Nekrasov Backgrounds from N=2 String Amplitudes

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Based on work with

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Introduction

Topological strings have played a significant role in recent developments of String Theory

- "Twisted" versions of String Theory
- Topological (-semi) dynamics on the worldsheet & spacetime
- Topological amplitudes : subsector of physical string amplitudes
- Insight into non-perturbative physics
- Applications in supersymmetric gauge theories (microstate counting for BPS black holes, wall-crossing, etc)
- Relation to Matrix Models
- ...

Topological String Theory

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3(g-1)} |G^-(\mu_i)|^2 \rangle$$

Topological String partition function

What is the relation of F_g to the full String Theory ?

Certain N=2 F-terms of Type II String Theory on a CY_3 at genus g are precisely computed by the topological string partition function F_g

Bosonize U(1) current $J = i\sqrt{3}\,\partial H$

Twisting the energy-momentum tensor is equivalent to deforming the worldsheet action by

$$\frac{-i\sqrt{3}}{8\pi} \int d^2 z \, R^{(2)} H$$

On a genus-g Riemann surface pick a metric such that $R^{(2)} = -\sum_{i=1}^{2g-2} \delta^{(2)}(z - z_i)$

$$F_g = \left\langle \prod_{i=1}^{2g-2} e^{i\frac{\sqrt{3}}{2}(H+\bar{H})} \times \prod_{i=1}^{3(g-1)} |G^-|^2 \times \dots \right\rangle_{\text{untwisted}}$$

graviphoton vertex operator in (-1/2,-1/2) ghost picture

Same as computing an amplitude in the untwisted string theory involving 2(g-1) graviphoton vertex operators

Topological String Theory

Antoniadis, Gava, Narain, Taylor 1993

Topological string partition function computes 1/2 BPS F-terms in the effective action:

$$\int d^4x\ d^4\theta\ F_g(X)\,(W^{ij}_{\mu\nu}W^{\mu\nu}_{ij})^g = \int d^4x\ F_g(\phi)R_{(-)\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}_{(-)}\ (F^G_{(-)\lambda\tau}F^{G,\lambda\sigma}_{(-)})^{g-1} + \ldots$$

NO perturbative or non-perturbative corrections to the F_g in Type II

- F_0 is identified with the Seiberg-Witten prepotential
- For g>1 the F_g are interpreted as higher derivative gravitational corrections
- Relation to Nekrasov partition function for N=2 SU(2) gauge theory

$$\sum_{g=0}^{\infty} g_s^{2g-2}\,F_g\Big|_{\text{field theory}} = \log\, Z_{\text{Nek}}(\epsilon_+ = 0, \epsilon_- = g_s)$$

Refinement : one-parameter deformation that captures also ε_+

Omega background & Nekrasov partition function

Nekrasov's partition function is defined as the trace

$$Z_{\rm Nek}(\epsilon_+,\epsilon_-) = {\rm Tr}\ (-)^F\, e^{-2\epsilon_- J^3_-}\, e^{-2\epsilon_+(J^3_+ + J^3_R)}$$

over the 4d Hilbert space of the N=2 theory

Can be regarded as the vacuum amplitude in the Omega background

- J^3_+ , J^3_- are the Cartan generators of $SU_+(2) \times SU_-(2)$ Lorentz group
- J^3_R is the Cartan current of SU(2) R-symmetry

The perturbative result is

$$Z_{\rm Nek}(\epsilon_+,\epsilon_-) = \int_0^\infty \frac{dt}{t}\, \frac{-2\cos(2\epsilon_+ t)}{\sin(\epsilon_- - \epsilon_+)t\ \sin(\epsilon_- - \epsilon_+)t}\, e^{-\mu t}$$

Nekrasov 2002
Nekrasov, Okounkov 2003

"refinement" of the Topological String !

$$\frac{1}{\epsilon_-^2 - \epsilon_+^2} \sum_{g,n=0}^{\infty} \epsilon_-^{2g}\epsilon_+^{2n}\, F_{g,n} = \log\ Z_{\rm Nek}(\epsilon_+,\epsilon_-)$$

Refinement of the Topological String

Ideally, a good proposal for refinement should satisfy the following requirements :

- ☑ Unrefined limit
- ☑ Exact sigma model
- ☑ Correct Field Theory limit

General Strategy

Exploit the connection of the unrefined Topological String to BPS-saturated string amplitudes

Consider perturbative string amplitudes as a definition of the worldsheet partition function of the refined topological string

No convincing proposal for refinement at the worldsheet level so far (twisted CFT)...

2 proposals in the literature...

- Antoniadis, Hohenegger, Narain & Taylor 2010
- Nakayama & Ooguri 2011

Proposals for Refinement so far

- Antoniadis, Hohenegger, Narain & Taylor 2010
- Nakayama & Ooguri 2011

Additional insertions of self-dual vertices $V_{(+)}$

AHNT '10

- Field strength of vector partners of the $\bar{S}$ modulus
- First computed by Morales & Serone (1996)
- Exact string amplitude
- Fails to reproduce the Nekrasov partition function (misses an ε_+ dependent phase)

NO '11

- Field strengths of vector partners of $\bar{T}, \bar{U}$ moduli + FI terms
- Claims to reproduce the Nekrasov partition function
- Cannot be evaluated exactly as a string amplitude
- Higher ε-corrections
- Non-compact only

A New Proposal for Refinement

Consider the refinement to be defined in terms of physical amplitudes that compute generalized N=2 F-terms of the form

$$I_{g,M} = \int d^4x\, d^4\theta\, (W^2)^g\, \Upsilon^M$$

Chiral Weyl superfield

Superfield defined as an N=2 chiral projection of the anti-chiral vector multiplet $\bar{T}$

$$\Upsilon = (\epsilon_{ij} \bar{D}^i \bar{\sigma}_{\mu\nu} \bar{D}^j)^2 \frac{f(\hat{X}^I, (\hat{X}^I)^\dagger)}{(X^0)^2}$$

Allows mixing of chiral and anti-chiral vector superfields in a supersymmetric way

In terms of component fields :

$$I_{g,M} = \int d^4x\; F_{g,M} \left[R^2_{(-)} \, (F^G_{(-)})^2 + B^4_{(-)} \right] \, (F^G_{(-)})^{2(g-2)} \, (F^{\bar{T}}_{(+)})^{2M}$$

gravitino field strength

N=2 Amplitude & Vertex Operators

Antoniadis, IF, Hohenegger, Narain, Zein Assi 2013

We will calculate the 1-loop amplitude in the Heterotic dual

$$\langle R^2_{(-)}\,(F^G_{(-)})^{2g-2}\,(F^{\bar{T}}_{(+)})^{2M}\rangle \longrightarrow \langle B^4_{(-)}\,(F^G_{(-)})^{2g-2}\,(F^{\bar{T}}_{(+)})^{2M}\rangle$$

- We only capture the perturbative part of the refined amplitudes
- 4d N=2 Heterotic string on K3xT^2 in the presence of Wilson lines
- Z^1 , Z^2 , χ^1 , χ^2 : complexified spacetime
- X , ψ : complexified T^2

$$V_{\psi^\pm}(\xi_{\mu,\alpha},p) = \xi_{\mu,\alpha}\, e^{-\varphi/2} S^\alpha e^{i\phi_3/2} \Sigma^\pm \bar{\partial} Z^\mu e^{ip\cdot Z}$$

$$V^G(\epsilon,p) = \epsilon_\mu\,(\partial X - i(p\cdot\chi)\psi)\bar{\partial} Z^\mu e^{ip\cdot Z}$$

$$V^{\bar{T}}(\epsilon,p) = \epsilon_\mu\,(\partial \bar{Z}^\mu - i(p\cdot\chi)\chi^\mu)\bar{\partial} X e^{ip\cdot Z}$$

$$S^1 = e^{\frac{i}{2}(\phi_1+\phi_2)} \qquad S^2 = e^{-\frac{i}{2}(\phi_1+\phi_2)} \qquad \Sigma^\pm = e^{\pm\frac{i}{2}(\phi_4+\phi_5)}$$

Generating Functions

Since only X appears in the correlator but not its complex conjugate, the T^2 currents only contribute zero modes

The bosonic correlator is encoded into the generating function :

$$G^{\rm bos}(\epsilon_-,\epsilon_+) = \Big\langle \exp\Big[-\epsilon_- \int d^2z\; \partial X (Z^1 \bar\partial Z^2 + \bar Z^2 \bar\partial \bar Z^1) - \epsilon_+ \int d^2 z\; (Z^1 \partial \bar Z^2 + Z^2 \partial \bar Z^1) \bar\partial X \Big] \Big\rangle$$

After spin-structure sum, one obtains non-trivial K3-fermionic correlator :

$$G^{\rm ferm}\left[{}^h_g\right](\epsilon_+) = \Big\langle \exp\Big[-\epsilon_+ \int d^2z\, (\chi^4\chi^5 - \bar\chi^4\bar\chi^5)\bar\partial X \Big] \Big\rangle_{h,g} = \frac{\theta\left[{}^{1+h}_{1+g}\right](\check\epsilon_+;\tau)\, \theta\left[{}^{1-h}_{1-g}\right](\check\epsilon_+;\tau)}{\eta^2}\; e^{\frac{\pi}{\tau_2}\check\epsilon_+^2}$$

The deformed action precisely mimics the Ω-background !

Absorb the dX zero modes into the deformation parameters ε via:

$$\begin{cases} \tilde\epsilon_\pm = \langle \partial X \rangle \epsilon_\pm = \lambda_i (M + \bar\tau N)^i \epsilon_\pm \\ \check\epsilon_\pm = \langle \bar\partial X \rangle \epsilon_\pm = \bar\lambda_i (M + \tau N)^i \epsilon_\pm \end{cases}$$

the path integral is effectively Gaussian !

Bosonic Generating Functions

Expand the bosonic fields into modes and perform the path integral

$$G^{\rm bos}(\epsilon_-,\epsilon_+) = G_{\rm hol}(\epsilon_-,\epsilon_+)\times G_{\text{non-hol}}(\epsilon_-,\epsilon_+)$$

Almost holomorphic part

Non-holomorphic part

The almost holomorphic part can be expressed in terms of special functions by ζ-function regularization

$$G_{\rm hol}(\epsilon_-,\epsilon_+) = \frac{(2\pi)^2(\epsilon_-^2-\epsilon_+^2)\ \bar\eta(\bar\tau)^6}{\bar\theta_1(\tilde\epsilon_- - \tilde\epsilon_+;\bar\tau)\,\bar\theta_1(\tilde\epsilon_- + \tilde\epsilon_+;\bar\tau)}\ e^{-\frac{\pi}{\tau_2}(\tilde\epsilon_-^2+\tilde\epsilon_+^2)}$$

A naive ζ-function regularization fails for the non-holomorphic part : special care is required !

It turns out that a modular invariant way to regularize this product is via the analytic continuation of Selberg-Poincaré series

$$\log[G_{\text{non-hol}}(\epsilon_-,\epsilon_+)] =$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-)^{\ell}\tau_2^{\ell-k}\sum_{\substack{r=0\\ k+r\in 2\mathbb{Z}}}^{\infty}\binom{k+r-1}{r}\tilde\epsilon_+^{\ell}\,\check\epsilon_+^{k-\ell}\Big[(\tilde\epsilon_- - \tilde\epsilon_+)^r + (-\tilde\epsilon_- - \tilde\epsilon_+)^r\Big]\Phi^*_{k-\ell,r+k}$$

Bosonic Generating Functions

In all relevant cases, the Selberg-Poincaré series is absolutely convergent and we can consistently remove the regulator

Fourier expansion

$$\tau_2^{-\alpha}\Phi_{\alpha\beta}(0;\tau,\bar\tau) = 2\zeta(\beta) + 2\tau_2^{1-\beta}\left\{ C_0^{\alpha,\beta} + \sum_{n>0}\left[C_n^{\alpha,\beta}(\tau_2) q^n + I_n^{\alpha,\beta}(\tau_2)\bar q^n \right] \right\}$$

$$C_n^{\alpha,\beta}(\tau_2) = \frac{(2\pi)^\beta(-i)^{\beta-2\alpha}}{\Gamma(\beta-\alpha)}(n\tau_2)^{\beta-1}\sigma_{1-\beta}(n)(4\pi n\tau_2)^{-\frac{\beta}{2}} e^{2\pi n\tau_2} W_{\frac{\beta}{2}-\alpha,\frac{\beta-1}{2}}(4\pi n\tau_2)$$

$$I_n^{\alpha,\beta}(\tau_2) = \frac{(2\pi)^\beta(-i)^{\beta-2\alpha}}{\Gamma(\alpha)}(n\tau_2)^{\beta-1}\sigma_{1-\beta}(n)(4\pi n\tau_2)^{-\frac{\beta}{2}} e^{2\pi n\tau_2} W_{\alpha-\frac{\beta}{2},\frac{\beta-1}{2}}(4\pi n\tau_2)$$

$$C_0^{\alpha,\beta} = 2^{2-\beta}\pi(-i)^{\beta-2\alpha}\frac{\Gamma(\beta-1)\zeta(\beta-1)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}$$

Field Theory Limit

Including all contributions, extract the field theory limit around SU(2) Wilson-Line enhancement point

$$\mathcal{F}(\epsilon_-, \epsilon_+) \sim (\epsilon_-^2 - \epsilon_+^2) \int_0^\infty \frac{dt}{t} \frac{-2\cos(2\epsilon_+ t)}{\sin(\epsilon_- - \epsilon_+)t \ \sin(\epsilon_- + \epsilon_+)t} e^{-\mu t}$$

- BPS mass parameter $\mu \sim \bar{Y} \cdot Q$
- Leading singularity power of $F_{(g,n)}$ correctly given as $\mu^{2-2g-2n}$
- Reproduces precisely the perturbative part of Nekrasov's partition function in 4d !
- Holomorphic dependence on the complexified Wilson line Y
- Even powers of ε_+, ε_- : Lorentz invariance
- Asymmetry under exchange of ε_+, ε_- : phase / R-symmetry twist

Dual Type I Theory

Antoniadis, Partouche, Taylor 1998

Heterotic-Type I duality in D=4 on K3xT^2 : weakly coupled regimes

$$T = B_{45} + iG^{1/2} \longrightarrow S' = B_{45} + iG^{1/4}V^{-1/2}e^{-\phi_4}$$

- Realize K3 ~ T^4/Z_2 orbifold
- 16 D9 and 16 D5 branes at orbifold fixed point : U(16) x U(16)
- Wilson Lines for the D9's : study enhancement points
- S, S' associated with gauge couplings of the D9 and D5 branes

Bianchi, Sagnotti 1991

Anti-self-dual graviphoton

$$\left[(\partial X + ip\cdot\chi\Psi)(\bar\partial Z^\mu + ip\cdot\tilde\chi\tilde\chi^\mu) - p_\nu(\sigma^{\mu\nu})^{\alpha\beta}e^{-(\phi+\tilde\phi)/2}S^\alpha\tilde S_\beta e^{i(\phi_3+\tilde\phi_3)/2}\Sigma^+\tilde\Sigma^-\right]e^{ip\cdot Z} + [\mathrm{L}\leftrightarrow\mathrm{R}]$$

Self-dual S'-vector (b=+1) or S-vector (b=-1)

$$\left[(\partial X + ip\cdot\chi\Psi)(\bar\partial Z^\mu + ip\cdot\tilde\chi\tilde\chi^\mu) + bp_\nu(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}e^{-(\phi+\tilde\phi)/2}S^{\dot\alpha}\tilde S_{\dot\beta}e^{i(\phi_3+\tilde\phi_3)/2}\hat\Sigma^+\hat{\tilde\Sigma}^-\right]e^{ip\cdot Z} + [\mathrm{L}\leftrightarrow\mathrm{R}]$$

Dual Type I Theory

One loop effective coupling, expanded around SU(2) enhancement point, correctly reproduces the perturbative part of Nekrasov's partition function

What about the non-perturbative part ?

- Realize gauge theory instantons as D-brane configurations (point-like in 4d)
- D9 gauge theory effective action contains $\mu_5 \int C_6 \wedge \mathrm{Tr}[\, F \wedge F \,]$
- Instanton configuration on D9 carries C_6 charge : D5 instanton
- N D9's and k D5-instantons wrapping entirely internal space
- 9-9 : perturbative N=2 SYM gauge theory U(N)
- 5-5 : unmixed instanton moduli
- 9-5 and 5-9 : mixed instanton moduli

Sector	Field	R / NS
9-9	A^μ	NS
	$\Lambda^{\alpha A}$	R
	$\Lambda_{\dot\alpha A}$	R
	ϕ^a	NS
5-5	a^μ	NS
	χ^a	NS
	$M^{\alpha A}$	R
	$\lambda_{\dot\alpha A}$	R
5-9/9-5	$\omega_{\dot\alpha}, \bar\omega_{\dot\alpha}$	NS
	μ^A	R

Instanton corrections

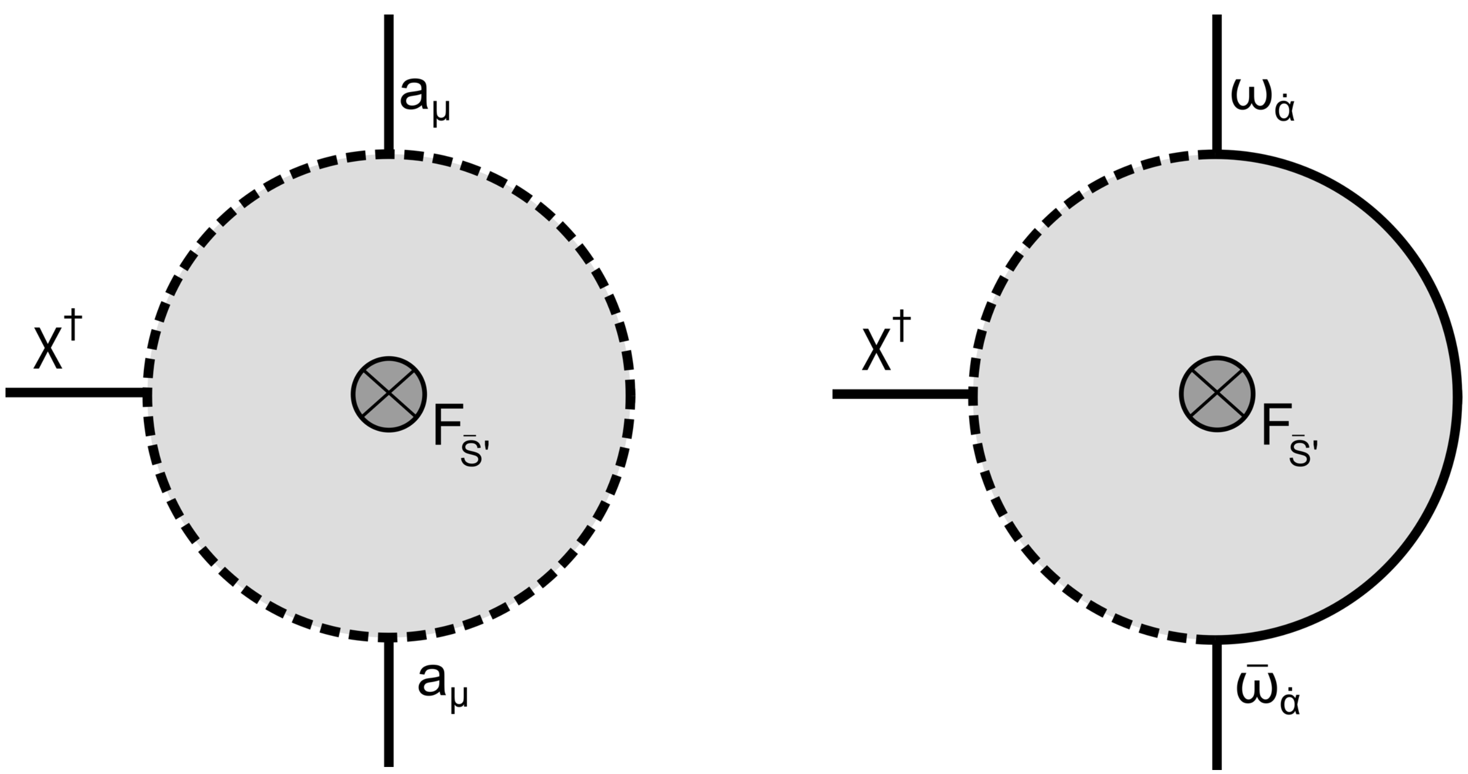
Study instanton configurations in the closed string background of graviphotons and S'-vectors

In pure graviphoton background (ε_+=0) reproduces Ω-deformed ADHM action

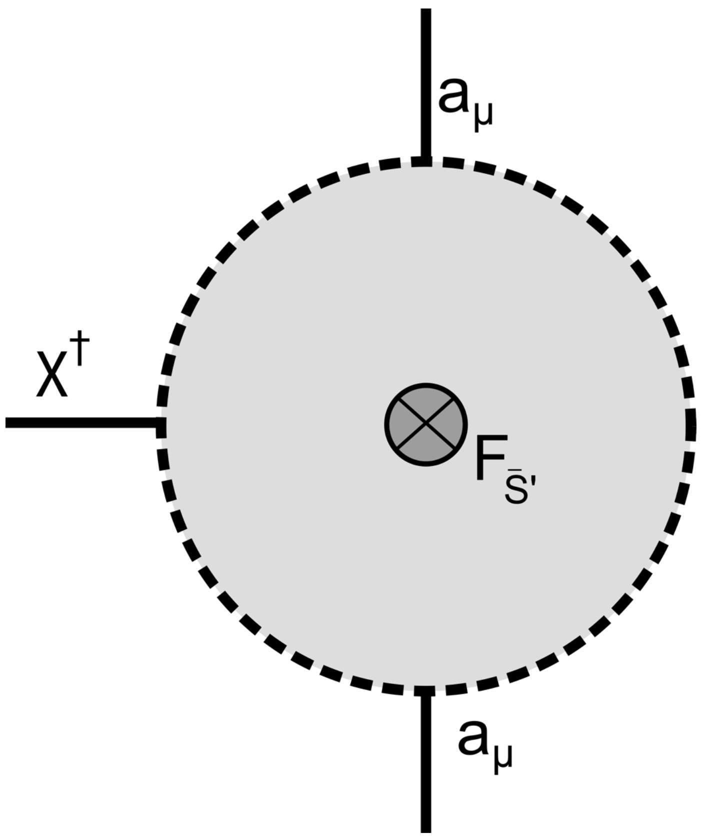
Billó, Frau, Fucito, Lerda 2006

Consider now full background with self-dual S'-vectors of field strength ~ ε_+

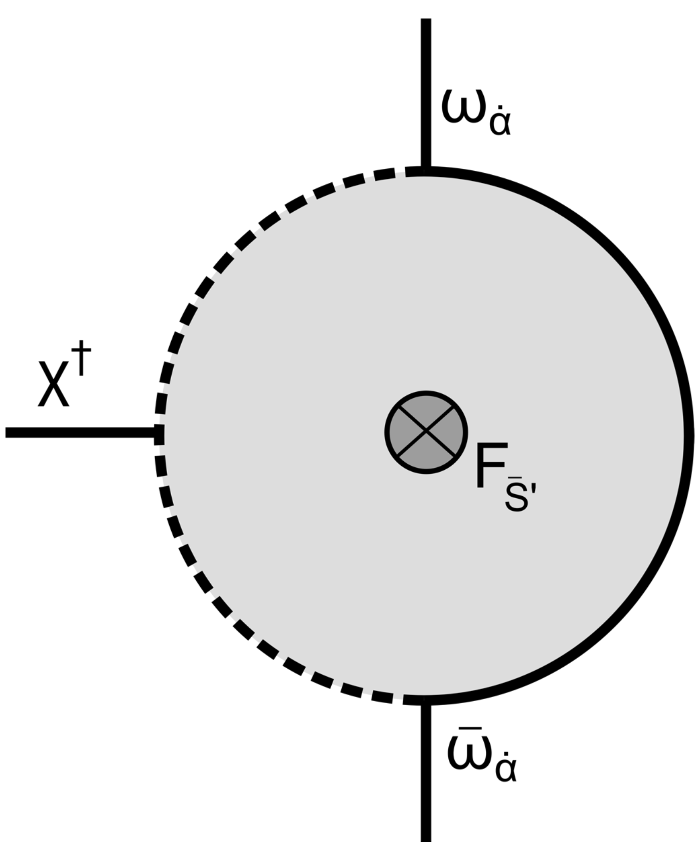
- Calculate all disc diagrams and take field theory limit
- D9-D9 diagrams : N=2 SYM action
- D5-D5 diagrams
- D9-D5 diagrams



Instanton corrections



$$= -4i\,\mathrm{Tr}\Big([\chi^\dagger, a_\mu]\, a_\nu\, F^{\mu\nu}_{\bar{S}'}\Big)$$



$$= \frac{i}{2}\,\mathrm{Tr}\Big(\bar{\omega}_{\dot{\alpha}}\,\chi^\dagger \omega_{\dot{\beta}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} F^{\bar{S}'}_{\mu\nu}\Big)$$

Instanton corrections

Identify constant field strength of self-dual S'-vectors with ϵ_+

$$F^{\bar{S}'}_{\mu\nu} = \bar{\eta}^{c}_{\mu\nu}\,\delta_{3c}\,\frac{\epsilon_+}{2}$$

Adding all disc diagrams together reproduces the full Ω-deformed ADHM action

$$S_{\mathrm{ADHM}} = -\mathrm{Tr}\Big([\chi^\dagger, a_{\alpha\dot\beta}]\big([\chi, a^{\dot\beta\alpha}] + \epsilon_-(a\tau_3)^{\dot\beta\alpha}\big) - \chi^\dagger\bar{\omega}_{\dot\alpha}(\omega^{\dot\alpha}\chi - \tilde{a}\omega^{\dot\alpha}) - (\chi\bar{\omega}_{\dot\alpha} - \bar{\omega}_{\dot\alpha}\tilde{a})\omega^{\dot\alpha}\chi^\dagger + \epsilon_+[\chi^\dagger, a_{\alpha\dot\beta}](\tau_3 a)^{\dot\beta\alpha} - \epsilon_+\bar{\omega}_{\dot\alpha}(\tau_3)^{\dot\alpha}{}_{\dot\beta}\chi^\dagger\omega^{\dot\beta}\Big)$$

unmixed: $+\epsilon_+[\chi^\dagger, a_{\alpha\dot\beta}](\tau_3 a)^{\dot\beta\alpha}$

mixed: $-\epsilon_+\bar{\omega}_{\dot\alpha}(\tau_3)^{\dot\alpha}{}_{\dot\beta}\chi^\dagger\omega^{\dot\beta}$

Integrating over instanton moduli yields the non-perturbative Nekrasov partition function

$$Z^{\mathrm{Nek}}(\epsilon_-, \epsilon_+) = Z_{\mathrm{pert}}(\epsilon_-, \epsilon_+)\, Z_{\text{non-pert}}(\epsilon_-, \epsilon_+)$$

Nekrasov 2002
Nekrasov, Okounkov 2003

Conclusions

- We proposed a new class of N=2 amplitudes
- Generalized F-terms of the type $F_{g,n}W^{2g}Y^{2n}$ in the string effective action
- Y is a vector superfield defined as an N=2 chiral projection of a particular anti-chiral vector multiplet, identified with $\bar{T}$
- Calculable exactly at the 1-loop string level in Heterotic and Type I theories compactified on K3xT^2 in the orbifold limit
- The field theory limit near a point of SU(2) enhancement in the string moduli space reproduces exactly the full Nekrasov partition function
- Two deformation parameters ε_+, ε_- : interpreted as constant field-strength backgrounds for self-dual gauge field of T vector multiplet and anti-self-dual graviphoton, respectively
- Radius deformation of Nekrasov-Okounkov's partition function, associated to Ω-background (6d, 5d)
- For n=0, the couplings reduce to the well-known topological string amplitudes, computing partition functions of the (unrefined) topological string
- String realization of the full Ω-background

Outlook

- ☑ Generalized Holomorphic anomaly equation ?
- ☑ Type II dual on K3-fibered CY threefold : twisted correlator ?

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Thank You !

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