The disk entanglement entropy of free Maxwell

Daniel L. Jafferis

Harvard University

Cesar Agon, Matthew Headrick, DLJ, Skyler Kasko
(to appear soon)

DESY

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- Motivation: 3d F-theorem
- Free 3d Maxwell is not a CFT
- Calculation using replica trick
- Hilbert space interpretation

F-theorem

- Finite term in the disk entanglement entropy:
  - works away from conformality
  - proof of monotonicity
  - hard to calculate and even define

[Myers Sinha]

- Log of the S^3 partition function:
  - inequality of values at UV and IR conformal fixed points
  - much easier to calculate

[DLJ Klebanov Pufu Safdi]

c-theorems in various dimensions

- A measure of the number of degrees of freedom in interacting field theories. It should decrease along rg flow.

- Most obvious conjecture is the thermal free energy. Not constant along conformal manifolds. Also, in 3d, the critical O(N) model is a counter-example.

c-theorems in various dimensions

- In 2d, the coefficient of the trace anomaly has this property. RG flow is the gradient flow for this quantity.

Zamolodchikov

- In 4d, $16\pi^2 \langle T^\mu{}_\mu \rangle = c(\text{Weyl})^2 - 2a(\text{Euler})$, and it is a that plays this role.

Komargodski Schwimmer

- In odd dimensions, there are no anomalies, so this has long been an open problem.

Sphere partition function

- Any conformal field theory can be put on the sphere – it is conformal to flat space.
- IR finite observable in odd dim, but non-local.
- Weyl invariance is maintained, hence 1-point functions vanish, and Z is constant along conformal manifolds.

S^3 partition function well-defined

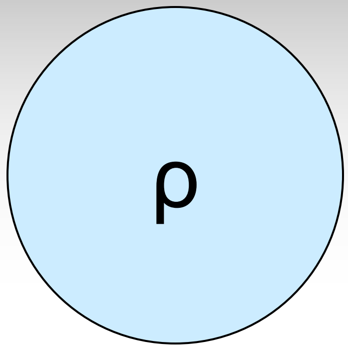
- In general, a calculation in an effective theory with a lower cutoff $\Lambda' < \Lambda$ differs by a local effective action for the background fields.

$$\int \sqrt{g}, \qquad \int \sqrt{g}\, \mathcal{R}$$

- In odd dimensions, no such counter term can integrate to a number, so the finite term is well-defined.

- Would be interesting to develop the analogous theory to deal with divergences in entanglement entropy.

Entanglement entropy



$$S_{ent} = -\mathrm{Tr}(\rho \log \rho)$$

- Shown that $-S + r\partial_r S$ is smaller for the IR fixed point than in the UV, *when* the Hilbert factorizes on the lattice.

[Casini Huerta]

- Equal to the sphere partition function for conformal field theories

[Casini Huerta Myers]

Counts topological d.o.f

- The original motivation for studying entanglement entropy in QFT was in condensed matter physics as a way to characterize topological order in gapped systems.
- For example, WZW edge states contribute in pure Chern-Simons.
- Therefore, F cannot be determined only from local correlation functions (likewise for S^3 free energy).

Replica trick

- There is a path integral description of the Renyi entropies, $$S_n(r) = -\frac{\log(\mathrm{tr}\rho^n)}{n-1}$$

- ϱ is given in the semi-classical wavefunctional of fields basis as the path integral on R^3 with boundary conditions on across the disk.

- One obtains an n-sheeted branched cover.

$$\mathrm{tr}\rho^n = Z_n / Z_1^n$$

From entanglement to spheres

[Casini Huerta Myers]

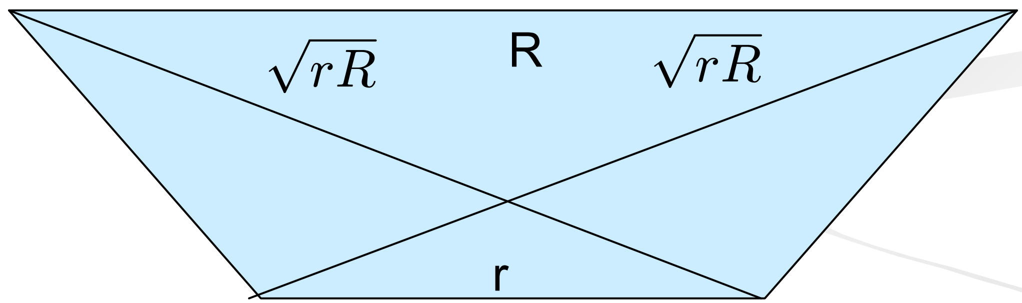
- The ball in flat space is conformal to the hemisphere in $S^{d-1} \times R$. Its casual diamond can be mapped by a time dependent Weyl rescaling to the static patch in de Sitter. The CFT vacuum maps to the euclidean vacuum.
- The reduced density matrix of any QFT in the static patch is thermal at the dS temperature.
- Analytic continuation of the static patch is S^d.

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \qquad -S = \mathrm{tr}(\rho \log \rho) = \mathrm{tr}(\rho(-\beta H - \log Z)) = -\log Z$$

Sketch of the proof of the monotonity of entenglement entropy

- Strong sub-additivity

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$



$$2S(\sqrt{rR}) \geq S(R) + S(r)$$

[Casini Huerta]

$$rS(r) + S(r) \leq 0$$

Known values

- For massless fields, can compute via the sphere partition function. Massive fields known in an expansion in 1/(r m).

$$F_{\text{scalar}} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2} \qquad F_{\text{fermion}} = \frac{\log 2}{8} + \frac{3\zeta(3)}{16\pi^2}$$

[Klebanov Pufu Safdi]

- For abelian Chern-Simons theory at level k.

$$F_{\text{CS}} = \frac{1}{2}\log k$$

Exotic rg flows?

- Consider a weakly coupled abelian Chern-Simons-matter CFT. $F \approx \frac{1}{2}\log k + N_f^{(UV)}$

- The F-theorem permits flows to IR theories with $N_f^{(IR)} > N_f^{(UV)}$.

- No examples are known. Maybe there a stronger constraint? Or we should find such flows.

Non-conformality of Maxwell

- The stress tensor is $T_{\mu\nu} = F_{\nu\rho}F^{\rho}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F^2$
- This is not traceless $T^{\mu}_{\mu} = \frac{4-d}{4}F^2$
- Nor is the virial current $V_\mu = \frac{d-4}{2}A^\nu F_{\mu\nu}$ gauge invariant.
- But charge associated to $\Delta_\mu = x^\nu T_{\mu\nu} + V_\mu$ exists.

[El-Showk Nakayama Rychkov]

Dual photon description

- It's useful to dualize to a free compact scalar with radius 2πg.

$$\partial_\mu \phi = \epsilon_{\mu\nu\rho}\partial^\nu A^\rho$$

- This is gauging the shift symmetry of the scalar with a Z-gauge field.

$$\int (\partial_\mu \phi + 2\pi g A_\mu)(\partial^\mu \phi + 2\pi g A^\mu) + 4\pi i \ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho$$

- The monopole operator is $e^{i\phi/g}$.

Spontaneous symmetry breaking

- In d > 2 dimensions, the shift symmetry of the compact scalar theory is spontaneously broken. The field takes a definite value, say 0, at infinity in the superselection sector of the vacuum.

- Otherwise cluster decomposition would be violated for correlation functions of $e^{i\phi/g}$

- Corresponds to monopole condensation.

S^3 partition function

- The theory cannot be conformally coupled because the scalar is compact (no improvement term, it's not a CFT, nor does virial current exist).

- There is a zero mode integral, proportional to the circle size, g. Together with the one loop determinant, one finds a logarithmic dependence

$$F = -\frac{1}{2}\log(g^2 r) + \frac{\zeta(3)}{4\pi^2}$$

- Can't see the symmetry breaking on the sphere.

[Klebanov Pufu Sachdev Safdi]

Maxwell-Chern-Simons rg flow

- Consider Maxwell-Chern-Simons at level k. This represents an rg flow from free Maxwell to the gapped Chern-Simons theory.

- The crossover scale is g^2k, so $F_{\mathrm{Maxwell}}(r = \frac{1}{g^2 k}) \sim \frac{1}{2}\log k$

- This is valid for any large k (weak coupling), so for large r, one expects $F_{\mathrm{Maxwell}}(r) \sim -\frac{1}{2}\log(rg^2)$

Instanton sum

- In the n-fold branched cover of R^3, there are n asymptotic regions. The field must go to 0 mod 2πg there.

- This results in a sum over winding sectors $(0, w_1, , w_{n-1})$. The fluctuation determinants are all equal.

$$Z_n = Z_{\text{scalar}}^{\text{1-loop}} \sum_w e^{-S(w)} \qquad S(w) = (2\pi g)^2 r \sum_{jk} (M_n)_{jk} w^j w^k$$

UV limit

- At short distances, the sum becomes an integral

$$Z_n^{\text{inst}}(r) = \sum_w e^{-4\pi^2 g^2 r (M_n)_{jk} w^j w^k} \sim \left(\det(\frac{4\pi^2 g^2 r M_n}{\pi})\right)^{-1/2}$$

- The Renyi entropies become

$$\Delta S_n(r) = \tfrac{1}{2}\log(g^2 r) + \tfrac{1}{2(n-1)}\log\det M_n$$

- Thus,

$$F \approx \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2} + \frac{1}{2} + \frac{\gamma}{2} - \frac{1}{2}\log(rg^2)$$

Finding the instanton solutions

- It’s convenient to regard the field on the n-sheeted cover as an n-vector with monodromy.

- One can then go to a diagonal complex basis.

$$(M_n)_{jk} = \frac{1}{n}\sum_{m=1}^{n-1} e^{2\pi i(j-k)m/n} J(k/n)$$

where $J(\beta) = \frac{1}{2}\int d^3x \partial_\mu \phi^*_\beta \partial^\mu \phi_\beta$ is the action for a solution with multiplicative cut $e^{2\pi i\beta}$

On-shell action

- The on-shell action is equivalent to a boundary integral, which equals the flux since the field approaches 1 at infinity.

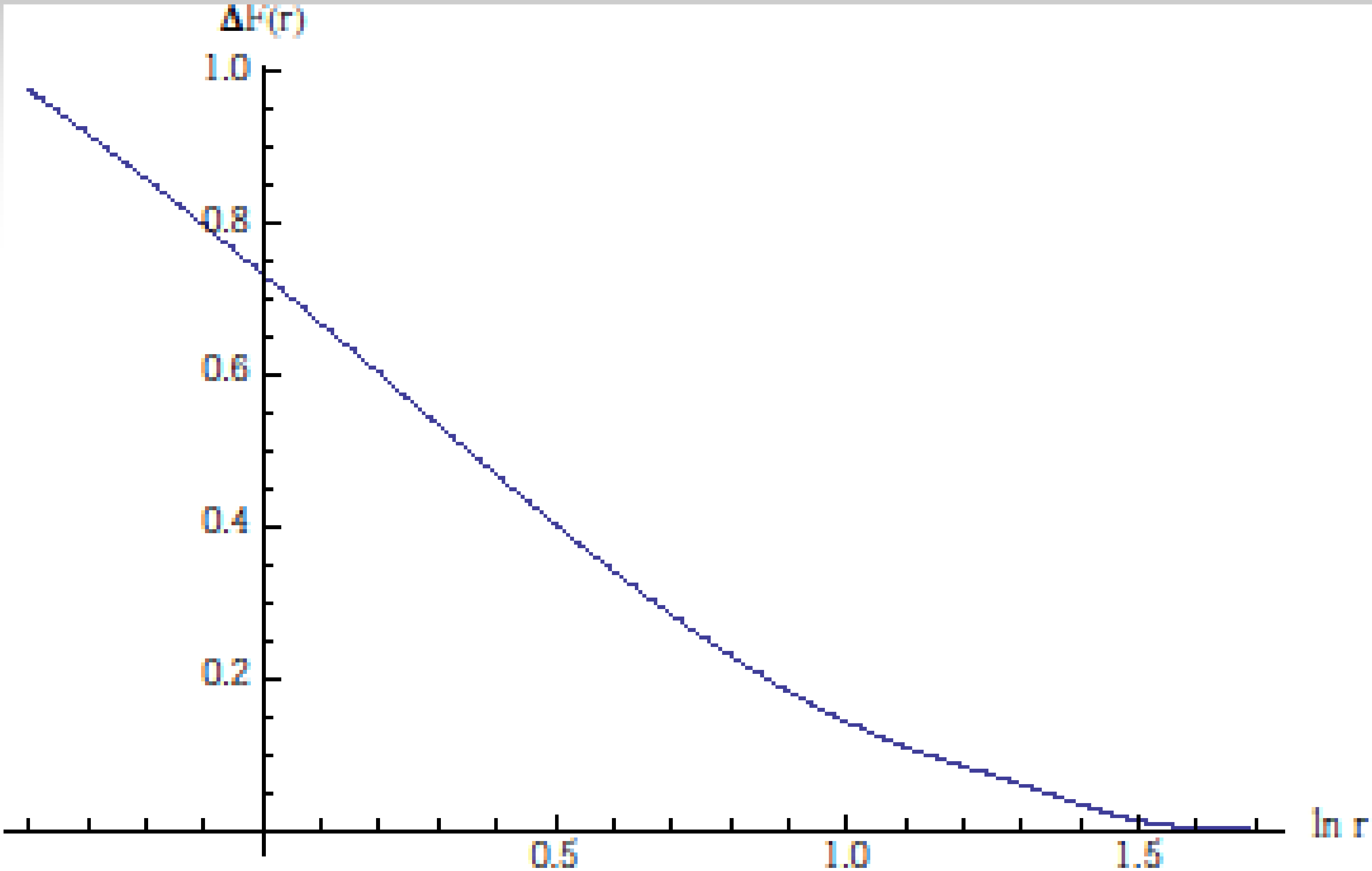
- Actually found the solutions in a convenient basis of harmonic functions numerically.

$$J(\beta) = 2\pi(1 - 2\beta)\tan \pi\beta$$

Numerical extension

- Unknown how to continue Θ function in n.
- Fitting with rational functions works much better than polynomials.
- Check that convergence is good by approximating the known $S_2(r)$ from higher $S_n(r)$, and checking at small r limit.

F(r)



Hilbert space structure

- The Hilbert space of the compact scalar is the same as the ordinary scalar in flat space, due to spontaneous symmetry breaking by the vacuum.

- However, the Hilbert spaces in the disk (really a sum of selection sectors with different boundary conditions) differ between the two theories.

Disk Hilbert space

- In continuum QFT, one has continuity across the disk, moreover the field value there defines boundary conditions, $\mathcal{H}_{\text{physical}} = \oplus_f \, \mathcal{H}_f^{\text{in}} \otimes \mathcal{H}_f^{\text{out}}$

- Suggests that the reduced density matrix is block diagonal in the boundary field basis (superselection).

- Can check this using the fact that $\rho = e^{-K}$ for conformal field theories. K leaves scalar primaries at the boundary invariant.

$$[\rho, \Phi|_{S^1}] = 0$$

Averaging versus interfering

- In the compact scalar, the wavefunctionals must be invariant under the shift, in other words one first sums the states and then forms the density matrix.

- This gives a purer density matrix than summing afterward, hence the EE deficit, F, is larger.

$$\langle \phi_+ | \rho_{\rm c} | \phi_- \rangle = \sum_n \langle \phi_+ + 2\pi g n | \rho_{\rm nc} | \phi_- + 2\pi g n \rangle$$

Why does F depend on such subtleties?

- It might seem strange that to obtain an rg monotonic quantity one needs something that depends on more data than the Hilbert space and Hamiltonian in flat space.
- However, in principle more rg flows (in the space of local QFTs) are allowed if fewer operators in the UV are considered to be local.

Punchlines

- The 3d compact Maxwell theory flows from the noncompact R gauge theory to the free noncompact scalar.
- Found the interpolating F(r). It diverges in the UV.
- F depends on more data than H and the Hilbert space in Minkowski space.