Workshop Seminar: Skyrmions & composite Higgs

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1 Introduction

Even thought QCD has been well understood at high energies, the low-energy descriptions still have several problems, which are mainly resulting from the absence of a small expansion parameter. It has been shown by 't Hooft that in the large N_c limit of $SU(N_c)$ gauge theories one can use $\frac{1}{N_c}$ as an expansion parameter. Those models (assuming the existence of a confined phase) reduce to a weakly coupled phase of mesons and glue balls, which decouple at leading order. One low-energy description of mesons is the non-linear σ-model. Since this model just describes the lightest meson states and no baryons, Skyrme's idea was to extend it and introduce baryons as solitons[1]. Solitons are defined as classical static solutions of the Euler-Lagrange equations with nontrivial topology and finite energy.

2 Non-linear σ-model of mesons

The elementary fields of the non-linear σ-model are the Nambu-Goldstone bosons associated with the spontaneous breaking of chiral symmetry of QCD $SU(2)_R \times SU(2)_L \rightarrow$

[1]The solitons in the Skyrme model will be called Skyrmions

$SU(2)_V$. As we want to be as simple as possible we neglect the s-quark in our considerations.

One of the simplifications of the $SU(2)_f$ is that the Wess-Zumino-Witten term vanishes. This can be easily shown by taking

$$n\Gamma \propto \mathrm{Tr}(A\partial_\mu A\partial_\nu A\partial_\rho A\partial_\sigma A) \tag{1}$$

and expressing the A in terms of $SU(2)$-generators $A = a_a\tau^a$. This leads to

$$n\Gamma \propto \epsilon_{\mu\nu\rho\sigma} a_a\partial_\mu a_b\partial_\nu a_c\partial_\rho a_d\partial_\sigma a_e \,\mathrm{Tr}(\tau_a\tau_b\tau_c\tau_d\tau_e) \ . \tag{2}$$

As this term is completly anti-symmetric in the Lorentz indices, it must be also anti symmetrical in the isospin variables b, c, d, e. Since there are only three independent generators of $SU(2)$ this term must vanish.

Because of the symmetry it is convenient to parametrize the light mesons by an $SU(2)$-matrix $U(x)$

$$U(x) = \exp\left[\frac{i}{f_\pi}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right] \ , \tag{3}$$

where $\boldsymbol{\tau}$ are the Pauli matrices, $\boldsymbol{\pi}$ the pion fields and $f_\pi = \frac{F_\pi}{2}$ is the chiral radius.
The relation of U to the left current L_μ is given by

$$L_\mu = U^\dagger\partial_\mu U \ . \tag{4}$$

In terms of the field $U(x)$ and respectively the left current L_μ the Lagrangian can be expressed as

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) = \frac{f_\pi^2}{4}\,\mathrm{Tr}(L_\mu L^{\mu^\dagger}) = -\frac{f_\pi^2}{4}\,\mathrm{Tr}(L_\mu L^{\mu^\dagger}) \ . \tag{5}$$

A soliton configuration of $U(x)$ can be regarded as a mapping: $\mathbb{R}^3 \to SU(2) \sim S^3$. The restriction to finite energy can be fullfilled by the condition

$$U(|x| \to \infty) = 1 \ . \tag{6}$$

Since at spatial infinity the field must approach the vacuum, the mapping is topological equivalent to $U(x) : S^3 \to S^3$, which is non-trivial under the third homotopy group $\pi_3(S^3) \sim \mathbb{Z}$. The topology of the mapping is classified by an integer (winding) number

$$B := \int d^3x B_0 \ , \tag{7}$$

where the B_0 is the time component of a conserved topological current

$$B_\mu := \frac{1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}(L^\nu L^\alpha L^\beta)\ . \tag{8}$$

Skyrme proposed to identify the winding number as the baryon number and thus the soliton as a baryon. In the simple non-linear σ-model this is impossible, because any static field configuration of the Euler-Lagrange equations is unstable (Derrick's theorem). This can easily be proven by looking at the energy in D spatial dimensions

$$E = \int d^D x \frac{f_\pi^2}{4}\,\mathrm{Tr}(\partial^i U^\dagger \partial^i U)\ , \tag{9}$$

which scales as

$$E_\lambda = \lambda^{2-D} E\ . \tag{10}$$

In $D = 3$ it is obvious that the most favorable configurations have zero energy and thus making all possible solitons unstable against scale transformations.

3 The Skyrme Model

To avoid the collapse of the solitons Skyrme added a higher derivative term, which is called Skyrme term

$$\mathcal{L}_S = -\frac{f_\pi^2}{4}\,\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32e^2}\,\mathrm{Tr}([L_\mu, L_\nu])^2\ , \tag{11}$$

where e is the Skyrme parameter.

Now take a look at the energy properties of the static solutions of the Skyrme Langrangian. The Skyrme term is a quartic term in the current and will prevent the collapse. The energy scaling of the Skyrme Lagrangian behaves as

$$E_{\lambda_S} = E_2 \lambda^{2-D} + E_4 \lambda^{4-D}\ , \tag{12}$$

which got a true minimum for $D \geq 3$ since

$$\frac{dE_{\lambda_S}}{d\lambda} = 0 \Rightarrow \frac{E_2}{E_4} = \frac{-D+4}{D-2}\ , \tag{13}$$

$$\frac{d^2 E_{\lambda_S}}{d\lambda^2} \Rightarrow 2(D-2)E_2 > 0\ . \tag{14}$$

In three dimensions obviously $E_2 = E_4$ holds. This also shows that the total energy of the static solutions is bounded from below, which can be estimated by taking

$$E = \int dx \left\{ -\frac{f_\pi^2}{4} \operatorname{Tr}(L_i^2) - \frac{1}{32e^2} \operatorname{Tr}([L_i, L_j])^2 \right\} \tag{15}$$

and applying the Cauchy-Schwartz inequality

$$E = -\frac{f_\pi^2}{4} \int dx \operatorname{Tr}(L_i^2 + \frac{1}{8f_\pi^2 e^2}(\sqrt{2}\epsilon_{ijk}L_jL_k)^2) \geq \frac{f_\pi^2}{4} \int dx \left| \operatorname{Tr}\left(\frac{1}{ef_\pi}\epsilon_{ijk}L_iL_jL_k\right)\right| \ . \tag{16}$$

This inequality expressed in the topological charge is given by

$$E \geq \frac{6\pi^2}{e} f_\pi |B| \ . \tag{17}$$

This relation is often called the Bogomolny bound. The energy will always be larger than this estimate because the fields that would reach the lower boundary (a self-dual chiral field) $L_{sd_i} = (\epsilon\sqrt{2}/f_\pi)\epsilon_{ijk}L_jL_k$ are not compatible with the Maurer Cartan equation

$$\partial_{[\mu}L_{\nu]} + [L_\mu, L_\nu] = 0 \ , \tag{18}$$

which follows from eq. (4).

Now that we have shown that there are finite energy solutions let us discuss the meaning of the Skyrme term. Skyrme himself just introduced the term by hand to make the solitons stable, but it can also be regarded as a higher-order correction to the effective chiral description. In fact the Skyrme term can be related to a $\rho - \pi - \pi$-coupling. The non-linear σ-model term is unique to order $\mathcal{O}(p^2)$, but the Skyrme term is not unique to order $\mathcal{O}(p^4)$. It is just unique in the sense that it is the only order $\mathcal{O}(p^4)$ term with a positive Hamiltonian (and also in the sense of two time-derivatives).

In order to find Skyrmions we take the so called hedgehog (or sometimes Skyrme) ansatz

$$U_0(x) = \exp(iF(r)\boldsymbol{\tau} \cdot \hat{\boldsymbol{x}}) \ , \tag{19}$$

where the function $F(r)$ has to satisfy the boundary condition that the winding number equal to one can be identified with a baryon. With this ansatz baryon (winding) number is given by

$$B = \int d^3x B_0 = \frac{1}{\pi}(F(0) - F(\infty)) \ . \tag{20}$$

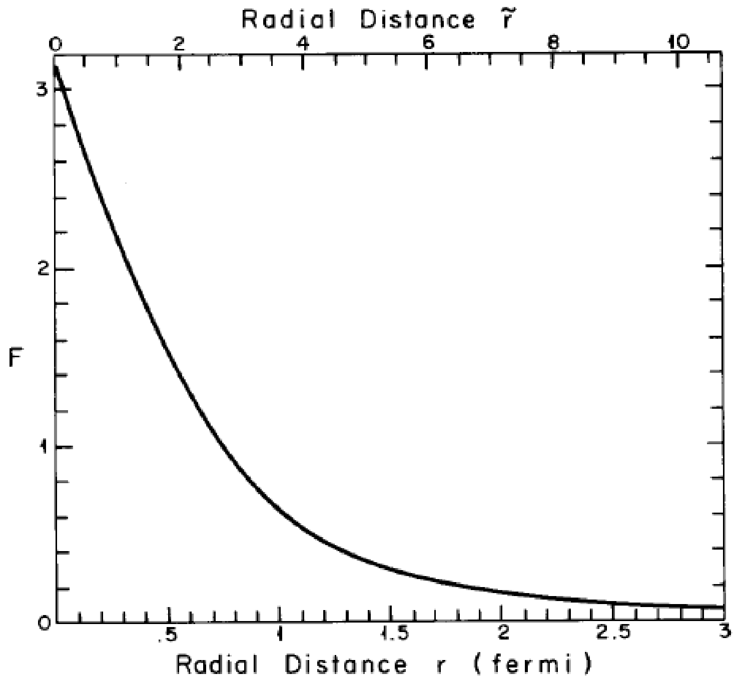

Figure 1: Numerically solution for $F(r)$ in the hedgehog ansatz [2].

Since we know that (6) must hold one finds as boundary conditions for F

$$F(r=0) = \pi \ , \qquad F(r \to \infty) = 0 \ . \tag{21}$$

This leads to the following equation for the energy (basically its mass) of the skyrmion

$$M = \int_0^\infty (4\pi r^2) \left[\frac{f_\pi^2}{8} \left(\left(\frac{\partial F}{\partial r} \right)^2 + 2\frac{\sin^2 F}{r^2} \right) + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left(\frac{\sin^2 F}{r^2} + 2 \left(\frac{\partial F}{\partial r} \right)^2 \right) \right] dr \tag{22}$$

This mass term is often referred to as the hedgehog mass.
The variational equation from this integral is

$$\left(\frac{\boldsymbol{r}^2}{4} + 2\sin^2(F) \right) F'' + \frac{\boldsymbol{r}}{2} F' + \sin(2F) F'^2 - \frac{1}{4}\sin(2F) - \frac{\sin^2(F)\sin(2F)}{\boldsymbol{r}^2} = 0 \ , \tag{23}$$

where the dimensionless variable $\boldsymbol{r} = \epsilon F_\pi r$ has been introduced. This equation has to be solved numerically. The shape of $F(r)$ is relatively model-independent. One example is shown in figure 1.

It is important to notice that if U_0 is a skyrmion solution also a constant $SU(2)$ transformation should be a solution

$$U_A = AU_0A^{-1} \ , \tag{24}$$

where A is an arbitrary constant $SU(2)$-matrix. A solution of an arbitrary A is not an eigenstate of spin and isospin, therefor we have to treat A as a quantum mechanical variable. The easiest way to accomplish that is to write the Lagrangian in terms of a time-dependent $A(t)$. This procedure allows to write a Hamiltonian which has to be diagonalized. The eigenstates of proper spin and isospin will correspond to the nucleons and the delta-baryon. With the the substitution $U = A(t)U_0A^{-1}(t)$ Adkins et al. [2] find

$$\mathcal{L} = -M + \left\{ \underbrace{\frac{4\pi}{6e^3 f_\pi} \int \boldsymbol{r}^2 \sin^2(F) \left[1 + 4F'^2 + \frac{4\sin^2 F}{\boldsymbol{r}^2} \right] d\boldsymbol{r}}_{\lambda} \right\} \mathrm{Tr}(\partial_0 A \partial_0 A^{-1}) \tag{25}$$

From this Lagrangian one can now construct Hamiltonian and the wave functions of the Skyrmions afterwards. Rather than doing the explicit steps we just give some aspects of the final output. One finds for the masses of the baryons

$$M_N = M + \frac{3}{8\lambda} \simeq 36.5\frac{F_\pi}{e} + 0.0035 F_\pi e^3, \tag{26}$$

$$M_\Delta = M + \frac{15}{8\lambda} \simeq 36.5\frac{F_\pi}{e} + 0.018 F_\pi e^3 \ . \tag{27}$$

The resulting masses for the baryons are plotted in figure (2).

There is one important feature concerning the spin statistics of the soliton wave functions. The solutions $U_A = AU_0A^{-1}$ are invariant under the replacement $A \to -A$. Naively one would expect that $\psi(A) = \psi(-A)$ would be the correct way to quantize the field, but in fact there are two consistent ways to quantize the soliton

$$\psi(A) = \psi(-A) \qquad \text{or} \qquad \psi(A) = -\psi(-A) \ . \tag{28}$$

The first choice corresponds to a boson and the second one to a fermion. As we are interested in a description of baryons we of course take the second choice.

In fact if one takes the strange quark into consideration(and thus extends the flavour symmetry to $SU(3)_f$), one does not have a choice anymore. Witten has proven that in that case the soliton can only be a fermion[2] [3].

[2]In fact the spin statistics depends on the gauge group $SU(N)$. Odd N leads to fermionic skyrmions, while even N yields bosonic skyrmions.

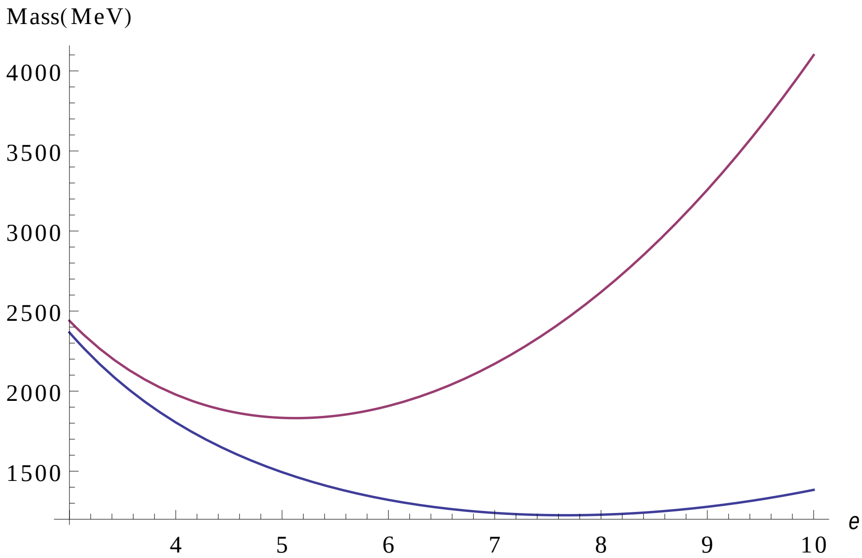

Figure 2: Approximate nucleon and delta masses for different values of the Skyrme parameter.

3.1 properties and limitations of this model

The numerical results for Skyrmions of two different groups as well as experimental results are shown table 1. As one can see some values are significantly off. In more complicated models one can do a lot better. One of the problems of the $SU(2)_f$-model is that the Skyrme-Lagrangian is more symmetric than the QCD-Lagrangian. The Skyrme model is invariant under the transformations

$$P_0 : x \leftrightarrow -x, t \leftrightarrow t, U \leftrightarrow U, \tag{29}$$

$$J : x \leftrightarrow x, t \leftrightarrow t, U \leftrightarrow U^{-1} \; . \tag{30}$$

QCD does not respect both symmetries separately but only the combination of both. A solution would be to take a $SU(3)_f$-symmetry, because with this flavour symmetry the Wess-Zumino-Witten term does not vanish and the additional symmetry is lost.

Another problem of the chosen Lagrangian is that it oversimplifies the interactions of mesons and quarks. The complete reduction to mesons is only true in the large N limit and therefor one has to expect large corrections of order $1/N_c \simeq 30\%$[3].

A more modern study of the skyrmion parameters including corrections from $N_c = 3$, $SU(3)_f, \ldots$ yields much better results as shown in table 2.

[3]Witten claims the expansion parameter contains another factor $\frac{1}{4\pi}$, which would reduce this factor.

Table 1 Single baryon properties in the Skyrme model from Adkins, Nappi & Witten (ANW) (20) as compared with those in the Atiyah-Manton (AM) method

Quantity	ANW[a]	AM method	Experiment
Pion decay constant, F_π (MeV)	129	129	186
Skyrme parameter, e	5.45	5.45	—
Hedgehog mass, M_{H} (MeV)	864	871	—
Nucleon mass, M_{N} (MeV)	939*	927	939
Isobar mass, M_Δ (MeV)	1232*	1152	1232
Moment of inertia, $\mathcal{I}$ (fm)	—	1.32	1.01
Isoscalar r.m.s. radius, $\langle r^2\rangle_{\mathrm{B}}^{1/2}$ (fm)	0.59	0.61	0.79
Isoscalar g factor, $g_{I=0}$	1.12	0.89	1.76
Isovector g factor, $g_{I=1}$	6.38	8.36	9.40
Axial coupling constant, g_{A}	0.61	0.48	1.25
Pion-nucleon coupling constant, $g_{\pi\mathrm{NN}}$	8.9	9.2	13.5

[a] The asterisks indicate that the nucleon and delta masses are used to fit the parameters F_π and e.

Table 1: Table with numerical results for (simple) skyrmion models [5]

Physical parameter	Skyrme Model	experimental value
M_N	946 MeV	939 MeV
$\mu_{I=1}$	2.24	2.35
$r^2_{E,I=0}$	0.51 fm^2	0.62 fm^2
$r^2_{M,I=0}$	0.64 fm^2	0.73 fm^2
g_A	0.66	1.26

Table 2: Comparison of the results from [7] with experiment

4 Skyrmions in composite Higgs models

Skyrmions may appear not only appear in low-energy QCD, but in many other models which can be described by a non-linear σ-model. The presently most discussed class of those models are the composite Higgs models.

The question is if and in which cases one can write down an analogue of the Skyrme term and also find stable particles.

If these models contain skyrmions depends on the third homotopy group of the mapping $H \to G$, where $G, (H)$ are the (un-)broken symmetry groups. If it is non-trivial, the model contains skyrmions. It can be shown that there are several models which satisfy this condition $((SU(N)\times SU(N)) \to SU(N), (SO(N)\times SO(N)) \to SO(N), SU(N) \to SO(N))$ while others do not $(SU(N) \to SU(N-1), SO(N) \to SO(N-1), SU(2N) \to Sp(2N))$.

Since there are very many free parameters in the composite Higgs models it is not possible to make general statements about skyrmions. In order to be able to make general statements we restrict to the class of QCD-like models $SU(N) \times SU(N) \to SU(N)$ with a gauge group $SU(N_c)$. As mentioned before the spin statistics is entirely determined by the integer N_c. From this behaviour one also can estimate the masses of the lightest skyrmion excitations. For fermions the same formula as for QCD is valid for any $2 \leq \frac{N_c+1}{2} \in \mathbb{N}$. For bosons the story is even simpler: The lightest mass just equals the hedgehog mass. The reason for this universality is that the lowest energy skyrmions always live in $SU(2)$ subsets.

One important check is the charge of the lightest skyrmion. Models with charged skyrmions are are in conflict with cosmology ! If one checks the charge it turns out that it is given by the number of colors and the hypercharge of the fermion doublet

$$q = y_0 N_c \ , \tag{31}$$

but for fermionic skyrmions there is an additional contribution from the Wess-Zumino-Witten term. This results in the following restriction on the hypercharge of the fermion doublet

$$\text{bosons} : q = y_0 N_c \to y_0 = 0 \ , \tag{32}$$

$$\text{fermions} : q = y_0 N_c \pm \frac{1}{2} \to y_0 = \pm\frac{1}{2N_c} \ . \tag{33}$$

From this equation one can see that the skyrmion charges really does coincide with the ones of the nucleons.

4.1 Skyrmions as dark matter candidate

As we have seen there are several different composite Higgs models that naturally yield skyrmions. The question is if those can be viable dark matter candidates. First of all one has to address the question of stability. On classical level these field configurations are exactly stable, but quantizing these leads to a different result due to the coupling to the gauge fields. This instability originates in instanton effects, where the winding of the Higgs field is transferred into a winding of the gauge field . This mechanism leads to an approximate lifetime of the Skyrmion of order

$$\tau_{sky} = \frac{1}{\Gamma_{sky}} \sim \frac{e^{16\pi^2/g^2}}{M_0} \gg \tau_{universe} \ . \tag{34}$$

The next thing one has to check is if one can arrive at the correct relic abundance. As shown in [6] this is possible for a big parameter space ($10 \lesssim e \lesssim 100$). Actually all models with smaller values of the Skyrme parameter are also viable from the cosmological point of view, just the origin of dark matter has to be a different one. Models with a higher value for e are somehow excluded, since the skyrmions mass would be too light and thus they would be overproduced in the early universe.

5 Conclusions

- We have seen that in the non-linear σ-models there exist soliton solutions, which can be stabilized by the introduction of a higher derivative term e.g. the Skyrme term.
- The stability of the skyrmions arise from the conservation of the topological winding number, which can be identified with the baryon number.
- Also in other models represented by a non-linear σ-model it is possible to find skyrmion solutions. We briefly discussed some of the properties of skyrmions in composite Higgsa models.
- In those models we have shown that for a large parameter space it is possible to explain the observed dark matter content of the universe as skyrmions.

References

[1] T.H.R. Skyrme, Nuclear Physics 31 (1962), 556-569

[2] G. S. Adkins & C.R. Nappi & E. Witten, Nuclear Physics B228 (1983), 552-566

[3] E. Witten, Nuclear Physics B223 (1983), 422-432

[4] I. Zahed & G.E.Brown, Physics Reports 142, Nos. 1 & 2 (1986)

[5] M. Oka & A. Hosaka Annu. Rev. Part. Sci. 1992. 42:333-365

[6] M. Gillioz,A.v. Manteuffel, P. Schwaller, D. Wyler, arxiv.org:1012.5288

[7] F.Meier & H. Walliser, Phys.Rept. 289 (1997) 383-450