# Monopoles and Domain Walls

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T. KALAYDZHYAN

## Plan of the talk:

1. Kinks and domain walls
2. Dirac monopole
3. 't Hooft – Polyakov monopole
4. General statement
5. QCD monopoles and confinement (briefly)

## Literature:

1. D. Gorbunov, V. Rubakov "Introduction to the Theory of the Early Universe".
2. V. Rubakov "Classical theory of gauge fields".
3. A. Di Giacomo "Monopole condensation and colour confinement", hep-lat/9802008.
4. See also Refs. in Misha's talk.

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## ① Domain walls.

Let's consider the simplest situation:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2 .$$

It is invariant under $\mathbb{Z}_2$, which can be broken

Spontaneously:

$\langle \phi \rangle = \pm v$ (space of vacua consists of 2 points)

Field configurations interpolating between the two vacua are known as domain walls (DW)

Our Ansatz: $\phi = \phi(z)$,

$$\text{EOM}: \begin{cases} \partial_z^2 \phi - \lambda(\phi^2 - v^2)\phi = 0 \\ \partial_x^2 \phi = \partial_y^2 \phi = 0 \end{cases}$$

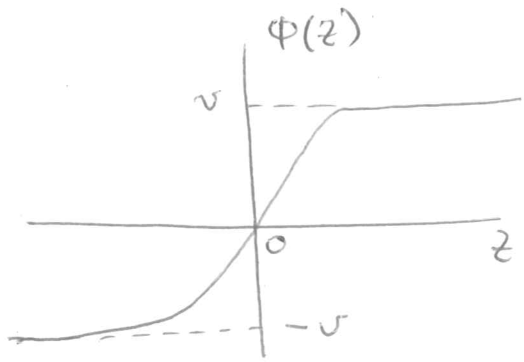
\+ boundary conditions

$$\phi(z \to \pm\infty) = \pm v \quad \text{(kink)}$$

$$\phi(z \to \pm\infty) = \mp v \quad \text{(antikink)}$$

Solutions:
$$\phi(z) = v \tanh\frac{z}{\Delta} \quad \text{(kink)}$$
$$\phi(z) = -v \tanh\frac{z}{\Delta} \quad \text{(antikink)},$$

where $\Delta \equiv \sqrt{\frac{2}{\lambda}} \cdot \frac{1}{v}$ – DW thickness.




$$T^{\text{scalar}}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \mathcal{L}\eta_{\mu\nu}$$

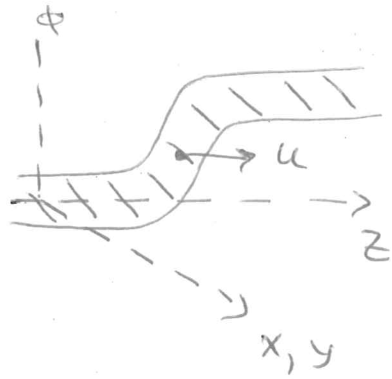
for the DW:

$$T^{DW}_{\mu\nu} = \frac{\lambda v^4}{2\cosh^4 \frac{z}{4\Delta}} \cdot \text{diag}(1,-1,-1,0),$$

So $\Delta$ is indeed the thickness.

we can also boost the solution:

$$\phi(z) = v \tanh\left( v \sqrt{\tfrac{\lambda}{2}} \, \frac{(z-z_0)-ut}{\sqrt{1-u^2}} \right),$$




where $u$ is the velocity of DW.

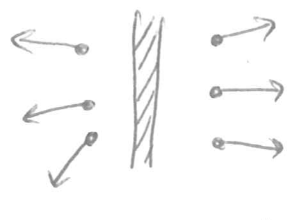
Stability of this solution in (1+1) (in Liapunov sense) can be easily proven, read Rubakov's text book.

Comment: what if we turn on gravity?

We can then linearize the Einstein's equations and find the Newtonian potential for DW:

$$\Delta \Phi = -4\pi G \, T_{00}^{DW} \qquad (g_{00} = 1 + 2\Phi),$$

So, it produces anti-gravity, i.e. non-relativistic particles would be repelled by DW.



## (2) U(1) monopoles and their topology.

Maxwell's equations in the presence of both electric, $j^\nu$, and magnetic $j_M^\nu$ currents:

$$\partial_\mu F^{\mu\nu} = j^\nu, \qquad \partial_\mu F^{*\mu\nu} = j_M^\nu,$$

$F^{*\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$ – dual tensor.

1) if $j_M^\nu \equiv 0$, then $\partial_\mu F^{*\mu\nu} = 0$ –

Bianchi identity, solved by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

2) what if $j_M^\nu \neq 0$ ?

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho_e \\ \vec{\nabla}\cdot\vec{B} = \rho_m \\ -\vec{\nabla}\times\vec{E} = \dot{\vec{B}} + \vec{j}_m \\ \vec{\nabla}\times\vec{B} = \dot{\vec{E}} + \vec{j}_e \end{cases}$$

Coulomb gauge $\text{div}\vec{A} = 0$

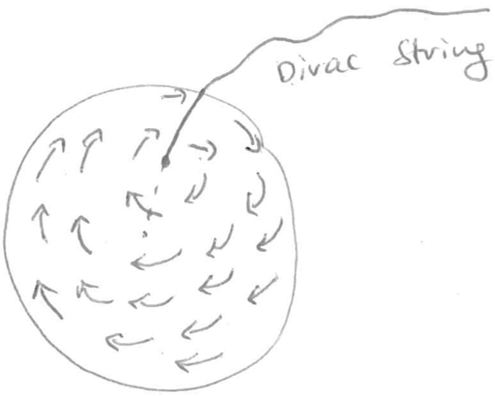
$\vec{B} = \text{rot}\vec{A}$ = const on $S^2$

$\text{div}\vec{B} = \rho_m$

Consider $\rho_m = M\,\delta(\vec{x})$.

"Hary ball theorem":

You can't comb a hairy ball without creating a cowlick, crown. (Brouwer, 1912)




Euler characteristic:

$\chi = 2 - 2g$ = vertices – edges + faces = $\Sigma$ of indices (zeros) of special points of the vector field.

$g$ = genus = number of handles.

Examples: $\chi_{S^2} = 2$, $\chi_{T^2} = 0$

i.e. one can comb a torus, but not a sphere.

$\Rightarrow$ our vector field should be singular $\Rightarrow$ Dirac string

By allowing the Dirac string to exist, we preserve the Bianchi identity.

The Dirac string is a thin solenoid and will be invisible (physically) if for any particle with charge $e$ the parallel transport around it is trivial:

$$\exp\left\{ie \oint \vec{A}\, d\vec{x}\right\} = 1, \qquad \text{i.e. } \Phi \cdot e = 2\pi n, \quad n \in \mathbb{Z}.$$

Magnetic flux $\Phi$ is related to $M$: $\Phi = M$.

$$eM = 2\pi n, \quad \text{hence} \quad \boxed{e = \frac{2\pi}{M} n} \text{ quantization.}$$

③ 't Hooft - Polyakov monopoles

Consider the Georgi-Glashow model:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2,$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, \epsilon^{abc} A^b_\mu A^c_\nu$

$$D_\mu \phi^a = \partial_\mu \phi^a + g\, \epsilon^{abc} A^b_\mu \phi^c.$$

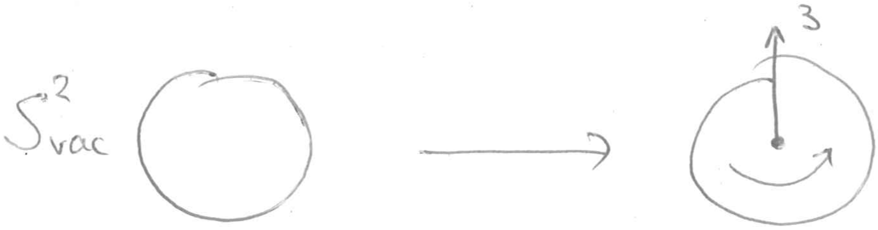
(i.e. triplet of scalars in adjoint of $SU(2)$).

the vacuum manifold $\langle \phi^a \phi^a \rangle = v^2$ is $S^2_{vac}$.

We can choose one particular vacuum as

$\langle \phi^a \rangle = \delta_3^a v$ and, by doing so, we break

$SU(2) \to U(1)$




$SU(2) \simeq SO(3) \times \mathbb{Z}_2$ $\qquad$ $U(1) \simeq SO(2)$

One can easily show that $A_\mu^3$ remains massless (gauge field of unbroken $U(1)$, a photon), while

$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \pm i A_\mu^2)$ and $h \equiv \phi^3 - v$.

obtain masses $m_v \equiv m_W = g v$ and $m_h = \sqrt{2\lambda}\, v$.

Let's study a static field of the form

$$A_0^a = 0, \quad A_i^a = A_i^a(\vec{x}), \quad \phi^a = \phi^a(\vec{x})$$

and require finiteness of energy

$$E = \int d^3x \left[ \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} D_i \phi^a D_i \phi^a + \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2 \right].$$

then at $r \equiv \sqrt{\vec{x}^2} \to \infty$ we get minimum (vacuum)

$(\phi^a)^2 = v^2$ and $A_\mu^a = 0$ (up to gauge transform.)

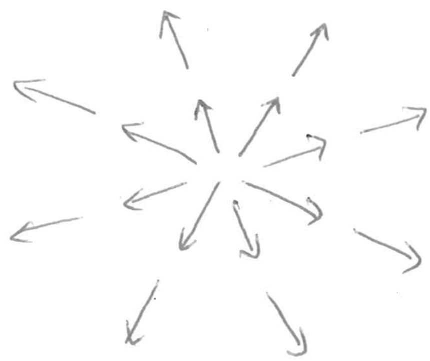
$E < \infty \Rightarrow F_{ij}^a, \ D_i \phi^a, \ (\phi^a)^2 - v^2$ should decay fast enough.

Let's now find a solution. In analogy with vortices, where solutions mapped $S^1_\infty \to S^1_{vac}$, here we have similar situation: $S^2_\infty \to S^2_{vac}$.

These solutions, again, can be classified by their topological numbers, since $\pi_2(S^2) = \mathbb{Z} \neq 0$.

the simplest non-trivial mapping:

$$\begin{cases} \phi^a \big|_{r\to\infty} = v\, n^a, & n^a \equiv \dfrac{x^a}{r} \quad \text{(hedgehog)} \\ A_i^a(\vec{x}) \big|_{r\to\infty} = \dfrac{1}{gr} \epsilon^{aij} n_j \end{cases}$$



the Ansatz:

$$\begin{cases} \phi^a = v\, n^a (1 - f(r)) \\ A_i^a = \dfrac{1}{gr} \epsilon^{aij} n_j (1 - a(r)) \end{cases}$$

this Ansatz is invariant under spatial rotations supplemented by global $SU(2)$ transformations.

Asymptotics for $a(r)$ and $f(r)$:

$f\big|_{r\to\infty} = a\big|_{r\to\infty} = 0$,

$(1-f)\big|_{r\to 0} \propto r$, $\quad (1-a)\big|_{r\to 0} \propto r^2$.

(from the absence of singulality at the center of the monopole and finiteness of energy).

$$E_i^a \sim \partial_0 A_i - \partial_i A_0 = 0$$
$$B_i^a = -\frac{1}{2}\epsilon_{ijk} F_{jk}^a = \frac{1}{gr^2} n_i n_a$$

Asymptotic electric and magnetic fields

- direction of $B_i^a$ in the internal space coincides with the Higgs field $\varphi^a$, so the $\varphi^a$ corresponds to the unbroken-$U(1)$ magnetic field.

the gauge-invariant field strength

$B_i = B_i^a \frac{\varphi^a}{v} = \frac{1}{g}\frac{n_i}{r^2}$ equals to the one of magnetic monopole of charge $g_M = 1/g$.

The massive vector and Higgs fields decay exponentially, and the field of this ('t Hooft - Polyakov) monopole coincides with the Dirac monopole field far from the origin.

Comments: 1) Antimonopole : $\varphi^a = -v n^a (1 - f(r))$

2) for $m_v \sim m_h$ we can estimate mass of the monopole:

we change the variables,

$$X = (gv)^{-1}\xi; \quad A_\mu^a = v\,\mathcal{A}_\mu^a; \quad \phi^a = v\varphi^a$$

then

$$E = \frac{m_v}{g^2}\int d^3\xi \left[\frac{1}{4}\mathcal{F}_{ij}^a\mathcal{F}_{ij}^a + \frac{1}{2}D_i\varphi^a D_i\varphi^a + \right.$$
$$\left. + \frac{m_h^2}{8m_v^2}\left(\varphi^a\varphi^a - 1\right)^2\right],$$

where $D_i\varphi^a = \partial_i\varphi^a + \epsilon^{abc}\mathcal{A}_i^b\varphi^c$ and so on.

the monopole configuration minimizes the energy functional. the integrand doesn't contain small/large parameters and is dimensionless, hence

$$m_M \simeq \frac{4\pi m_v}{g^2} = \frac{4\pi v}{g} \qquad \left(4\pi = \int d\Omega\right).$$

So, the monopole mass exceeds the scale $v$ of the symmetry breaking $SU(2) \to U(1)$

(4) Existence of monopoles is a general feature of the Higgs models.

$G$ – group of symmetries of the Lagrangian

$H \subset G$ – group of symmetries of the groud state.

Symmetry breaking pattern: $G \to H$, the vacua manifold is then $G/H$.

"Theorem": Stable monopoles are possible if and only if $\pi_2(G/H) \neq 0$.

(i.e. $G/H$ contains non-contractible 2-spheres and therefore the second homotopy group is not trivial)

Example 1: if $G$ is simple or semi-simple, and $H$ includes one factor $U(1)$, then

$\pi_2(G/H) = \pi_1(H) = \mathbb{Z}$, and monopoles exist.

Example 2: Standard Model

$G = SU(3) \times SU(2) \times U(1)$ is not semi-simple

$H = SU(3) \times U(1)$,

so $\pi_2(G/H) = 0 \Rightarrow$ no monopoles.

(you can also imagine $SU(2) = S^3$)

Example 3: in GUTs the $G$-group is (semi-) simple, for instance,

$$G = SU(5) \longrightarrow SU(3)_c \times SU(2)_w \times U(1)_Y \rightarrow \underbrace{SU(3)_c \times U(1)_{em}}_{H}$$

So $\pi_2(G/H) = \mathbb{Z} \Rightarrow$ monopoles.

(5) In pure Yang-Mills there are no Higgs fields, but one can still construct monopoles in Abelian gauge:

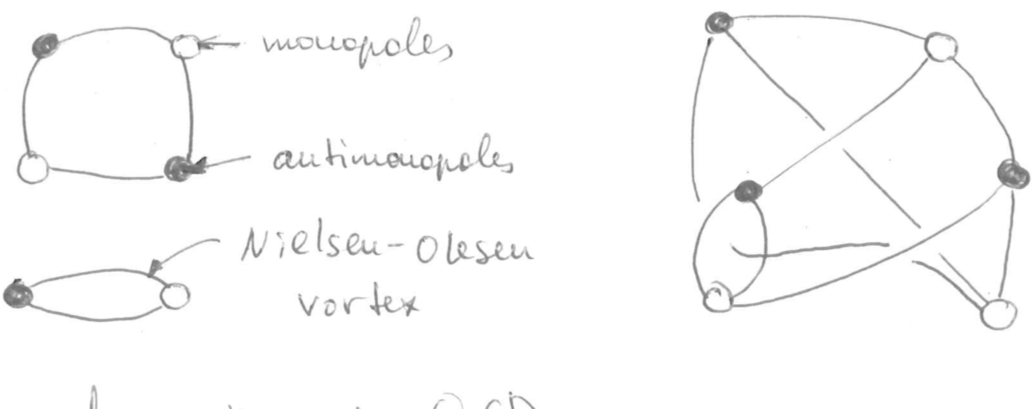
$$\int d^4x \left( \left[A_\mu^1(x)\right]^2 + \left[A_\mu^2(x)\right]^2 \right) = \min$$

$A_\mu^3(x)$ is then an Abelian field.

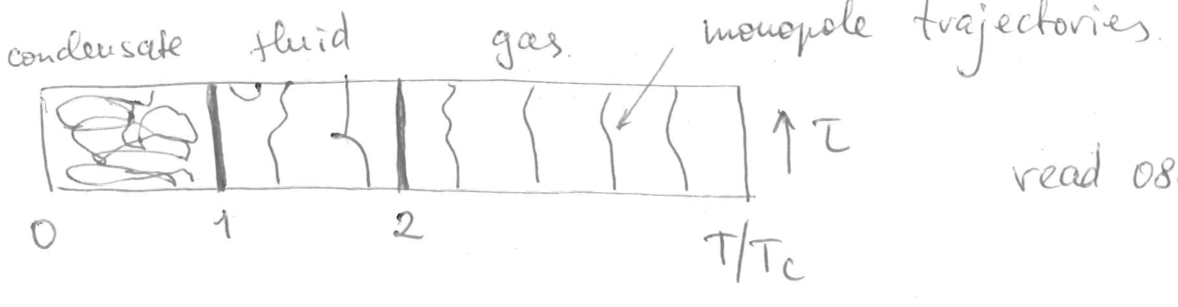
1) these monopoles play a role in quark confinement, i.e. the confinement manifests itself in condensation of monopoles, and they squeeze inter-quark fields into flux tubes (like the magnetic field is squeezed in type-II superconductors).

2) Vacuum structure

SU(2)




3) Thermodynamics of QCD

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