Two-dimensional S-matrices from unitarity cuts

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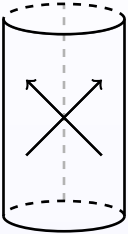


DESY Theory Workshop
25th August 2013

Based on work with L. Bianchi and V. Forini [1304.1798]

see also work by O. Engelund, R. McKeown and R. Roiban [1304.4281]

Motivation

- In 4 dimensions unitarity methods have been used extensively, to compute
  - scattering amplitudes
  - form factors
  - correlation functions . . .
- However never really been applied in 2 dimensions.
- Two-dimensional field theories of interest for many reasons. Today:
  - Integrability.
    - Can unitarity methods test integrability?
    - Unitarity methods should give information about the phase.
  - String world-sheet theories.
    - Example: $\mathbf{AdS_5} \times \mathbf{S^5}$. Light-cone gauge fix & and expand around BMN.
    - Integrable world-sheet theory for $8 + 8$ massive excitations.
    - World-sheet S-matrix is input for computation of string spectrum.

Basics of 2-dimensional scattering

Aim: 4-point amplitude at 1-loop $\longrightarrow$ $2 \to 2$ S-matrix at 1-loop.

$$\langle \Phi^P(p_3)\Phi^Q(p_4)|\mathbb{S}|\Phi_M(p_1)\Phi_N(p_2)\rangle$$
$$= (2\pi)^2\delta^{(2)}(p_1+p_2-p_3-p_4)\mathcal{A}_{MN}^{PQ}(p_1,p_2,p_3,p_4)$$

Assumptions:
- Asymptotic states: all massive with equal mass ($\epsilon_i^2 = \mathrm{p}_i^2 + 1$).
- Fix ordering on incoming states: $\mathrm{p}_1 > \mathrm{p}_2$.

Kinematics: (special to 2 dimensions)

$$\delta^{(2)}(p_1+p_2-p_3-p_4)$$
$$= \frac{\epsilon_1\epsilon_2}{\epsilon_2\mathrm{p}_1 - \epsilon_1\mathrm{p}_2}\Big(\delta(\mathrm{p}_1-\mathrm{p}_3)\delta(\mathrm{p}_2-\mathrm{p}_4) + \delta(\mathrm{p}_1-\mathrm{p}_4)\delta(\mathrm{p}_2-\mathrm{p}_3)\Big)$$

S-matrix:

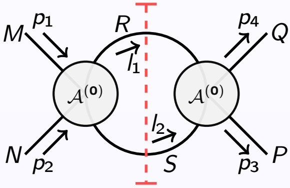
$$S_{MN}^{PQ}(\mathrm{p}_1,\mathrm{p}_2) = \frac{1}{4(\epsilon_2\mathrm{p}_1 - \epsilon_1\mathrm{p}_2)}\mathcal{A}_{MN}^{PQ}(p_1,p_2,p_1,p_2)$$

$\rightsquigarrow$ quantities defined in terms of just the amplitude (no Jacobian factor).

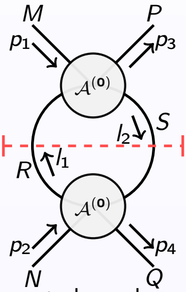
Unitarity cuts in two dimensions

$$S^{PQ}_{MN}(\mathrm{p}_1,\mathrm{p}_2)=\delta^P_M\delta^Q_N+\zeta\,\underset{\text{KNOW}}{\underbrace{T^{(0)}{}^{PQ}_{MN}(\mathrm{p}_1,\mathrm{p}_2)}}+\zeta^2\,\underset{\text{WANT}}{\underbrace{T^{(1)}{}^{PQ}_{MN}(\mathrm{p}_1,\mathrm{p}_2)}}+\dots$$

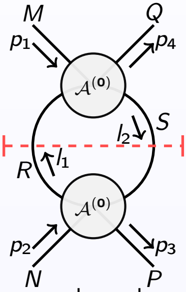
Possible 2-particle cuts:



s-channel



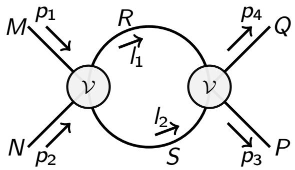
t-channel



u-channel

Unitarity cuts in two dimensions

- 2-particle cuts are maximal at 1-loop in 2 dimensions.
- Completely freezes loop momenta (analogous to quadrupole cuts in 4 dimensions).
- Therefore, putting loop momenta on-shell, we can pull the tree-level amplitudes out of the integral.
- Returning the loop momenta off-shell then gives an expression in terms of scalar integrals that gives the "cut-constructible" part of the amplitude.



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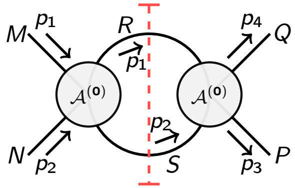

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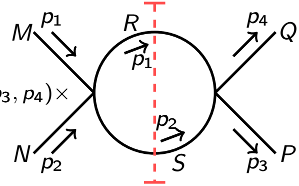
$$\mathcal{A}_0{}^{RS}_{MN}(p_1,p_2,p_1,p_2) \times \mathcal{A}_0{}^{PQ}_{RS}(p_1,p_2,p_3,p_4) \times$$



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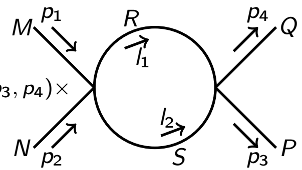

Unitarity cuts in two dimensions - scalar integrals

In the kinematical configuration where $p_3 = p_1$ and $p_4 = p_2$, the scalar integrals are given by:

$$I_u = \frac{\mathrm{arsinh}(\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2)}{4\pi i(\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2)}$$

$$I_s = \frac{1}{\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2} - \frac{\mathrm{arsinh}(\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2)}{4\pi i(\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2)}$$

$$I_t = \frac{1}{4\pi i}$$

Logarithmic terms

Rational terms

Unitarity cuts in two dimensions - t-channel

- **t-channel is subtle.**
- If we first set $p_3 = p_1$ and $p_4 = p_2$ before doing integrals it leads to problems – zero momentum flowing across the cut.
- Furthermore, depending on which vertex we choose to use to freeze the loop momenta we end up with different expressions (just for the t-channel).
- Leads to a **consistency condition:**

$$\tilde{T}^{(0)}{}^{SP}_{MR}(\mathrm{p}_1, \mathrm{p}_1)\, \tilde{T}^{(0)}{}^{RQ}_{SN}(\mathrm{p}_1, \mathrm{p}_2) = \tilde{T}^{(0)}{}^{PS}_{MR}(\mathrm{p}_1, \mathrm{p}_2)\, \tilde{T}^{(0)}{}^{QR}_{SN}(\mathrm{p}_2, \mathrm{p}_2)$$

Unitarity cuts in two dimensions - final formula

Assuming the consistency condition is satisfied, all this leads to an expression for the cut-constructible part of the 1-loop 2-particle S-matrix:

$$
\begin{aligned}
T^{(1)}{}_{MN}^{PQ}(\mathrm{p}_1,\mathrm{p}_2) = \frac{1}{4(\epsilon_2\,\mathrm{p}_1-\epsilon_1\,\mathrm{p}_2)}\Big[\,&\tilde{T}^{(0)}{}_{MN}^{RS}(\mathrm{p}_1,\mathrm{p}_2)\,\tilde{T}^{(0)}{}_{RS}^{PQ}(\mathrm{p}_1,\mathrm{p}_2)\,I_s\\
&+\tilde{T}^{(0)}{}_{MR}^{SP}(\mathrm{p}_1,\mathrm{p}_1)\,\tilde{T}^{(0)}{}_{SN}^{RQ}(\mathrm{p}_1,\mathrm{p}_2)\,I_t\\
&+\tilde{T}^{(0)}{}_{MR}^{SQ}(\mathrm{p}_1,\mathrm{p}_2)\,\tilde{T}^{(0)}{}_{SN}^{PR}(\mathrm{p}_1,\mathrm{p}_2)\,I_u\Big]
\end{aligned}
$$

Logarithms better be correct, but are these all the rational terms?

Unitarity cuts in two dimensions - comments

- **Caveats:**
  - Have ignored contributions from tadpoles.
  - Result is manifestly finite – lose information about renormalizability/finiteness.
- Generalized sine-Gordon models ($SO(N)/SO(N-1)$ gauged WZW model plus integrable potential).
  - $N = 1$: sine-Gordon, $N = 2$: complex sine-Gordon, ...
  - Unitarity agrees with perturbation theory up to a shift in the coupling.
- Supersymmetric generalizations. Minus signs!
  - $\mathcal{N} = 1, 2$ supersymmetric sine-Gordon ($\mathcal{N} = 4, 8$ extensions).
  - Unitarity reproduces full result.
- In both cases, reproduces the result consistent with integrability.
- **STRING THEORY** ...

Spectrum of string theories in curved space with RR flux?

- Interested in $AdS \times S$ backgrounds.
  - Important for gauge-string duality.
  - Related to quantum black hole models.

- Certain examples are integrable (for example $AdS_5 \times S^5$),
  and we can use this to help us.

Strategy

Starting point: Superstring world-sheet action

$$\frac{T}{2}\int d^2x\sqrt{-h}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \text{ fermions}$$

RR flux $\longrightarrow$ use GS action to include fermions.

Expansion around background: As we have GS action we need to expand around a bosonic background to be able to do perturbation theory (fermion kinetic terms).

Gauge-fixing: Conformal gauge leads to non-unitary world-sheet S-matrix
$\longrightarrow$ how do we implement Virasoro conditions?
Alternative is to consider physical "light-cone" gauge
$\longrightarrow$ introduces mass scale ($x^+ = p^+\tau$) $\longrightarrow$ massive modes.
For the theories under consideration these modes all have equal mass.

Decompactification limit: to define world-sheet S-matrix. (Input into computation of spectrum.)

Unitarity methods

- $AdS_5 \times S^5$:
  - Tree-level S-matrix computed. [Klose, McLoughlin, Roiban, Zarembo, 2006]
  - Exact result from integrability. [Beisert, 2005; Beisert, Eden, Staudacher, 2006]
  - One-loop unitarity matches integrability result, including rational terms. [Bianchi, Forini, BH, 2013]
  - Two-loop generalized unitarity correctly reproduces logs. [Engelund, McKeown, Roiban, 2013]
- For $AdS_3 \times S^3 (\times T^4 / S^3 \times S^1)$
  - Logs work fine, but rational terms are not so clear – consistency condition?
- Logs work fine for $AdS_2 \times S^2 (\times T^6)$ [Abbott, Murugan, Sundin, Wulff, 2013]
- Two-loop rational terms?
- Interplay between integrability and unitarity?
- Non-integrable backgrounds?
- Form factors, correlation functions, . . .

Thank you!