DESY Theory Workshop: Nonperturbative QFT

Parallel Session: Strings and Mathematical Physics

Closed formulae for superstring tree amplitudes: Multiple zeta values and the Drinfeld associator

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based on arXiv:1304.7304: J. Brödel, OS, St. Stieberger, T. Terasoma

26.09.2013

I. The N point disk amplitude

Color stripped tree amplitude for scattering N massless open string states

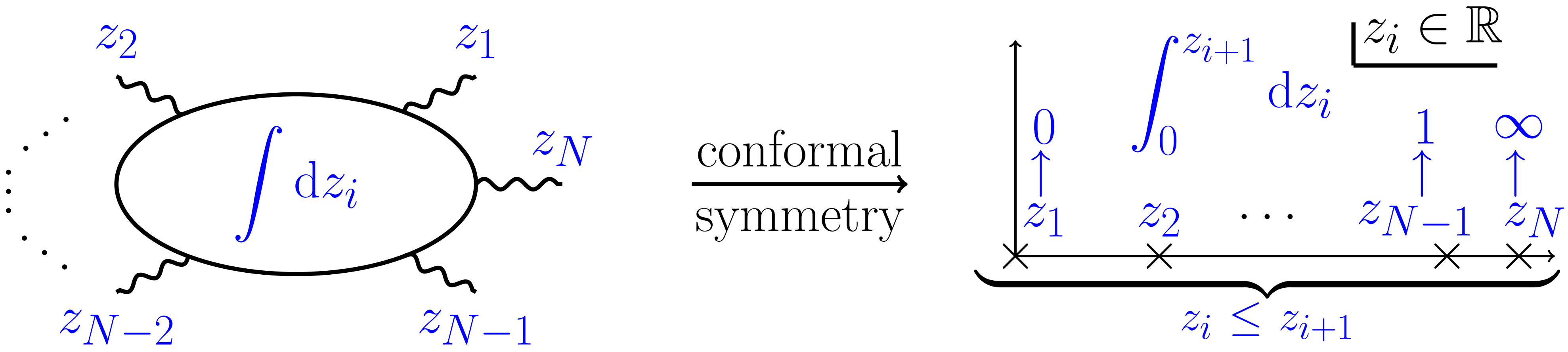
$$\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\rm YM}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\, F^\pi(\alpha')$$

[Mafra, OS, Stieberger 1106.2645, 1106.2646]

- all polarization dependence in $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\rm YM}_{\pi\in S_{N-3}}$
- valid for states of $\mathcal{N}=1$ SYM in $D=10$ (all gluon and gluino helicities)
- string effects (α' dependence) from generalized Euler integrals $F^\pi(\alpha')$
- consistent with field theory limit: $F^\pi(\alpha'\to 0) \;=\; \delta^\pi_{(2,3,\ldots,N-2)}$

This talk: Low energy expansion of F^π in α' $\Rightarrow$ closed form expressions

String corrections $F^{\pi}(\alpha')$ $\equiv$ iterated integrals along disk boundary



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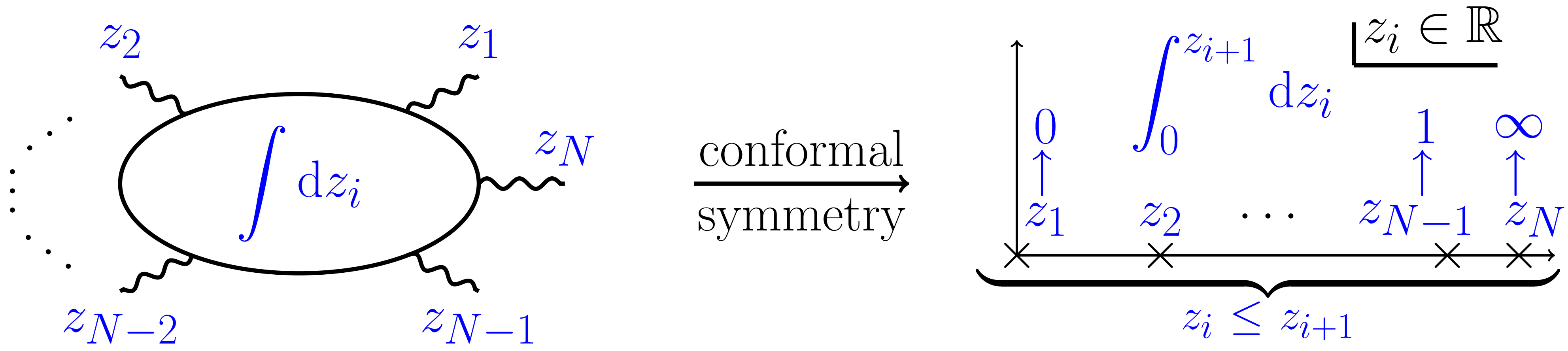

α' enters F^π through dimensionless Mandelstam invariants

$$s_{ij} \;:=\; \alpha' (k_i + k_j)^2$$

Explicit form of F^π with shorthand $z_{ij} := z_i - z_j$.

$$F^\pi(s_{ij}) \;=\; \int\limits_{0 \leq z_2 \leq z_3 \leq \ldots \leq z_{N-2} \leq 1} dz_2 \ldots \int dz_{N-2} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \prod_{k=2}^{N-2} \sum_{j=1}^{k-1} \frac{s_{\pi(j),\pi(k)}}{z_{\pi(j),\pi(k)}} \Bigg|_{z_{N-1}=1}^{z_1=0}$$

[Mafra, OS, Stieberger 1106.2645, 1106.2646]

II. Multiple zeta values: iterated integrals vs. nested sums

Let $v \in \{0,1\}^{\times}$ $\equiv$ non-commutative words $v_1 v_2 \ldots$ in letters $v_i \in \{0,1\}$,

$$\zeta_{\{v\}} \quad := \quad \pm \int_{0 \le z_i \le z_{i+1} \le 1} \frac{\mathrm{d}z_1}{z_1 - v_1} \frac{\mathrm{d}z_2}{z_2 - v_2} \ldots \frac{\mathrm{d}z_j}{z_j - v_j}$$

Reproduces multiple zeta values (MZV)

$$\zeta_{n_1,n_2,\ldots,n_r} \quad := \quad \sum_{0<k_1<\ldots<k_r} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \ , \qquad n_r \le 2$$

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Dictionary between nested sums and iterated integrals:

$$\zeta_{n_1, n_2, \ldots, n_r} \quad = \quad \zeta_{\{\underbrace{10\ldots0}_{n_1}\,\underbrace{10\ldots0}_{n_2}\ldots\underbrace{10\ldots0}_{n_r}\}}$$

Transcendental weight $\equiv \sum_{j=1}^{r} n_j \equiv \mathrm{length}(v)$

$\exists$ rich network of $\mathbb{Q}$ relations among MZVs:

- shuffle $\zeta_{\{v\}} \cdot \zeta_{\{u\}} \; = \; \zeta_{\{v ⧢ u\}}$ with $v ⧢ u \equiv \sum$(shuffles of u and v)

- stuffle $\underset{\sum_k}{\zeta_m} \cdot \underset{\sum_l}{\zeta_n} \; = \; \underset{\sum_{k<l}}{\zeta_{m,n}} + \underset{\sum_{k>l}}{\zeta_{n,m}} + \underset{\sum_{k=l}}{\zeta_{m+n}}$

Preserve the weight $w = \sum_j n_j$ and lead to (conjectural) $\mathbb{Q}$ bases

w	0	1	2	3	4	5	6	7	8
basis	1	$\emptyset$	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6	$\zeta_7, \;\; \zeta_2\zeta_5$	$\zeta_8, \;\; \zeta_3\zeta_5$
MZV						$\zeta_3\zeta_2$	ζ_3^2	$\zeta_4\zeta_3$	$\zeta_{3,5}, \;\; \zeta_2\zeta_3^2$
dim	1	0	1	1	1	2	2	3	4

Explicit basis reductions up to $w = 22$ collected in the MZV "data mine"

[Blümlein, Broadhurst, Vermaseren arXiv:0907.2557]

III.1 The Drinfeld associator as a MZV generating function

Drinfeld associator Φ $\equiv$ series in non-commutative variables e_0, e_1

$$\Phi(e_0, e_1) = \sum_{v\in\{0,1\}^\times} \zeta_{\{v_1 v_2 \ldots v_j \ldots\}}\, e_{\{\ldots v_j \ldots v_2 v_1\}}$$

$$= 1 \;+\; \zeta_2\,[\,e_0, e_1\,] \;+\; \zeta_3\,\big[\,e_0 - e_1\,,\,[e_0, e_1]\big] \;+\; \ldots$$

with shorthand $e_{\{v\}} := e_{v_1} e_{v_2} \ldots e_{v_j}$ for $v \in \{0,1\}^\times$

[Le, Murakami 1996]

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with shorthand $e_{\{v\}} := e_{v_1} e_{v_2} \ldots e_{v_j}$ for $v \in \{0,1\}^\times$

[Le, Murakami 1996]

Divergent MZVs $\zeta_{\{0\ldots\}} = \int_0 \frac{\mathrm{d}z}{z}$ and $\zeta_{\{\ldots 1\}} = \int^1 \frac{\mathrm{d}z}{1-z}$ are shuffle-regularized:

$$\zeta_{\{0\}} = \zeta_{\{1\}} = 0$$

e.g. $0 = \zeta_{\{0\}} \cdot \zeta_{\{1\}} = \zeta_{\{01\}} + \zeta_{\{10\}}$ such that $\zeta_{\{01\}} = -\zeta_{\{10\}} = \zeta_2$.

III.2 The Drinfeld associator as a KZ-equation monodromy

Consider the Knizhnik–Zamolodchikov (KZ) equation in $z \in \mathbb{C} \setminus \{0, 1\}$

$$\frac{\mathrm{d}f(z)}{\mathrm{d}z} = \left(\frac{e_0}{z} + \frac{e_1}{1-z} \right) f(z)$$

At singular points $z = 0, 1$, define regularized boundary values

$$C_0 := \lim_{z \to 0} z^{-e_0} f(z) \,, \qquad C_1 := \lim_{z \to 1} (1-z)^{e_1} f(z)$$

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The Drinfeld associator Φ relates the two

$$C_1 = \Phi(e_0, e_1) \, C_0$$

[Drinfeld 1989, 1991]

Connection to string amplitude firstly noticed in ...

[Drummond, Ragoucy arXiv:1301.0794]

IV.1 Main result

String corrections F^π to the disk amplitude $\sum_{\pi\in S_{N-3}} \mathcal{A}^{\rm YM}(\pi)F^\pi$ satisfy

$$\begin{matrix} (N-3)! \left\{ \right. \\ (N-2)! \\ -(N-3)! \left\{ \right. \end{matrix} \begin{pmatrix} F^\pi \\ \hline \vdots \\ \vdots \\ \vdots \end{pmatrix} = \Big[\Phi(e_0,e_1)\Big] \begin{pmatrix} F^\pi \,\big|_{k_{N-1}\to 0} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \left. \right\} (N-2)!$$

[Brödel, OS, Stieberger, Terasoma arXiv:1304.7304]

- e_0, e_1 in $(N-2)! \times (N-2)!$ matrix rep. linear in $s_{ij} = \alpha'(k_i+k_j)^2$

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N \text{ pts} \left\{ \begin{pmatrix} F^\pi \\ \hline \vdots \\ \vdots \\ \vdots \end{pmatrix} = \Big[\Phi(e_0, e_1) \Big] \begin{pmatrix} F^\pi \big|_{k_{N-1} \to 0} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \right\} N-1 \text{ points}
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[Brödel, OS, Stieberger, Terasoma arXiv:1304.7304]

- e_0, e_1 in $(N-2)! \times (N-2)!$ matrix rep. linear in $s_{ij} = \alpha'(k_i + k_j)^2$
- soft limit $k_{N-1} \to 0$ acts recursively:

$$
F^{\pi(2,3,\ldots,N-2)} \big|_{k_{N-1} \to 0} = \begin{cases} F^{\pi(2,3,\ldots,N-3)} & : \ \pi(N-2) = N-2 \\ 0 & : \ \text{otherwise} \end{cases}
$$

IV.2 Examples

Simplest $N=4$ string correction $F^{(2)}$ from constant 3-vertex $F^{\{\emptyset\}}=1$:

$$\begin{pmatrix} F^{(2)} \\ \\ 0 \end{pmatrix} = \Big[\Phi(e_0, e_1) \Big]_{2\times 2} \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}$$

with 2×2 matrices $e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix}$.

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$\Rightarrow$ reproduces the Veneziano amplitude with single ζ_n only

$$F^{(2)} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} \left[s_{12}^n + s_{23}^n - (s_{12} + s_{23})^n \right] \right) = \Big[\Phi(e_0, e_1) \Big]_{1,1}$$

Higher depth MZVs $\zeta_{n_1,n_2,\ldots}$ cancel for e_0, e_1 above since

$$[e_0, e_1] \, , \quad [e_0, [e_0, e_1]] \, , \quad [e_1, [e_0, e_1]] \, , \quad \ldots \sim \left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right) \text{ nilpotent}$$

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At $N=5$, first closed form expression with manifest MZV structure

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \hline \vdots \\ \vdots \\ \vdots \end{pmatrix} = \Big[\Phi(e_0,e_1)\Big]_{6\times 6} \begin{pmatrix} F^{(2)} \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}$$

i.e. $F^{(23)} = \big[\Phi(e_0,e_1)\big]_{1,1} F^{(2)}$ as well as $F^{(32)} = \big[\Phi(e_0,e_1)\big]_{2,1} F^{(2)}$.

IV.2 Examples

At $N = 5$, have 6×6 matrices

$$e_0 = \begin{pmatrix} s_{12}+s_{13}+s_{23} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{12}+s_{13}+s_{23} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{23}-s_{24}-s_{34} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{23}-s_{24}-s_{34} \end{pmatrix}$$

@ higher $N \leq 9$, corresp. $(N-2)! \times (N-2)!$ matrices e_0, e_1 available at

`http://mzv.mpp.mpg.de`

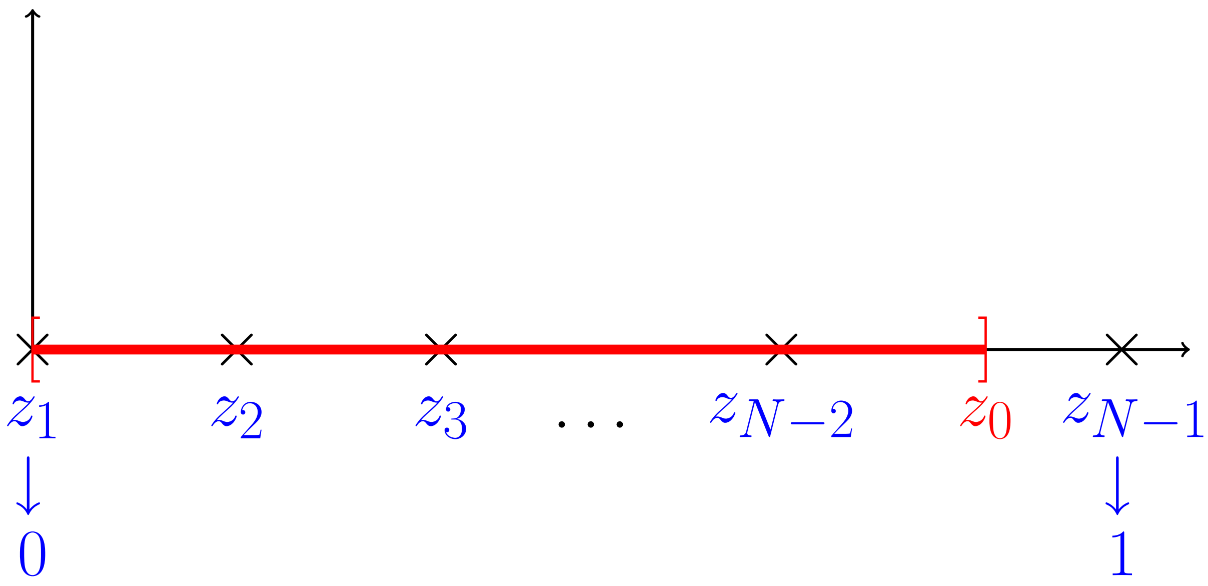
IV.3 Proof

Consider deformations $F^\pi \mapsto \hat{F}^\pi_\nu$ by auxiliary $\begin{cases} \text{worldsheet position } z_0 \\ \text{Mandelstam var's } s_{0,k} \end{cases}$

$$\hat{F}^\pi_\nu(s_{ij};z_0) = \int\limits_{0\le z_2\le z_3\le\ldots\le z_{N-2}\le z_0} dz_2 \ldots \int dz_{N-2} \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \prod_{k=2}^{N-2} (z_{0,k})^{s_{0,k}}$$

$$\times \prod_{k=2}^{\nu} \sum_{j=1}^{k-1} \frac{s_{\pi(j),\pi(k)}}{z_{\pi(j),\pi(k)}} \prod_{m=\nu+1}^{N-2} \sum_{n=m+1}^{N-1} \frac{s_{\pi(m),\pi(n)}}{z_{\pi(m),\pi(n)}} \Bigg|_{z_{N-1}=1}^{z_1=0}$$

With $\nu = 1,2,\ldots,N-2$ and $\pi \in S_{N-3}$, have $(N-2)!$ cpts. $\hat{F}^\pi_\nu$ ...



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with KZ equation $\frac{\mathrm{d}\hat{F}^\pi_\nu}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} + \frac{e_1}{1-z_0}\right) \hat{F}^\pi_\nu$

$$C_0 = \lim_{z_0\to 0} z_0^{-e_0} \hat{F}^\pi_\nu = \left(F^\pi \big|_{k_{N-1}\to 0}, 0 \ldots 0\right)$$

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$$C_0 = \lim_{z_0 \to 0} z_0^{-e_0} \hat{F}^\pi_\nu = \left(F^\pi \big|_{k_{N-1} \to 0}, 0 \ldots 0\right)$$

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$\Bigg\}$ $C_1 = \Phi(e_0, e_1)\, C_0$

... don't forget to send $s_{0,k} \to 0$ at the end.

V. Conclusion & Outlook

- derived recursion $F_N = \Phi(e_0, e_1) F_{N-1}$ for N point disk integrals

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- Drinfeld associator $\Phi(e_0, e_1)$ has well-known expansion in terms of MZVs

- beautiful transcendentality pattern in F^π revealed by "motivic" MZVs

[OS, Stieberger arXiv:1205.1516]

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Thank you for your attention !