# Orbifolds and topological defects

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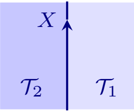
based on arXiv:1307.3141 with I. Brunner and N. Carqueville

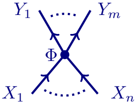
# Motivation

- orbifolds with symmetry group $G$ can be described via **defects** $A_G$
- **generalization**: allow any defect $A$ with appropriate algebraic structure
- standard results on orbifolds recovered in simpler/more conceptual way
  $\rightsquigarrow$ carries over to the generalized setting
- case study: $\mathcal{N} = 2$ **Landau-Ginzburg** models:
  - description of many $\mathcal{N} = 2$ CFTs, stringy regime of **CY compactifications**
  - explicit description of defects via matrix factorizations
  - compute **arbitrary topological correlators** in the generalized orbifold theory: bulk/boundary correlators ($\rightsquigarrow$ **eff. superpotentials, D-brane charges**), defect actions on bulk fields, ...

# Defects in 2D field theories



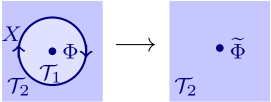

- A **defect** $X$ is a 1D interface between two theories $\mathcal{T}_1$ and $\mathcal{T}_2$ together with a gluing condition on fields
- defects can form **junctions** $\rightsquigarrow$ junction fields:



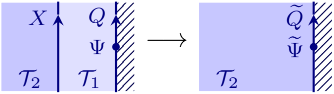

- usually want to preserve some symmetry (e.g. conformal, SUSY, ...)
- $X$ is a **topological defect** if all correlators inv. under deformations of $X$.

  use this to map objects of $\mathcal{T}_1$ to objects of $\mathcal{T}_2$ (and vice versa):

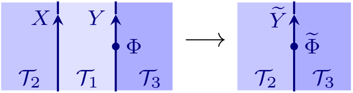
bulk fields:




boundary (fields):




defect (junction fields):



# Defect description of orbifolds

- ordinary orbifolds: theory $\mathcal{T}$ with finite symmetry group $G$
  $\rightsquigarrow$ orbifold theory $\mathcal{T}/G$:

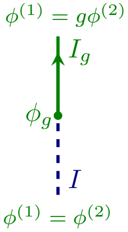
  twisted fields

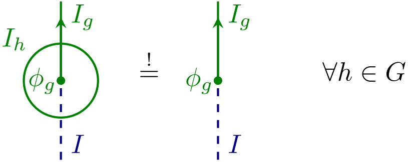
  $$\phi_g(e^{2\pi i} z) = g\phi_g(z)$$

  orbifold projection

  $$P_{\mathsf{orb}}|\phi_g\rangle \equiv \tfrac{1}{G}\sum_{h\in G} h|\phi_g\rangle \stackrel{!}{=} |\phi_g\rangle$$

- **defect perspective**:



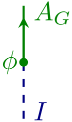




- Note: $I_g$ is always topological since $g \in G$ is a symmetry of $\mathcal{T}$

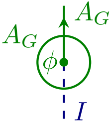
# Generalized orbifolds

Assemble $I_g$ to $A_G = \bigoplus_{g \in G} I_g$

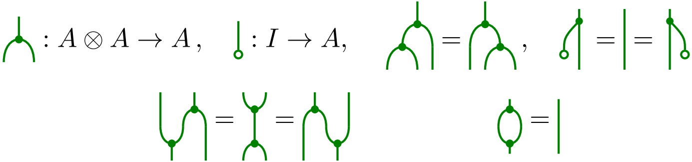
twisted fields – elements in $\mathrm{Hom}(I, A_G)$



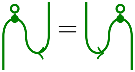

orbifold projector

$P_{\mathrm{orb}}(\phi) =$ 


**Generalized orbifolds** – allow any (symmetric) separable Frobenius algebra $A$:
[Fröhlich, Fuchs, Runkel, Schweigert '09], [Carqueville, Runkel '12]

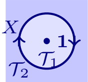


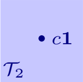

$\rightsquigarrow$ associative OPE, unique vacuum, non-degenerate bulk pairings, $P_{\mathrm{orb}}$, ...

(symmetry): 

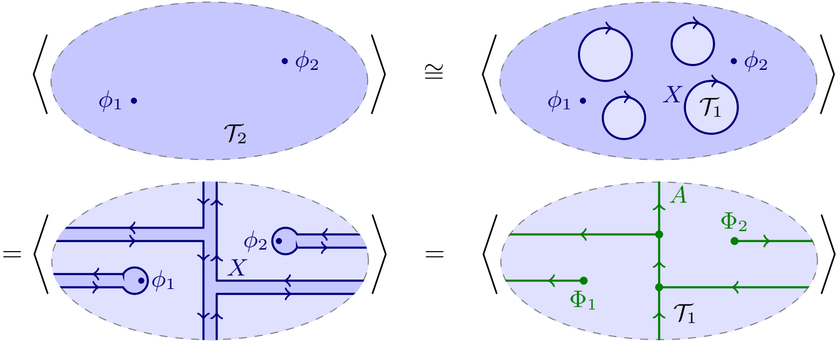
$\rightsquigarrow$ spectral flow operator

# Generalized orbifolds

- **important class** $A = X^\dagger \otimes X$, where $X : \mathcal{T}_1 \to \mathcal{T}_2$ is a top. defect with invertible $\dim(X)$, i.e. 

$=$ 

$, c \in \mathbb{C} \setminus \{0\}$
- $\mathcal{T}_1$ and $\mathcal{T}_2$ are related via generalized orbifold:




- "twisted fields" in $\mathcal{T}_1 \leftrightarrow$ bulk fields in $\mathcal{T}_2$
- similarly with D-branes (defects) – twisted branes in $\mathcal{T}_1 \leftrightarrow$ branes in $\mathcal{T}_2$ $\rightsquigarrow$ equivalence of D-brane categories

# Landau-Ginzburg models

- $\mathcal{N} = 2$ LG models are 2D QFTs with action:

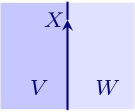
$$S = \int d^2z\, d^4\theta\, K(X_i, \bar{X}_i) + \frac{1}{2}\left(\int d^2z\, d^2\theta\, W(X_i)|_{\bar{\theta}^{\pm}=0} + c.c.\right)$$

  $W(X_i)$ – **superpotential**

- LG not conformal, but for $W$ quasi-homogeneous $\rightsquigarrow$ flow to an IR fixed pt.
- **CFT in IR** characterized solely by $W$. Can extract information about the CFT from properties of $W$ , e.g. chiral primary fields $\leftrightarrow \frac{\mathbb{C}[X_i]}{(\partial W)}$
- Many $\mathcal{N} = 2$ CFTs described as IR fixed pts. of LG orbifolds, in particular, **CY compactifications** in stringy regime of Kähler moduli space.
- from here on: topologically B-twisted LG models

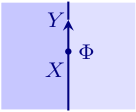
# Defects in LG models

- Defects between LG models with superpotentials $W(x_1, .., x_n)$, $V(z_1, .., z_m)$ described by **matrix factorizations** of $V - W$. [Brunner, Roggenkamp '07] [Khovanov, Rozansky '04] [Kapustin, Li '02]




- A matrix factorization of a polynomial $p(u_i)$ is a pair $(X, d_X)$, where $X$ is a $\mathbb{Z}_2$-graded $\mathbb{C}[u_i]$-module and $d_X$ is an odd operator s.t.

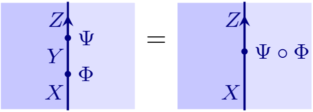
$$d_X^2 = p(u_i) \cdot 1_X$$

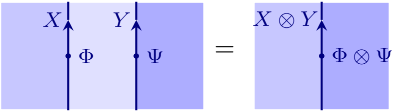



- Junction fields $\Phi$ from $X$ to $Y$ given by maps in the **cohomology** $\mathrm{Hom}(X, Y)$ of the operator

$$D_{XY}\Phi = d_Y\Phi - (-1)^{|\Phi|}\Phi\, d_X$$

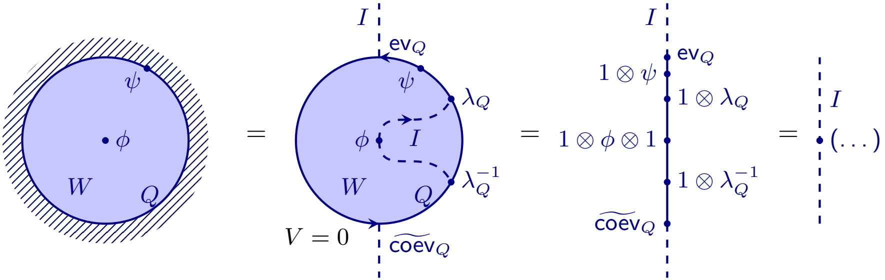
- Product (chiral ring) – composition of maps, defect fusion – $\otimes$ of m.f.



# Defects in LG models

- one can compute **arbitrary topological correlators** using defects and m.f.
- first express everything in terms of defects, then evaluate using m.f. by composing horizontally ($\otimes$) and vertically ($\circ$)
- e.g. disk correlator:




$$= \ \mathrm{Res}\left[\frac{\phi\ \mathrm{STr}\big[\partial_{x_1} d_Q \dots \partial_{x_m} d_Q\ \psi\big]\underline{\mathrm{d}x}}{\partial_{x_1} W \dots \partial_{x_m} W}\right]$$

- $\lambda$, ev, coev are canonical maps known explicitly for any defect [Carqueville, Murfet '12]
- natural language for top. defects – bicategories with adjoints

# LG orbifolds – bulk sector

study twisted **RR g.s.** and **(c,c)-fields** [Intriligator, Vafa '90]

unprojected RR g.s. in g-th sector

$|\phi_g\rangle = \prod_{\Theta_i^g \in \mathbb{Z}} (X_i)^{l_i} |0\rangle_{RR}^g$

action of $h \in G$

$hX_ih^{-1} \equiv h_i^j X_i = e^{2\pi i \Theta_i^h} X_i$

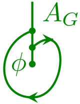
$h|0\rangle_{RR}^g = \det(h)\, e^{2\pi i \sum_{\Theta_i^h \in \mathbb{Z}} \Theta_i^g} |0\rangle_{RR}^g$

**defect/m.f. perspective**:

compute cohomology Hom$(I, I_g)$

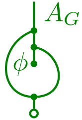
$\phi_g = \prod_{\Theta_i^g \in \mathbb{Z}} (X_i)^{l_i} \prod_{\Theta_i^g \notin \mathbb{Z}} \omega_i^g$
[Brunner, Roggenkamp '07]

evaluate the diagram

$P_{\text{orb}}^{RR}(\phi) =$ 

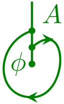
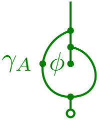
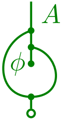

$\rightsquigarrow$ reproduces the spectrum and phases above [Brunner, Carqueville, DP '13]

(c,c)-fields similarly – unprojected spectrum iso to RR g.s. via spectral flow,

but different representation of $G$ – reproduced by $P_{\text{orb}}^{(c,c)}(\phi) =$ 

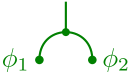
# Generalized LG orbifolds – bulk sector

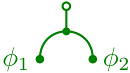
- generalization: allow any defect $A$ with separable Frobenius algebra structure
- the RR g.s. and (c,c) fields are then images of the projectors

$P^{RR}_{\text{orb}}(\phi) =$  $=$  $\qquad P^{(c,c)}_{\text{orb}}(\phi) =$ 

$\gamma_A =$  **Nakayama automorphism** $\Rightarrow \gamma_A = 1_A$: {RR g.s.} $\cong$ {(c,c)}

(spectral flow)

- (c,c) fields endowed with a commutative product $\quad \phi_1 \cdot \phi_2 =$ 

- topological bulk pairing: $\langle \phi_1, \phi_2 \rangle = \Big\langle$  $\Big\rangle_W$

  ($\langle - \rangle_W$ ordinary bulk pairing in unorbifolded theory)
  computes pairing on RR ground states

  consistency check: nondegenerate in the generalized orbifold theory

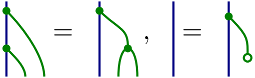
# LG orbifolds – boundary sector

- a boundary $Q$ in LG orbifold is described by a $G$-**equivariant matrix factorization**, i.e. there is a representation $\gamma$ of $G$ on $Q$, s.t.

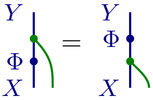
$$\gamma\, d_Q(gx_i)\, \gamma^{-1} = d_Q(x_i)$$

- for generalized orbifolds, boundaries are $A$-**modules**


$: X \otimes A \longrightarrow X$ s.t.



boundary fields **module maps**




- for $A = A_G$ this reproduces $G$-equivariant matrix factorizations [Carqueville, Runkel '12]

# LG orbifolds – topological disk correlators

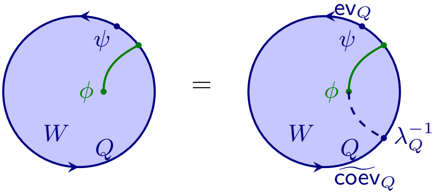
- for ordinary LG orbifolds proposal by [Walcher '04]

$$\langle \phi_g \rangle_Q = \mathrm{Res}\left[ \frac{\phi_g^{\text{inv.}}\, \mathrm{STr}[\gamma\, \partial_1 d_{\bar{Q}} \ldots \partial_r d_{\bar{Q}}]}{\partial_1 \overline{W} \ldots \partial_r \overline{W}} \right]$$

  with $\phi_g^{\text{inv.}}$ the polynomial part of $\phi_g$, and $\overline{W}$, $d_{\bar{Q}}$ are $W$, $d_Q$ with non-invariant variables set to zero.

  checks: Cardy condition, comparison with known D-brane charges

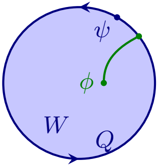
- defect approach:

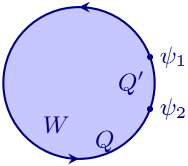



- reproduces above proposal for $A = A_G$

# Generalized LG orbifolds – boundary sector

- topological disk correlators:

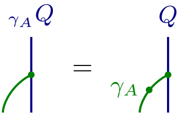
$\langle \phi\psi \rangle_Q =$ 


$\langle \psi_1, \psi_2 \rangle_{Q,Q'} =$ 


- consistency checks: boundary paring nondegenerate:

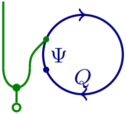
  boundary chiral sector paired with Ramond sector

  $\langle -, - \rangle_{Q,Q'} : \mathrm{Hom}(Q, Q') \times \mathrm{Hom}(Q', {}_{\gamma_A}Q[n]) \longrightarrow \mathbb{C}$

  nontrivial Serre functor $S_A = {}_{\gamma_A}(-)$ twisting the $A$-action: 

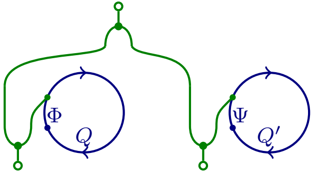

  where again $\gamma_A$ is the Nakayama automorphism $\gamma_A =$ 

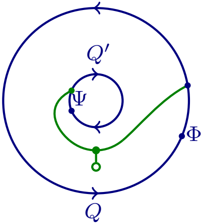
# Cardy condition

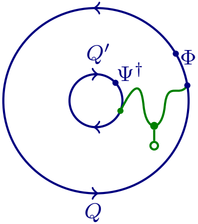
- **boundary-bulk map** $\beta_{\mathrm{orb}}^{Q}(\Psi) :=$ 

- Theorem: The **Cardy condition** holds for (generalized) LG orbifolds, i.e. $\left\langle \beta_{\mathrm{orb}}^{Q}(\Phi), \beta_{\mathrm{orb}}^{Q'}(\Psi) \right\rangle_{(W,A)} = \mathrm{str}({}_{\Psi}m_{\Phi})$

  Proof:

$\left\langle \beta_{\mathrm{orb}}^{Q}(\Phi), \beta_{\mathrm{orb}}^{Q'}(\Psi) \right\rangle_{(W,A)} = \Big\langle$  $\Big\rangle_W =$


$=$ 

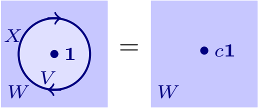
$=$ 

$= \mathrm{str}({}_{\Psi}m_{\Phi})$

using only the general (algebraic) properties of $A$ and $Q, Q'$

# Orbifold equivalences between LG models

- Recall: if there is a top. defect $X : V \to W$ with




  s.t. $c \equiv \mathrm{dim}(X) \in \mathbb{C} \setminus \{0\}$, one can describe $W$ as a generalized orbifold of $V$ with $A = X^\dagger \otimes X$

- in LG models amounts to finding matrix factorisation $X$ of $W(x) - V(z)$ s.t.

$$\mathrm{dim}(X) = \mathrm{Res}\left[\frac{\mathrm{STr}\left[\prod_i \partial_{x_i} d_X \; \prod_j \partial_{z_j} d_X\right] \underline{\mathrm{d}z}}{\partial_{z_1} V, \dots, \partial_{z_m} V}\right] \neq 0$$

- constructed explicitly between A-$\leftrightarrow$D-type and A-$\leftrightarrow$E-type singularities [Carqueville, Runkel '12], [Carqueville]

- Task: classify defects with invertible $\mathrm{dim}(X) \rightsquigarrow$ new equivalences between LG models and their D-brane categories ($\mathrm{mod}(X^\dagger \otimes X, V) \cong \mathrm{mf}(W)$) beyond the rational case

# Summary & Outlook

- describe orbifolds via **defects** $\rightsquigarrow$ **generalized orbifolds**
- one can recover standard results on LG orbifolds in this approach
- in particular, rigorously derive expressions for all topological correlators (e.g. **RR-charges** of D-branes, **eff. superpotentials**) and a new, simpler proof of Cardy condition
- all this can be carried over to the generalized setting
- several consistency checks: nondegeneracy of bulk and boundary pairings, Cardy condition

Outlook:

- understand discrete torsion and its effect on D-branes/defects from the generalized perspective [Brunner, Carqueville, DP in progress]
- find **new equivalences** between theories via a generalized orbifold construction (e.g. between different **CY** compactifications)