Workshop
Seminar
2013

"Nielsen - Olesen vortex"

Plan:

1. Introduction. Definition of the soliton
2. Bounds on soliton existence in simplest field theories
3. Derivation of the Nielsen-Olesen vortex
4. Properties
5. Applications, related sujets and further developments (if we have time)

Literature

Original papers:

Abrikosov (1966) – condensed matter

Nielsen, Olesen (1973) – field theory

Text books:

Rubakov, „Classical theory of gauge fields" (p. II, §7.3)

Rajaraman, „Solitons and instantons" (§3.6)

Ryder, „Quantum field theory" (§10.2)

Schwarz, „Quantum field theory and topology" (§2.3)

Weinberg, „Quantum theory of fields", II, §21.6 (Superconductivity)

Soliton is any stable particle-like solution of the classical equations of motion (EOM) which is essentially non-linear.

In what follows we restrict ourselves to static solutions.

In cosmological context finding solitons is interesting, because we could possibly see them in Early Universe (monopoles, vortex strings, e.t.c.)

Derrick (1964)

Consider first a pure scalar theory in $(d+1)$ dimensions:

$$\mathcal{L} = \frac{1}{2} F_{ab}(\varphi)\, \partial_\mu \varphi^a \partial_\mu \varphi^b - V(\varphi) \qquad a = \overline{1,n} \quad \mu = \overline{0,d}$$

$F_{ab}(\varphi)$ - positive definite

$V(\varphi)$ - bounded below $(V \geq 0)$

Minkowski signature $(+,-,-,-,\dots -)$

Let $\tilde{\varphi}^a(\vec{x})$ be a static solution of EOM. It has to extremize static energy:

$$E_s[\varphi] = \int d^dx \left[\tfrac{1}{2} F_{ab}(\varphi)\, \partial_i \varphi^a \partial_i \varphi^b + V(\varphi) \right], \quad i = \overline{1,d}.$$

In particular, it should be the case for any one-parametric family of field configurations:

$\varphi_\lambda(\vec{x}) \overset{\text{def}}{=} \tilde{\varphi}(\lambda x)$ // a is irrelevant

(if λ is close to 1, $\delta_\lambda \tilde{\varphi} = \tilde{\varphi}(\lambda\vec{x}) - \tilde{\varphi}(\vec{x})$ is a small field variation vanishing at infinity)

$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=1} = 0, \qquad E(\lambda) \equiv E[\varphi_\lambda(\vec{x})].$$

Performing simple rescaling in integrand, we obtain:

$E(\lambda) = \lambda^{2-d} + \lambda^{-d}\, \Pi$, where

$\Gamma \overset{\text{def}}{=} \int d^dx\, \tfrac{1}{2} F_{ab}(\tilde{\varphi})\, \partial_i \tilde{\varphi}^a \partial_i \tilde{\varphi}^b > 0$

$\Pi \overset{\text{def}}{=} \int d^dx\, V(\tilde{\varphi}) > 0$

don't depend on λ

Extremum condition:

$$(2-d)\Gamma - d\,\Pi = 0$$

$\Downarrow$

NO solitons for theory in $(\underset{\overset{\|}{d}}{2}+1)$ dimensions with non-trivial potential

(solitons are present only in „free" theories with some complicated kinetic term, e.g., O(3) model – Belavin, Polyakov, 1975)

Physically, this means that for particle-like non-linear configurations in (2+1) pure scalar theory with $V(\varphi) \neq 0$ it is energetically more preferable to shrink down:

$$\begin{array}{llccc} & & \varphi^a(\vec{x}) & & \varphi^a(\lambda\vec{x}) \\ \text{If } \lambda > 1 : & \text{Size} \sim & R & > & R/\lambda \\ & \text{Energy} & \Gamma + \Pi & > & \Gamma + \frac{\Pi}{\lambda^2} \end{array}$$

What can one do to evade this NO-GO argument?

Either add some terms with higher derivative powers (Skyrme, 1961), then

$$\frac{dE}{d\lambda} = (4-d)\,\Gamma^{(4)} + (2-d)\,\Gamma^{(2)} - d\,\Pi = 0$$

or add one gauge field:

$$E(\lambda) = \lambda^{4-d}\, G + \lambda^{2-d}\, \Gamma^{(\text{covariant})} + \lambda^{-d}\, \Pi ,$$

where $G = \int d^d x \left(-\frac{1}{2g^2} \operatorname{Tr} F_{ij}(\vec{x}) F_{ij}(\vec{x}) \right) > 0$

Then, for $d = 2$: $G = \Pi$

NO boundaries on soliton existing in (2+1) scalar + gauge field theory.

LET US FIND ONE!

③ Consider an Abelian Higgs model in $(2+1)$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \varphi|^2 - V(\varphi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \mu, \nu = 0,1,2$$

$$D_\mu \varphi = (\partial_\mu - ie A_\mu)\varphi$$

$$V(\varphi) = \frac{\lambda}{2}\left(|\varphi|^2 - v^2\right)^2$$

$U(1)$ gauge transformation:

$$\varphi(x) \to e^{i\alpha(x)} \varphi(x)$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$

Vacuum:

$$A_\mu = 0$$

$$\varphi = v$$

Linear content:

Higgs mechanism ensures us to have the following spectrum of linear excitations above vacuum

1 vector Boson (massive) with mass $m_V = \sqrt{2}\, e v$

1 scalar Higgs with mass $m_H = \sqrt{2\lambda}\, v$

Let's find a static field configuration which is particle-like and stable (but is not a linear excitation above vacuum).

Choose a temporal gauge $A_0 = 0$ (we're still left with some gauge freedom of spatial components A_i which we'll exploit in what follows).

Static energy of a field configuration is:

$$E_s[A_\mu, \varphi] = \int d^2x \left[\frac{1}{4} F_{ij}F_{ij} + |D_i\varphi|^2 + V(\varphi)\right] \qquad i = 1,2$$

It has to be finite. First of all, we need:

$$\lim_{|\vec{x}|\to\infty} \varphi(\vec{x}) = v e^{if(\theta)}$$

// from the 3rd term in E_s

The simplest non-trivial function is $f(\theta) = \theta$.

Finiteness of the 2nd term implies:

$$\lim_{|\vec{x}|\to\infty} D_i\varphi = \lim_{|\vec{x}|\to\infty} (\partial_i\varphi - ieA_i\varphi) =$$

$$= -ive^{i\theta}\frac{\varepsilon_{ij}n_j}{r} - iev e^{i\theta} \lim_{|\vec{x}|\to\infty} A_i .$$

If $D_i\varphi \sim \frac{1}{r}$, $\int |D_i\varphi|^2 d^2x \sim \int \frac{dr\,d\theta}{r}$, diverges logarithmically.

$r^2 \equiv x_i x_i$

$n_i \stackrel{\text{def}}{=} \frac{x_i}{r}$

ε_{ij} - Levi-Civita symbol ($\varepsilon_{12} = 1$)

We need to compensate the first term in $D_i\varphi$ to have finite energy, which gives an asymptotic for A_i:

$$\lim_{|\vec{x}|\to\infty} A_i = -\frac{1}{er}\varepsilon_{ij}n_j = \frac{1}{e}\partial_i\theta \rightarrow$$

2nd term is finite
1st term is zero

Remark:
Sometimes people don't care about this divergence of $\partial_i \varphi$ and consider vortex-like „soliton" just for pure scalar theory (which, of course, then has infinite energy). It is called „logarithmic vortex" and has many common properties with usual Nielsen-Olesen vortex. We'll not consider it in what follows.

We have the asymptotics. What Ansatz we should make to find a solution of EOM with given asymptotics?

Answer („Coleman theorem"):
One has to use the <u>most general</u> Ansatz which is still compatible with the symmetries of the asymptotics. Then it will pass through EOM.

In our case, the symmetry is:

$$\varphi'(\theta) = e^{-i\alpha}\,\varphi(\theta+\alpha)$$

$$A'_i = \underbrace{\Omega_{ij}}_{SO(2)\ \text{matrix}} A_j$$

The corresponding Ansatz:

$$\varphi(r,\theta) = v e^{i\theta} F(r)$$

$$A_i(r,\theta) = -\frac{1}{er}\varepsilon_{ij} n_j A(r) + \underbrace{\cancel{n_i B(r)}}_{=\partial_i\left(\int^r B(r')\,dr'\right)}$$

We use residual gauge symmetry to fix a gauge in which $B(r)=0$.

Unknown functions A and F should have asymptotics like:

$$\lim_{r\to\infty} F(r) = \lim_{r\to\infty} A(r) = 1.$$

To prevent singularities at the origin, we must impose:

$$F(r) = O(r), \; A(r) = O(r^2) \text{ when } r \to 0.$$

It is straightforward to check that the Ansatz passes through EOM. Let's do it other way around: substitute the Ansatz to the E_S and vary w.r.t. $A(r)$, $F(r)$ (Maupertuis'-like principle):

$$E_S[A,F] = 2\pi \int dr \left[\frac{A'^2}{2e^2 r} + v^2 r (F')^2 + v^2 \frac{F^2(1-A)^2}{r} + \frac{\lambda v^4}{2} r (F^2-1)^2 \right].$$

$$\Downarrow$$

$$\begin{cases} \dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{dA}{dr}\right) = -\dfrac{2e^2 v^2}{r} F^2 (1-A) \\ \dfrac{d}{dr}\left(r \dfrac{dF}{dr}\right) = \dfrac{F}{r}(1-A)^2 + \lambda v^2 r F (F^2-1). \end{cases}$$

These equations, together with the asymptotics and the Ansatz, define

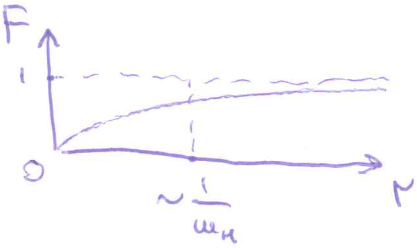
Abrikosov - Nielsen - Olesen vortex.

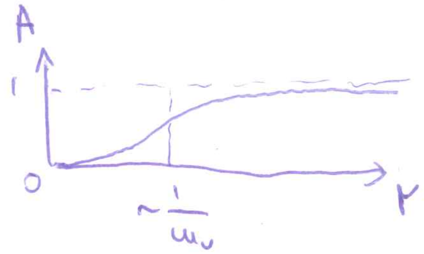
4 I. Does the solution of above system indeed exist?

II. If yes, why should it be stable?

III. What are vortex' size and mass?

I Full analysis of the existence of the solution: Taubes (1980).





We'll consider the more simple case $m_H < 2m_V$ (to outline some heuristic argument which is helpful by searching for solutions of such ODE systems).

One needs to show that the system has 2-parametric family of solutions both in a case when one uses only $r \to 0$ asymptotics or only $r \to \infty$ asymptotics. Then one can hope to glue these two families together at some intermediate point r_0:

$$\begin{cases} 4 \text{ equations for gluing } F(r_0), A(r_0), F'(r_0), A'(r_0) \\ \text{and } 2+2=4 \text{ parameters} \end{cases}$$

Usually such system then has a solution (it works for all such solitons).

Also,

$r \to \infty$:

Define $A(r) \stackrel{\text{def}}{=} 1 - a(r)$, $F(r) \stackrel{\text{def}}{=} 1 - f(r)$, $\lim_{r\to\infty} a = \lim_{r\to\infty} f = 0$.

$$\begin{cases} r\left(\frac{a'}{r}\right)' = \omega_V^2 a \\ \frac{1}{r}(rf')' = \omega_H^2 f \end{cases} \to \begin{aligned} a(r) &= c_a \sqrt{r}\, e^{-\omega_V r} \\ f(r) &= c_f \frac{e^{-\omega_H r}}{\sqrt{r}} \end{aligned} \to (c_a, c_f).$$

$r \to 0$:

Expand:

$$\begin{cases} F(r) = \alpha_f r + \beta_f r^3 + \dots \\ A(r) = \alpha_a r^2 + \beta_a r^4 + \dots \end{cases} \quad \left(\lim_{r\to 0} A = \lim_{r\to 0} F = 0. \right)$$

Substituting, we'll see that other parameters can be expressed using only two, e.g.:

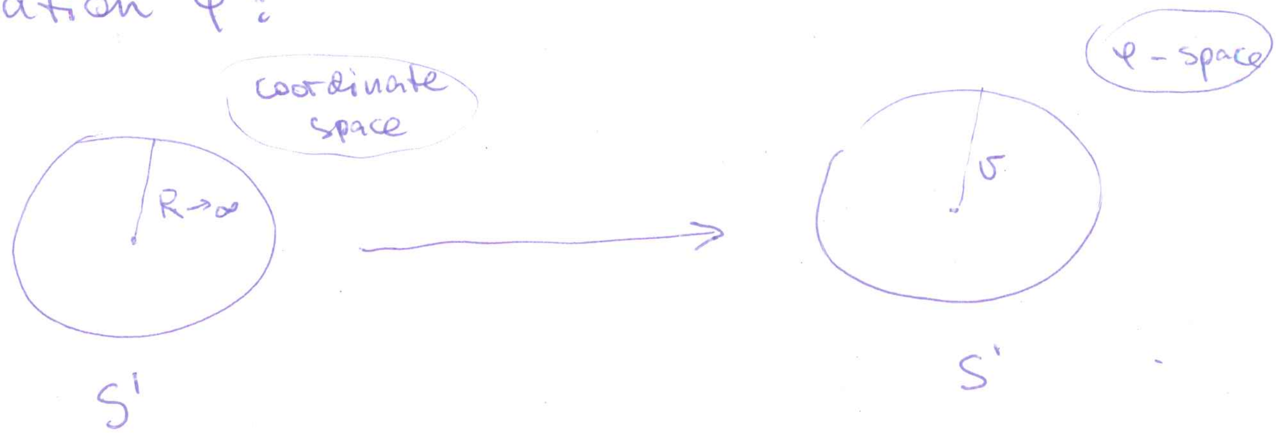
$$\begin{aligned} \beta_a &= -\frac{\omega_V^2}{8} \alpha_f^2 \\ \beta_f &= -\frac{\omega_H^2}{16} \alpha_f - \frac{1}{4} \alpha_a \alpha_f \end{aligned} \to (\alpha_a, \alpha_f).$$

II Consider again the asymptotic behaviour of the scalar field in our model:

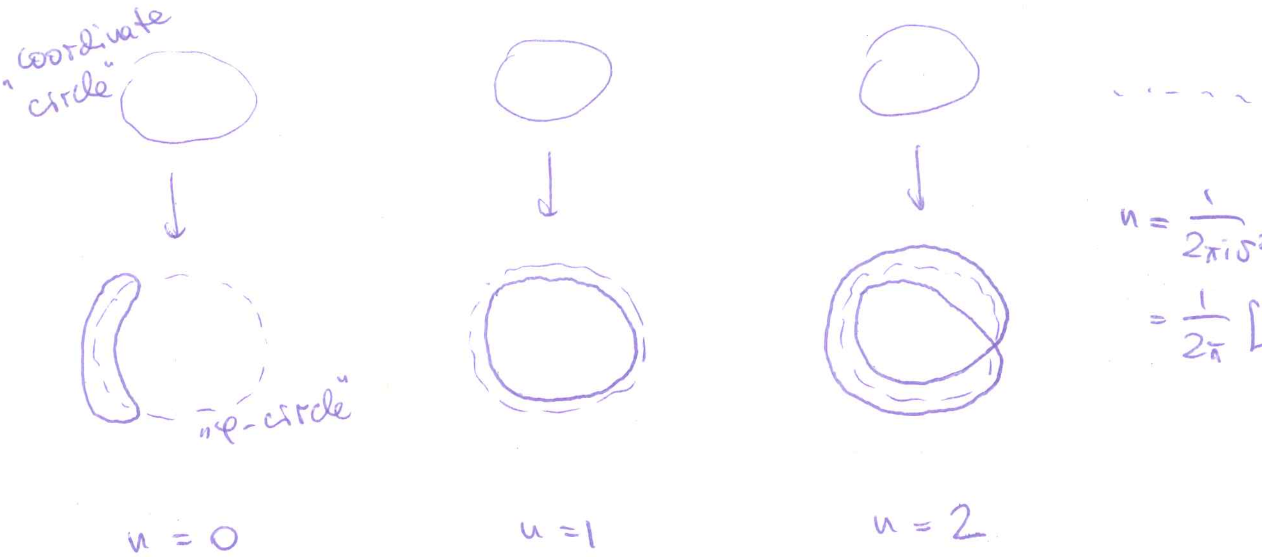
$$\varphi(\infty, \theta) = v e^{if(\theta)}$$

We see that it is governed by the way in which the angle coordinate θ is mapped onto

Space of possible asymptotics of field configuration φ:



This map can be characterized by an integer $n \in \mathbb{Z}$:



$$n = \frac{1}{2\pi i v^2} \oint dx^i \varphi^* \partial_i \varphi = \frac{1}{2\pi}\left[f(2\pi) - f(0)\right].$$

One cannot change winding number n using smooth gauge transformations (e.g., $e^{i\alpha(x)} = e^{-i\theta}$ is singular at the origin).

Mathematicians write it like:

$$\pi_1(S^1) = \mathbb{Z}.$$

fundamental group

Why is it useful?

vacuum $(n=0)$
vortex $(n=1)$
$(n=2)$
...
...

$\Longrightarrow$ The vortex is stable.

To dissolve into vacuum it needs infinite amount of energy (if our space has infinite size).

This is quite common mechanism for many solitons encountered in field theory:

$$\boxed{\text{TOPOLOGICAL SOLITONS}}$$

III. Estimate mass and size considering $\frac{m_H}{m_V} \sim 1$

(actually, M_{sol} depends on $\frac{m_H}{m_V}$ only logarithmically, so our approximation is not ~~so~~ bad).

Let us just go to dimensionless variables (designed in such manner that all three summands in E_s have the same order when $\Phi \sim 1$, $C_i \sim 1$, $y \sim 1$):

$$\vec{y} \overset{\text{def}}{=} m_V \vec{x} \qquad C_{ij} = \frac{\partial C_j}{\partial y_i} - \frac{\partial C_i}{\partial y_j}$$

$$\varphi(\vec{x}) \overset{\text{def}}{=} v\,\Phi(\vec{y})$$

$$A_i(\vec{x}) \overset{\text{def}}{=} \frac{m_V}{e} C_i(\vec{y}) \qquad D_i \Phi = \left(\frac{\partial}{\partial y_i} - i C_i\right)\Phi.$$

Static energy then looks like:

$$E_s = \frac{m_v^2}{e^2} \int d^2y \left[\frac{1}{4} C_{ij} C_{ij} + \frac{1}{2} |D_i \Phi|^2 + \frac{1}{8} \frac{m_H^2}{m_v^2} (|\Phi|^2 - 1)^2 \right].$$

If $m_H \sim m_V$, the expression in brackets has no dimensionful parameters, therefore its minimum is realized on configuration with

$\Phi \sim 1$, $C_i \sim 1$, and characteristic size of the soliton is $y \sim 1$. Which means that

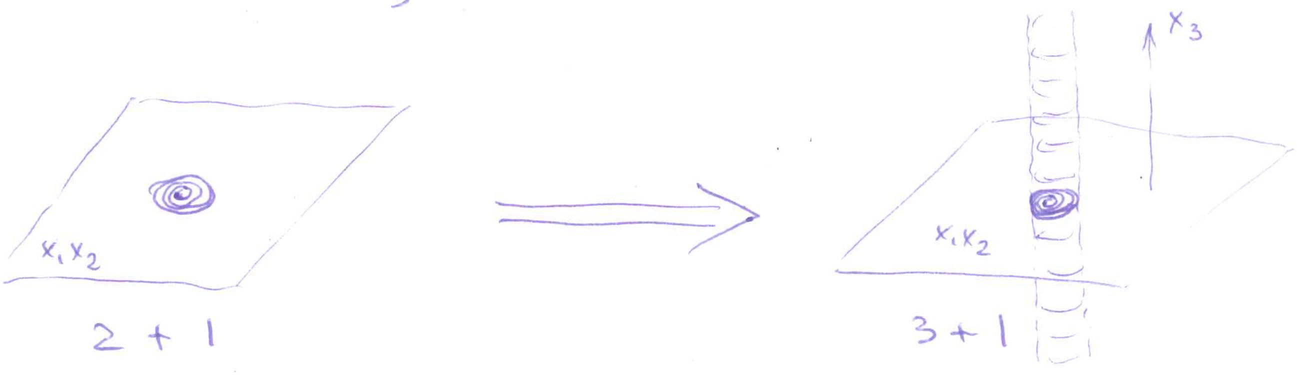
$$\boxed{\begin{aligned} R_{sol} &\sim \frac{1}{m_v} \\ M_{sol} &= E_s\big|_{min} \sim \frac{m_v^2}{e^2} \underbrace{M}_{\text{some function (actually, a logarithm)}}\left(\frac{m_H}{m_v}\right). \end{aligned}}$$

5

*) Vortex string

What is the use in (3+1)?

We can consider our vortex as a solution in (3+1) Abelian Higgs model which doesn't depend on x_3 and has $A_3 = 0$:



For a vortex with topological number n, we have:

$$A_i = \frac{n}{e}\partial_i\theta \qquad i=1,2$$

which means:

$$n = \frac{e}{2\pi}\oint A_i\, dx^i \xrightarrow{\text{Stokes' theorem}} \boxed{n = \frac{e}{2\pi}\int_{(x_1,x_2)} H_3\, d^2x}$$

Magnetic flux of a vortex string through a perpendicular plane is quantized! (already in the classical theory)

M_{sol} is then a vortex str. mass per unit length.

*) Superconductivity

$E_s(A_\mu, \varphi)$ is actually a free energy of the superconductor in Landau-Ginzburg model

In type II superconductors ($\lambda >> e^2$) there is a lattice of many $n=1$ vortices.

We can actually see them.

*) instability of $n > 1$ vortices under quantum fluctuations => Bogomolnyi (1976)

*) interaction of two vortices (effective potential) => => Schaposnik (1978)

*) interaction of vortices with matter (scattering of fermions on a vortex string) => Alford, Wilczek (1989)

*) vortex lines in pure SU(2) Yang-Mills => => Ambjørn, Olesen (1980)