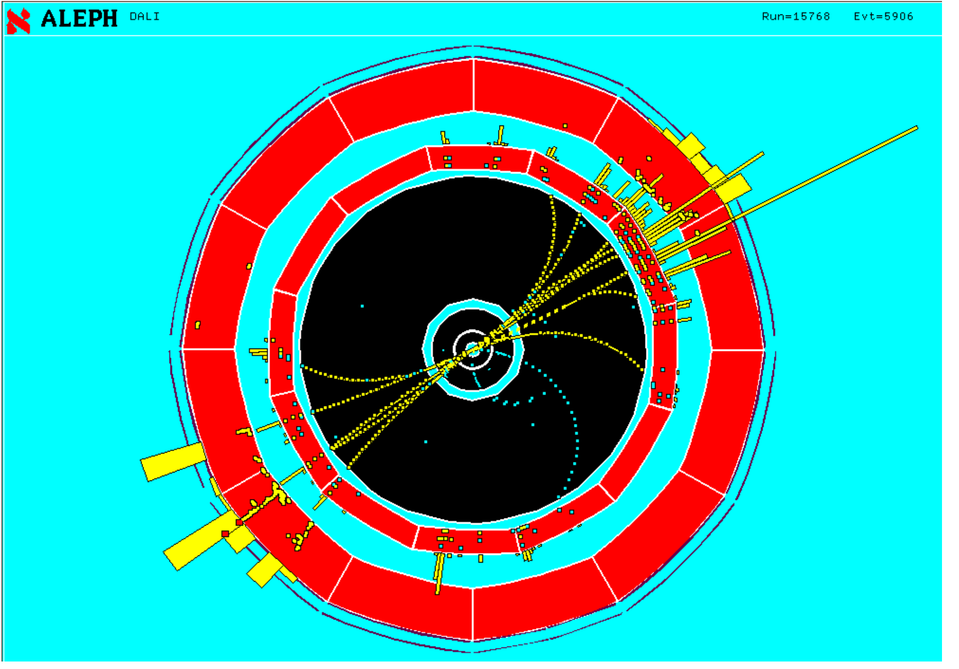
Energy flow at colliders

- ✔ e^+e^- annihilation at PETRA (1978-1986) and LEP (1989-2010)





- ✔ A virtual photon decays into an arbitrary number of quarks and gluons which go through hadronization process to become hadrons

- ✔ Final states can be described using the class of *infrared finite* observables (event shapes): energy-energy correlations (EEC), thrust, heavy mass, . . .

$$\mathrm{EEC} = \mathrm{EEC}_{\mathrm{pert}}(\alpha_s(q^2)) + \mathrm{EEC}_{\mathrm{nonpert}}(\Lambda^2_{\mathrm{QCD}}/q^2)$$

Can be computed in perturbative QCD, hadronisation corrections are ‘small’ at high energy

Conventional approach

- ✔ Event shapes are given by (an infinite) sum over the final hadronic states

$$\sigma_w(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - k_X) w(X) \left| \mathcal{M}_{\gamma^*(q)\to X} \right|^2$$

  Various event shapes correspond to different choices of the weight factor $w(X)$

- ✔ ‘Amplitude approach’ has the following disadvantages:
  - ✗ presence of intrinsic infrared divergences inside transition amplitudes $\mathcal{M}_{\gamma^*(q)\to X}$
  - ✗ integration over the phase space of the final states and subsequent intricate IR cancellations
  - ✗ necessity for summation over all final states
  - ✗ no analytical results beyond one loop

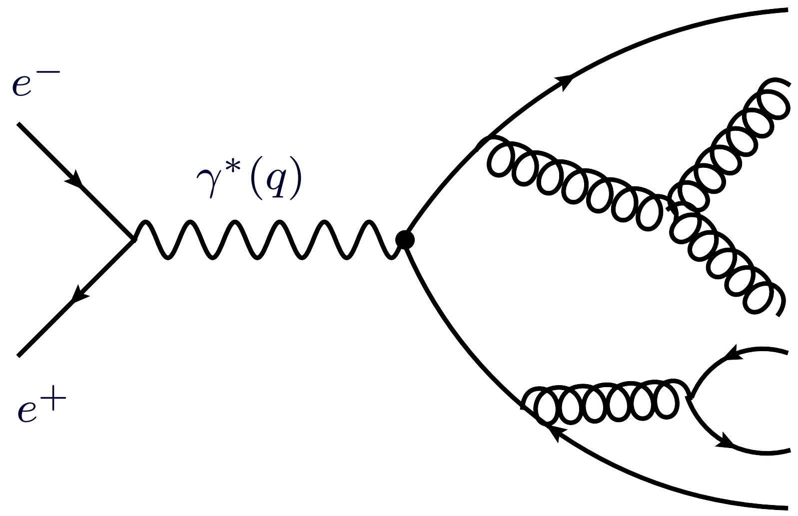
- ✔ New approach: event shapes (energy correlations) from *Wightman correlation functions*

$$\sigma_w(q) = \int d^4x\ \mathrm{e}^{iqx} \langle 0|O(x)\mathcal{E}[w]O(0)|0\rangle$$

  - ✗ no IR divergences are present in the correlation functions
  - ✗ no summation over all final states is needed
  - ✗ no integration over the phase space is required
  - ✗ strong coupling predictions (through AdS/CFT in $\mathcal{N}=4$ SYM)

e^+e^- annihilation in $\mathcal{N}=4$ SYM

- Define IR finite observables in $\mathcal{N}=4$ SYM and evaluate them both at weak/strong coupling
- Are closely related to the QCD weighted cross-sections for the final states in e^+e^-–annihilation



- From QCD to $\mathcal{N}=4$ SYM: introduce an analog of the electromagnetic current

  (protected) half-BPS operator built from the six real scalars

$$O^{IJ}_{\mathbf{20'}}(x) = \mathrm{tr}\left[\Phi^I\Phi^J - \tfrac{1}{6}\delta^{IJ}\Phi^K\Phi^K\right], \qquad (I,J=1,\dots,6)$$

$$O(x) = Y^I Y^J O^{IJ}_{\mathbf{20'}}(x) = Y^I Y^J \,\mathrm{tr}[\Phi^I(x)\Phi^J(x)]$$

  The null vector Y^I defines the orientation of the projected operator in the isotopic $SO(6)$ space

What are the properties of the final states created from the vacuum by the operator $O_{\mathbf{20'}}(x)$?

Final states in $\mathcal{N}=4$ SYM

- To lowest order in the coupling, $O(x)$ produces a pair of scalars out of the vacuum
- For arbitrary coupling, the state $O(x)|0\rangle$ can be decomposed into an infinite sum over on-shell states with an arbitrary number of scalars (s), gauginos (λ) and gauge fields (g)

$$\int d^4x\, \mathrm{e}^{iqx}\, O(x)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \dots$$

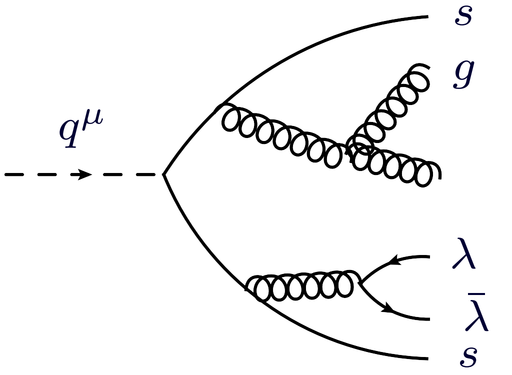
- The amplitude of creation of a particular final state $|X\rangle$ out of the vacuum

$$\langle X| \int d^4x\, \mathrm{e}^{iqx}\, O(x)|0\rangle = (2\pi)^4 \delta^{(4)}(q-p_X)\mathcal{M}_{O_{\mathbf{20'}}\to X}$$

  p_X is the total momentum of the state $|X\rangle$

- The amplitude $\mathcal{M}_{O\to X}$ has the meaning of a (IR divergent) form-factor

$$\mathcal{M}_{O_{\mathbf{20'}}\to X} = \langle X|O(0)|0\rangle$$



Total cross-section of $O_{\mathbf{20'}} \to \mathrm{everything}$

- Analog of the QCD process $\mathrm{e}^+\,\mathrm{e}^- \to$ everything

$$\sigma_{\mathrm{tot}}(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) |\mathcal{M}_{O_{\mathbf{20'}} \to X}|^2$$

- To lowest order in the coupling, the production of a pair of scalars

$$\sigma_{\mathrm{tot}}(q) = \frac{1}{2}(N^2-1) \int \frac{d^4k}{(2\pi)^4} (2\pi)^2 \delta_+(k^2) \delta_+((q-k)^2) + \dots$$

- To higher order in the coupling, each term in the sum $\sum_X$ has IR / collinear divergences
- How to avoid divergences? Use the completeness condition $\sum_X |X\rangle\langle X| = 1$

$$\sigma_{\mathrm{tot}}(q) = \int d^4x \ \mathrm{e}^{iqx} \sum_X \langle 0|O(0)|X\rangle \, \mathrm{e}^{-ixp_X} \, \langle X|O(0)|0\rangle$$

$$= \int d^4x \ \mathrm{e}^{iqx} \, \langle 0|O(x)O(0)|0\rangle$$

The operators are not time ordered!

Wightman correlation function (protected for half-BPS operators)

- All-loop result in $\mathcal{N}=4$ SYM [van Neerven]

$$\sigma_{\mathrm{tot}}(q) = \frac{1}{16\pi}(N^2-1)\theta(q^0)\theta(q^2)$$

Perturbative corrections cancel order by order

Weighted cross-section

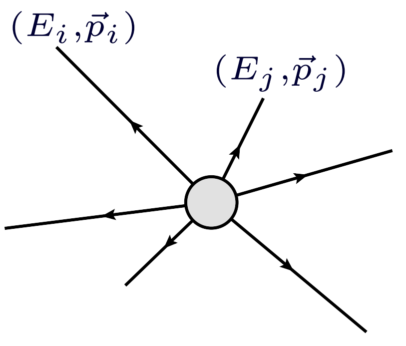
- More refined information about the final states in $O_{\mathbf{20'}} \to$ everything
- Assign a weight factor $w(X)$ to the contribution of each state $|X\rangle$

$$\sigma_W(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) w(X) |\mathcal{M}_{O_{\mathbf{20'}} \to X}|^2$$
$$= \int d^4x \ \mathrm{e}^{iqx} \sum_X \langle 0|O(x)|X\rangle w(X) \langle X|O(0)|0\rangle$$

- Less inclusive quantity as compared with the total cross section, no optical theorem
- Choose of the weight factors $w(X)$ gives an access to the flow of various quantum numbers of particles (energy, charge, etc) in the final state
- Popular choice – energy-energy correlations

[Basham,Brown,Ellis,Love]

$$w(X) = \sum_{i,j} E_i E_j \delta(\cos\theta_{ij} - \cos\chi)$$



Are known in QCD up to 2 loops numerically

Energy flow

- The total energy in the final state $|X\rangle = |k_1,\dots,k_\ell\rangle$ that flows into the detector located at spatial infinity in the direction of the vector $\vec n$.

$$w_{\mathcal E}(k_1,\dots,k_\ell) = \sum_{i=1}^{\ell} k_i^0\,\delta^{(2)}(\Omega_{\vec k_i} - \Omega_{\vec n})\,,$$

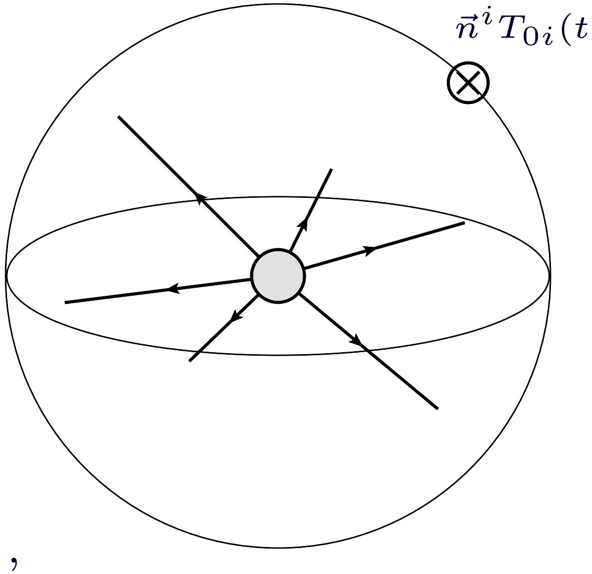
- Energy flow operator

$$\mathcal{E}(\vec n)|X\rangle = w_{\mathcal E}(X)|X\rangle\,.$$

- Is expressed in terms of the energy-momentum tensor in $\mathcal{N}=4$ SYM

[Sveshnikov,Tkachov],[GK,Oderda,Sterman]

$$\mathcal{E}(\vec n) = \int_0^\infty dt\,\lim_{r\to\infty} r^2\,\vec n^i T_{0i}(t, r\vec n)$$



- Representation for $\mathcal{E}(\vec n)$ in terms of creation and annihilation operators of on-shell states

$$\mathcal{E}(\vec n) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta_+(k^2)\,k^0\,\delta^{(2)}(\Omega_{\vec n} - \Omega_{\vec k}) \sum_{i=s,\lambda,\bar\lambda,g} a_i^\dagger(k)a_i(k)\,,$$

Energy correlations

- Single correlator

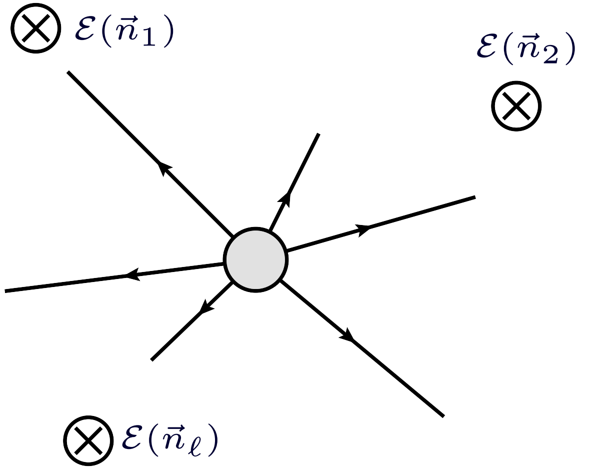
$$\sum_X \langle 0|O(x)|X\rangle w_{\mathcal{E}}(X)\langle X|O(0)|0\rangle = \sum_X \langle 0|O(x)\mathcal{E}(\vec{n})|X\rangle\langle X|O(0)|0\rangle = \langle 0|O(x)\mathcal{E}(\vec{n})O(0)|0\rangle$$

Wightman correlation function (no time ordering!) due to real-time evolution

- Single energy flow

$$\langle \mathcal{E}(\vec{n}_1)\rangle = \sigma_{\rm tot}^{-1}\int d^4x\ \mathrm{e}^{iqx}\langle 0|O(x)\,\mathcal{E}(\vec{n}_1)\,O(0)|0\rangle$$

- Multi-energy correlations [GK,Sterman],[Belitsky,GK,Sterman],[Hofman,Maldacena]



$$\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle$$
$$= \sigma_{\rm tot}^{-1}\int d^4x\ \mathrm{e}^{iqx}\langle 0|O(x)\,\mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\,O(0)|0\rangle$$

Energy flow in the direction of $\vec{n}_1,\ldots,\vec{n}_\ell$

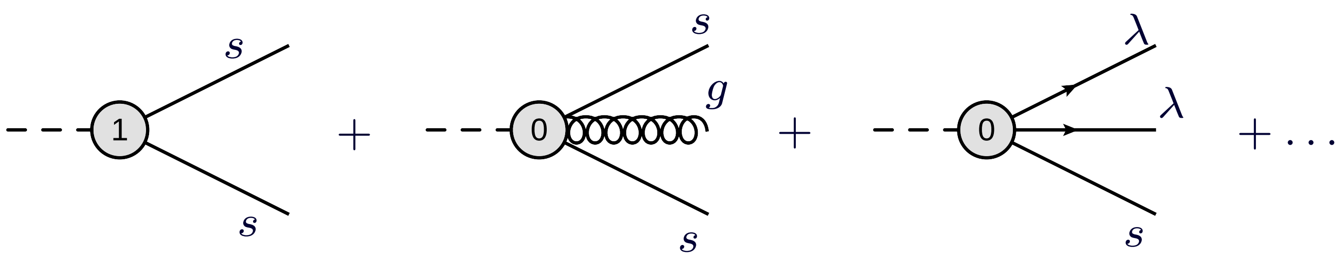
Depends on the relative angles $\cos\theta_{ij} = (\vec{n}_i\cdot\vec{n}_j)$

- The goal is to find $\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle$ for arbitrary coupling in $\mathcal{N}=4$ SYM

Energy correlations from amplitudes

✔ Transition amplitude at one loop

$$\mathcal{M}_{O_{\mathbf{20'}}\to X} =$$



✔ Energy correlations

$$\sigma_{\mathcal{E}}(q) = \int \mathrm{dPS}_2\, w_{\mathcal{E}}(1,2)\, |\mathcal{M}_{O_{\mathbf{20'}}\to ss}|^2 + \int \mathrm{dPS}_3\, w_{\mathcal{E}}(1,2,3) \Big(|\mathcal{M}_{O_{\mathbf{20'}}\to ssg}|^2 + |\mathcal{M}_{O_{\mathbf{20'}}\to s\lambda\lambda}|^2 \Big) + \dots$$

✗ Single detector correlation (protected from loop corrections)

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q_0}{4\pi}$$

✗ Two detectors oriented along $\vec{n}_i$ (unprotected quantity) [Zhiboedov],[Engelund,Roiban]

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle = -\frac{q_0^2}{(4\pi)^4} \left[-a \frac{\ln(1-z)}{2z^2(1-z)} + O(a^2) \right], \qquad (\vec{n}_1 \vec{n}_2) = \cos\theta_{12}$$

The scaling variable in the rest frame of the source $z = (1-\cos\theta_{12})/2$

✗ Two-loop corrections to $\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle$ are hard to compute ($\sim 10^2$ diagrams)

Energy correlations from correlation functions I

- Energy flow operator

$$\langle \mathcal{E}(\vec{n}_1)\rangle \sim \int d^4x\, \mathrm{e}^{iqx} \langle 0|O(x)\,\mathcal{E}(\vec{n}_1)\,O(0)|0\rangle$$

$$= \underbrace{\int d^4x\, \mathrm{e}^{iqx}}_{\text{Fourier}} \underbrace{\int_0^\infty dt \lim_{r\to\infty} r^2}_{\text{Detector limit}} \underbrace{\langle 0|O(x)\,T_{0\vec{n}_1}(x_1)\,O(0)|0\rangle}_{\text{Wightman corr. function}}\Big|_{x_1 = (t, r\vec{n}_1)}$$

- Generalization for ℓ detectors

$$\langle \mathcal{E}(\vec{n}_1)\ldots\mathcal{E}(\vec{n}_\ell)\rangle = \text{Fourier} \times \text{Limit} \left[\langle 0|O(x)\,T_{0\vec{n}_1}(x_1)\ldots T_{0\vec{n}_\ell}(x_\ell)\,O(0)|0\rangle\Big|_{x_i=(t_i,r_i\vec{n}_i)}\right]$$

- How to compute energy flow correlators:
  - Start with corr.function $\langle O(x)T(x_1)\ldots T(x_\ell)O(0)\rangle$ in Euclid
  - Continue to Minkowski with Wightman prescription
  - Take detector limit + perform Fourier
- Correlation functions in $\mathcal{N}=4$ SYM have a lot of symmetry :
  - $\langle O(x)T(x_1)O(0)\rangle$ is fixed by conformal symmetry $\to$ exact result for $\langle \mathcal{E}(\vec{n}_1)\rangle$ [Hofman,Maldacena]
  - $\langle O(x)T(x_1)T(x_2)O(0)\rangle$ is not fixed by conformal symmetry

Energy correlations from correlation functions II

Hidden beauty of $\mathcal{N}=4$ SYM:

- ✔ Quantum corrections to various correlation functions are determined by the same scalar function

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle_E = \frac{1}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}\Phi(u,v;a)$$

$$\langle O(x_1)T(x_2)T(x_3)O(x_4)\rangle_E = \frac{1}{(x_{12}^2x_{23}^2x_{34}^2)^2}P(\partial_u,\partial_v)\Phi(u,v;a)$$

Conformal ratios

$$u = x_{12}^2x_{34}^2/(x_{13}^2x_{24}^2)\,, \qquad v = x_{23}^2x_{41}^2/(x_{13}^2x_{24}^2)$$

- ✔ Universal function in $\mathcal{N}=4$ SYM at weak coupling [Eden,Schubert,Sokatchev],[Bianchi et al]

$$\begin{aligned}\Phi(u,v) =& a\,\Phi^{(1)}(u,v) + a^2\biggl(\frac{1}{2}(1+u+v)\left[\Phi^{(1)}(u,v)\right]^2 \\ &+ 2\left[\Phi^{(2)}(u,v) + \frac{1}{u}\Phi^{(2)}(v/u,1/u) + \frac{1}{v}\Phi^{(2)}(1/v,u/v)\right]\biggr) + O(a^3)\end{aligned}$$

$\Phi^{(1)}(u,v)$ 'box' integral, $\Phi^{(2)}(u,v)$ 'double' box integral

- ✔ AdS/CFT prediction for Euclidean $\Phi(u,v)$ at strong coupling [Arutyunov, Frolov]

From Euclid to Minkowski

- Brute force method: compute anew using Schwinger-Keldysh technique (too hard)
- Better method: analytically continue correlation functions from Euclid to Minkowski+Wightman
- Warm-up example: free scalar propagator $D_{\rm Euclid}(x) = \langle \phi(x)\phi(0)\rangle \sim 1/x^2$

$$\begin{aligned}\langle 0|\phi(x)\phi(0)|0\rangle &= \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle \\ &= \sum_{E_n>0} \mathrm{e}^{-iE_n(x^0-i0)+i\vec{p}\vec{x}}\langle 0|\phi(0)|n\rangle\langle n|\phi(0)|0\rangle \sim \frac{1}{(x^0-i0)^2-\vec{x}^2}\end{aligned}$$

- How to get Wightman correlation functions ('magic' recipe): [Mack]
  - Go to Mellin space:

$$\Phi_{\rm Euclid} = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1,j_2;a)\, u^{j_1} v^{j_2}\,, \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

  - Nontrivial Wick rotation

$$\Phi_{\rm Wightman} = \Phi_{\rm Euclid}\left(x_{ij}^2 \quad\to\quad x_{ij,+}^2 = x_{ij}^2 - i0\cdot x_{ij}^0\right)$$

- $M(j_1,j_2;a)$ is known both at weak and strong coupling in planar $\mathcal{N}=4$ SYM

All-loop prediction for energy correlations

✔ Energy correlations for arbitrary coupling

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle = \frac{1}{(4\pi^2)^2}\frac{q^2}{(n_1 n_2)^3}\mathcal{F}_{\mathcal{E}}(z;a)\,, \qquad z = (1-\cos\theta_{12})/2$$

✔ All-loop prediction

$$\mathcal{F}_{\mathcal{E}}(z;a) = \int_{-\delta-i\infty}^{-\delta+i\infty}\frac{dj_1 dj_2}{(2\pi i)^2}\underbrace{M(j_1,j_2;a)}_{\text{corr.function}}\underbrace{K_{\mathcal{E}}(j_1,j_2)}_{\text{detector}}\left(\frac{1-z}{z}\right)^{j_1+j_2}$$

Detector function is independent on the coupling

$$K_{\mathcal{E}}(j_1,j_2) \sim \frac{\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

$$M(j_1,j_2;a) = \underbrace{aM^{(1)}(j_1,j_2) + a^2 M^{(2)}(j_1,j_2)}_{\text{are known}} + \dots$$

✗ Weak coupling: $\mathcal{F}_{\mathcal{E}}(z;a<1) = \frac{a}{4}\frac{z\ln(1/(1-z))}{(1-z)} + a^2$*[Long expression]*$+ O(a^3)$

✗ Strong coupling: $\mathcal{F}_{\mathcal{E}}(z;a\to\infty) = 8\,z^3 + O(1/a)$ [Hofman,Maldacena]

✔ $\mathcal{F}_{\mathcal{E}}(z;a)$ is a regular, positive function of $0 \le z \le 1$ for any coupling, away from the planar limit !

Conclusions and open questions

- ✔ Energy correlations are good/nontrivial physical observables in $\mathcal{N}=4$ SYM
- ✔ Nontrivial constrains on the correlation functions in CFT coming from $\mathcal{F}_{\mathcal{E}}(z;a)>0$
- ✔ Relation to energy flow correlations in QCD (most complicated part)?
- ✔ All symmetries of $\mathcal{N}=4$ SYM are preserved, what is the manifestation of integrability?
- ✔ Interpolation between weak and strong coupling?
- ✔ Other proposals for 'good' observables?