Intro

CMB: cosmic microwave background.

- at early times the Universe is filled with a plasma of photons / baryons / dark matter
- superhorizont - fluctuations from inflation
- at $T \sim eV$ there are several important events
  - matter - radiation equality $\epsilon_d + \epsilon_b \sim \epsilon_r$
  - recombination: $p^+ + e^- \to H^0 \to$ the Universe becomes transparent
- fluctuations in the plasma have been produced in inflation and enter the horizont

today: only fluctuations that enter after recombination

Makhanov: CMB-slow 0308072

- hydrodynamics in an expanding Universe App B
- from hydrodynamics to temperature p. ~~4~~ 3 - 6
- Sachs-Wolfe effect
- multipoles

} p 7 - 9

Hydrodynamics in an expanding universe

We consider hydrodynamics in an expanding universe

$$ds^2 = a^2 \{ (1+2\phi)\, d\eta^2 - (1-2\phi)\, \delta_{ij}\, dx^i dx^j \} \qquad (7)$$

and several components with a energy momentum tensor of the form:

$$T^\alpha{}_\beta = (\epsilon + p) u^\alpha u_\beta - p\,\delta^\alpha{}_\beta$$

$$- \eta \left(P^\alpha{}_\gamma u_\beta{}^{;\gamma} + P^\gamma{}_\beta u^\alpha{}_{;\gamma} - \frac{2}{3} P^\alpha{}_\beta u^\gamma{}_{;\gamma} \right) \qquad (8)$$

[$P^\alpha{}_\beta = \delta^\alpha_\beta - u^\alpha u_\beta$ is a projection operator]

The equations of motion of this system are the energy-momentum conservation

$$T^\alpha{}_{\beta;\alpha} = 0$$

[seperately for the different components because

- DM only interacts grav.
- baryons are conserved]

and the Einstein equation.

In lowest order one finds the solutions.

$$\epsilon(x) = \bar{\epsilon} + \delta\epsilon(x) ..$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G a^2 (\bar{\epsilon}_r + \bar{\epsilon}_d + \bar{\epsilon}_b) \quad [\text{Einstein}]$$

$$a^4 \bar{\epsilon}_r = \text{const.} \quad ; \quad a^3 \bar{\epsilon}_d = \text{const} \quad ; \quad a^3 \bar{\epsilon}_b = \text{const}$$

$$u^\mu = \left[\frac{1}{a}(1 - \Phi), \vec{0} \right] \quad ; \quad u_\mu \left[a(1+\Phi), \vec{0} \right]$$

$$\Rightarrow \boxed{u^\mu u_\mu = 1}$$

this is the background solution we perturb around.

The fluctuations fulfill the equats. $\left(' = \frac{d}{d\eta} \right)$ (120)

$$\delta\epsilon' + 3H(\delta\epsilon + \delta p) - 3(\epsilon + p)\Phi' + a(\epsilon + p) u^i_{;i} = 0$$

$$\frac{1}{a^4}\left(a^5 (\epsilon + p) u^i_{,i}\right)' - \frac{4}{3}\eta \Delta u^i_{,i} + \Delta \delta p + (\epsilon + p)\Delta\Phi = 0$$

and the Einstein eq.

$$[\delta\epsilon_d + \delta\epsilon_b + \delta\epsilon_\gamma]$$

$$\Delta\phi - 3H\phi' - 3H^2\Phi = 4\pi G a^2 \left[\epsilon_d \delta_d + \frac{1}{3c_s^2} \epsilon_\gamma \delta_\gamma \right]$$

where we defined the energy contrast

$$\delta_x = \frac{\delta\epsilon_x}{\epsilon_x}$$

and the speed of sound $c_s^2 = \frac{\delta p}{\delta\epsilon} = \frac{\delta p_\gamma}{\delta\epsilon_\gamma + \delta\epsilon_b} = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\epsilon_b}{\epsilon_\gamma}\right)^{-1}$

dark matter:

$$p=0\,;\ \delta p=0\,;\ \eta=0 \qquad \delta_x = \frac{\delta\epsilon_x}{\epsilon_x}$$

$$\Rightarrow \quad (\delta_d - 3\Phi)' + a u^i_{d\,,i} = 0$$

$$\rightarrow \quad \boxed{\left(a(\delta_d - 3\Phi)'\right)' - a\,\Delta\Phi = 0} \quad (A)$$

baryons + photons: b: $p=\delta p = 0$; γ: $\epsilon = 3p$; $\delta\epsilon = 3\delta p$

$$(\delta_b - 3\Phi)' + a u^i_{,i} = 0$$

$$(\delta_\gamma - 4\Phi)' + \frac{4}{3} a u^i_{,i} = 0$$

} same u^i since coupled strongly

$$\hookrightarrow \quad \frac{3}{4}\delta_\gamma - \delta_b = \text{const} = \frac{\delta s}{s} = 0$$

$$\boxed{\delta_b = \frac{3}{4}\delta_\gamma} \quad (B) \quad \rightarrow \quad \epsilon_b \lesssim \epsilon_\gamma$$

and using this in the Euler eq.

$$(C) \quad \boxed{\left(\frac{\delta_\gamma'}{c_s^2}\right)' - \frac{3\eta}{\epsilon_\gamma a}\Delta\delta_\gamma' - \Delta\delta_\gamma = \frac{4}{3c_s^2}\Delta\Phi + \left(\frac{4\Phi'}{c_s^2}\right)' - \frac{12\eta}{\epsilon_\gamma a}\Delta\Phi'}$$

using (B) in the Einstein equations:

$$\delta\epsilon_d + \delta\epsilon_b + \delta\epsilon_\gamma \rightarrow \epsilon_d\delta_d + \frac{1}{3c_s^2}\epsilon_\gamma\delta_\gamma$$

Comments:

- these equations assume that the plasma is locally in equilibrium

  after recombination → photons stream freely

  → Boltzmann eq. without collision terms

- modes that enter at different times behave differently:

  $\eta > \eta_{eq}$: radiation domination

  Φ is mostly depending on $\delta\epsilon_\gamma$

  δ_d grows slowly

  Silk damping only for $\eta < \eta_{eq}$

$\eta_r < \eta < \eta_{eq}$: Φ is mostly depending on $\delta\epsilon_d$

$\delta_d \sim \eta^2$

Silk damping

$\eta < \eta_r$ free streaming (no damping)

· long wavelength : $\boxed{k << \eta_r^{-1}}$

Euler: $u^i{}_{;i} \to 0 \quad \Rightarrow \quad (\delta_\gamma - \cancel{4}\phi) = C$

$$-3H\phi' - 3H^2\phi = 4\pi G a^2 \left(\epsilon_d \delta_d + \frac{1}{3c_s^2} \epsilon_\gamma \delta_\gamma \right)$$

matter-dom: $\delta_d \simeq -2\Phi$

rad-dom: $\delta_\gamma \simeq -2\Phi$

$\Big\}$ 9/10 $\left[\delta_b = \frac{3}{4} \delta_\gamma \right]$

$$\Rightarrow C = -6\phi_0 \Rightarrow \delta_\gamma^{(\eta_r)} = -6\Phi_0 + 4\Phi_r \simeq \frac{\cancel{4}}{3}\delta_b \simeq -\frac{8}{3}\Phi_r$$

$$\boxed{\Rightarrow \phi_r = \frac{9}{10}\Phi_0}$$

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from hydro to temperature:

radiation that streams freely is well described by

$$f(\vec{x}, t, \vec{p}) \simeq \frac{1}{e^{u^\mu p_\mu / T} - 1} \qquad \begin{array}{l} T = \bar{T} + \delta T \\ u = u(x) \end{array}$$

we want to match this to the hydro system:

$$T^\alpha{}_\beta = \frac{1}{\sqrt{-g}} \int d^3p \, f(p) \frac{p^\alpha p_\beta}{p^0}$$

for the 00-component one finds:

$$T^0_{\ 0} = \epsilon_\gamma (1+\delta_\gamma) = \epsilon_\gamma + 4\epsilon_\gamma \int \frac{\delta T}{T} \frac{d^2 n}{4\pi}$$

for i0:

$$T^i_{\ 0} = 4\epsilon_\gamma \int \frac{\delta T}{T} n^i \frac{d^2 n}{4\pi} = \frac{4}{3} u^i \epsilon_d$$

$$\Rightarrow \quad \delta'_\gamma = -4 \int n^i \nabla_i \left(\frac{\delta T}{T}\right) \frac{d^2 n}{4\pi} \qquad (\Phi' = 0)$$

$$\hookrightarrow \quad \left(\frac{\delta T}{T}\right)(\eta, k) = \frac{1}{4}\left(\delta_{r,k} + \frac{3i}{k^2}(k \cdot n)\, \delta'_k\right)$$

After recombination:

$$\frac{\partial f}{\partial \eta} + n^i (1 + 2\Phi) \frac{\partial f}{\partial x^i} + 2p \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial p_i} = 0$$

$$\hookrightarrow \quad (\bar{T} a)' = 0$$

$$\left(\frac{\partial}{\partial \eta} + n^i \frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T} + \Phi\right) = 2 \frac{\partial \Phi}{\partial \eta}$$

$\frac{\partial \phi}{\partial \eta} = 0$ (no integrated SW effect)

$$\frac{\delta T}{T}(\eta_0, x_0, n^i) = \frac{\delta T}{T}(\eta_r, x^i(\eta_r); n_i) + \Phi(\eta_r, x^i(\eta_r)) - \Phi(\eta_0, x_0^i)$$

$\nearrow n_i$ indep

Fourier:

$$\frac{\delta T}{T}(\vec{x}_0, \eta_0, \vec{n}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(\Phi + \frac{\delta}{4} - \frac{3\delta_k'}{4k^2}\frac{\partial}{\partial \eta_0} \right) e^{ik(x_0 + \vec{n}(\eta_i - \eta_0))}$$

measure:

$$C(\Theta) = \left\langle \frac{\delta T}{T_0}(n_1) \frac{\delta T}{T_0}(n_2) \right\rangle \qquad n_1 \cdot n_2 = \cos\Theta$$

$$= \frac{1}{4\pi} \sum_{l=2}^{\infty} (2l+1)\, C_l\, P_l(\cos\Theta)$$

legendre polynomials

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for $k \ll \eta_r^{-1}$ $\qquad (l < \cancel{200}\ 30)$

$$\delta_k = -\frac{8}{3}\Phi_k \quad ; \quad \delta_k' \simeq 0$$

$$= -\frac{8}{3}\frac{9}{10}\Phi_0$$

inflation:

$$\langle \Phi_0(k)\, \Phi_0(k') \rangle = \delta(k+k')\, P_\Phi(k)$$

$$P_\Phi(k) = N_\Phi k^{n-4} \quad ;$$

$\underline{\underline{n=1}}$ $\qquad l(l+1)\, C_l \simeq \text{const} \longrightarrow$ no dependence on $\eta_r(\eta_0^-)$

no dependence on physics

however: $\qquad \frac{\Delta C_l}{C_l} \simeq \frac{1}{\sqrt{2l+1}}$ $\qquad$ cosmic variance