Thermal Pressure Waves in A Thin Target - Analytical Solution

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Introduction

The target for positron production needs to withstand induced pressure from the energy deposited by incident beam (see S. Riemann talk)

TASK:
To determine if the target will survive the impinged incident beam

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Previous Work

P. Sievers:

- Stress Waves Eqn.
- Rectangular Function for describing the beam

(see "Elastic Waves in Matter Due to Rapid Heating by An Intense High Energy Particle Beam", 1974)

T.A. Vsevolozhskaya (analytical)
and
A. Mikhailichenko (numerical -> flexPDE)

- Hydrodynamics Model
- Gaussian Distribution for describing the beam

Model

We use fluid model to simulate the target behavior

These involves:

- Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

- Equation of Motion

$$\rho \frac{\partial u}{\partial t} + \rho \cdot (u \nabla u) = -\nabla P$$

- Equation of State

$$P = \Gamma q(\mathbf{r}, t)$$

where ρ: density; u: velocity; P: pressure; Γ: Grüneisen coefficient and q: specific energy deposited

Linear Approximation

By linearizing those three equations, we arrives at:

$$\frac{\partial^2 P}{\partial t^2} - \nabla \cdot (c_0^2 \nabla P) = \Gamma \frac{\partial^2 q}{\partial t^2}$$

where c_0 is the sound speed in the material

In axially symmetric case, the Partial Differential Eqn. become:

$$\frac{\partial^2 P}{\partial t^2}(r,z,t) - c_0^2 \left[\frac{\partial^2 P}{\partial r^2}(r,z,t) + \frac{1}{r}\frac{\partial P}{\partial r}(r,z,t) + \frac{\partial^2 P}{\partial z^2}(r,z,t) \right] = \Gamma \frac{\partial^2 q}{\partial t^2}$$

As a first step, we assume to solve for a very thin cylinder:

$$\frac{\partial^2 P}{\partial t^2}(r,z,t) - c_0^2 \left[\frac{\partial^2 P}{\partial r^2}(r,z,t) + \frac{1}{r}\frac{\partial P}{\partial r}(r,z,t) \right] = \Gamma \frac{\partial^2 q}{\partial t^2}$$

Energy Deposition

Assuming instantaneous energy deposition:

$$q = \delta(t) Q(r, z)$$

where $Q(r, z)$ is defined as deposited energy density distribution on the target

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So, for $t > 0$:

$$\frac{\partial q}{\partial t} = 0$$

Therefore, the Partial Differential Equation become:

$$\frac{\partial^2 P(r,z,t)}{\partial t^2} - c_0^2 \left[\frac{\partial^2 P(r,z,t)}{\partial r^2} + \frac{1}{r} \frac{\partial P(r,z,t)}{\partial r} \right] = 0$$

Energy Density Distribution

By definition:

$$Q_{bunch} = \int_0^{vol} Q(r,z)\, dV$$

where:

$$Q(r,z) = Q_0 Q(z) \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

and

$$Q_0 = \frac{Q_{bunch}}{2\pi\sigma^2 \int_0^L Q(z)\, dz}$$

Q_{bunch} is the energy deposited per bunch; L is the target thickness; σ is the spot size

For $t > 0$ we have:

$$\frac{\partial^2 P(r,z,t)}{\partial t^2} - c_0^2 \left[\frac{\partial^2 P(r,z,t)}{\partial r^2} + \frac{1}{r} \frac{\partial P(r,z,t)}{\partial r} \right] = 0$$

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with the following initial conditions (at time, $t = 0$):

$$P(r,z,t) = \Gamma Q(r,z)$$

and

$$\frac{\partial P}{\partial t}(r,z,t) = 0$$

Since we assume axially symmetric case

and

$r(0,\infty)$ because $\sigma \ll R$, where R is the radius of the Target

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Hankel Integral Transform is desirable for solving this Partial Differential Eqn. below:

$$\frac{\partial^2 P}{\partial t^2}(r,z,t) - c_0^2 \left[\frac{\partial^2 P}{\partial r^2}(r,z,t) + \frac{1}{r}\frac{\partial P}{\partial r}(r,z,t) \right] = 0$$

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Infinite Hankel Transformation Definition

$$\tilde{P}(K, z, t) = \int_0^\infty rP(r, z, t)J_0(Kr)dr$$

By applying Hankel Integral Transform, the PDE can be reduce to:

$$\frac{\partial^2 \tilde{P}}{\partial t^2} + c_0^2 K^2 \tilde{P} = 0$$

Initial conditions at $t = 0$ become:

$$\tilde{P}(r, z, t) = \Gamma \tilde{Q}(K, z)$$

and

$$\frac{\partial \tilde{P}}{\partial t}(r, z, t) = 0$$

Analytical Solution

Inverse Hankel Transform of the solution to the above PDE give:

$$P(z,r,t) = \Gamma Q_0 \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(m+n)!2^{m+n}}{(2m)!(n!)^2} \left(\frac{r}{2\sigma}\right)^{2n} \left(\frac{c_0 t}{\sigma}\right)^{2m}$$

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can be simplified into:

$$P(r,z,t) = \Gamma Q_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2n-1)!!} \left(\frac{c_0 t}{\sigma}\right)^{2m} {}_1F_1\left[m+1; 1, -\frac{r^2}{2\sigma^2}\right]$$

OR

$$P(r,z,t) = \Gamma Q_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \left(\frac{r}{2\sigma}\right)^{2n} {}_1F_1\left[1+n; \frac{1}{2}, -\frac{c_0^2 t^2}{2\sigma^2}\right]$$

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Parameters

Below are the material and incident beam parameters for SLC and ILC:

Target Material Parameters			
Parameters	Units	SLC	ILC
Target Material	-	W25Re	Ti-Alloy
Target Thickness	mm	20.574	14.88
Radius	mm	63.5	15
Grüneisen constant	-	2.095	1.262
Sound Speed	ms^{-1}	4671.98	5072.83
Tensile Strength	MPa	1370	880

Beam Parameters			
Parameters	Symbol	SLC	ILC
Beam spot size (mm)	σ	0.8	1.2
Energy Deposited (J)	Q_{bunch}	41.67	0.72

At $r = 0$, both equations give:

$$P(r,z,t) = \Gamma Q_0 \left[1 - \sqrt{\frac{\pi}{2}}\frac{c_0 t}{\sigma}\exp\left(-\frac{c_0^2 t^2}{2\sigma^2}\right) Erfi\left(\frac{c_0 t}{\sqrt{2}\sigma}\right)\right]$$

At $r = \sigma$, both equations give:

$$P(\sigma,z,t) = \Gamma Q_0 \left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m-1)!!}\left(\frac{c_0 t}{\sigma}\right)^{2m} {}_1F_1\left(1+m; 1, -\frac{1}{2}\right)\right]$$

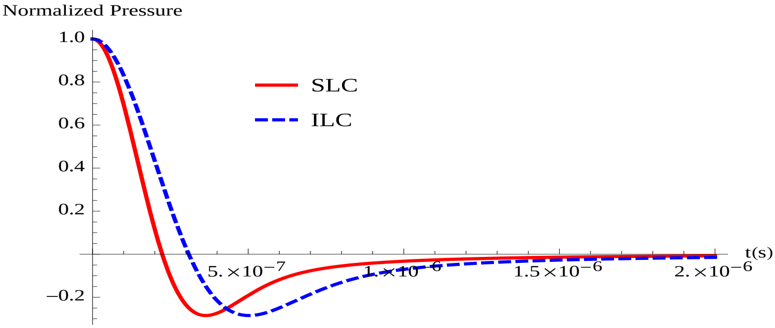
and

$$P(\sigma,z,t) = \Gamma Q_0 \left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!2^n}{}_1F_1\left(1+n; \frac{1}{2}, -\frac{c_0^2 t^2}{2\sigma^2}\right)\right]$$

respectively

Pressure Induced at the Center of the Target

At $r = 0$



Pressure (Pascal) $=$ Normalized Pressure $\times$ ΓQ_0

where ΓQ_0 for ILC is $6.75 \times 10^6 J/m^3$ and for SLC is $1.06 \times 10^9 J/m^3$

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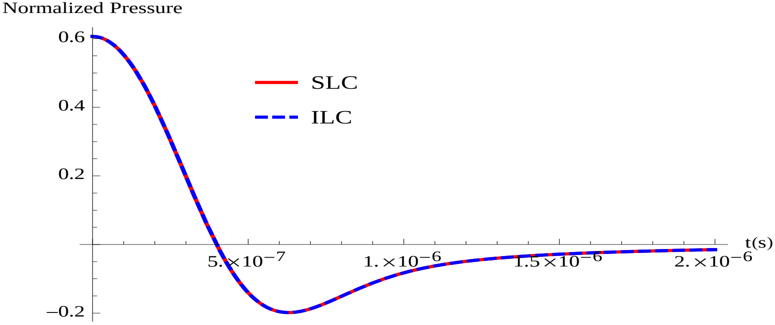
Pressure Induced at $r = \sigma$

Pressure vs. time

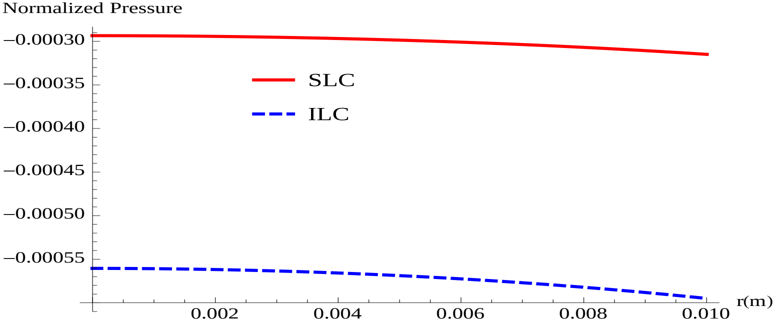


Pressure (Pascal) = Normalized Pressure $\times$ ΓQ_0

where ΓQ_0 for ILC is $6.75 \times 10^6 J/m^3$ and for SLC is $1.06 \times 10^9 J/m^3$

Pressure Induced at $t = 10\mu s$

Pressure vs. r



Pressure (Pascal) $=$ Normalized Pressure $\times\ \Gamma Q_0$

where ΓQ_0 for ILC is $6.75 \times 10^6 J/m^3$ and for SLC is $1.06 \times 10^9 J/m^3$

Summary and Outlook

SUMMARY:

1. We have a solution for pressure induced in a thin target by instantaneous energy deposition

2. The peak pressure is at the center of the beam spot

OUTLOOK:

1. Extend solution to thick cylindrical target;
2. Add damping effect and
3. Apply solution to both CLIC and ILC targets

Question Please