Nested (inverse) binomial sums and new iterated integrals for massive Feynman diagrams

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HELMHOLTZ GEMEINSCHAFT

Outline

1. Introduction
2. Integral representations
   - Mellin transform
   - Iterated integrals and their properties
3. Mellin convolution via differential equations
   - Construct ODEs by parametric integration
   - Solve ODEs in terms of basis of iterated integrals
4. Rewrite Rules
   - Patterns in convolution integrals
   - Rewrite rules for Mellin transforms and generating functions

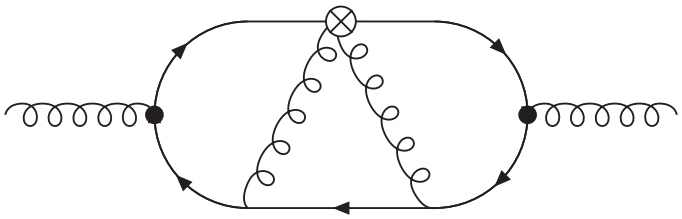
Introduction

Deep-inelastic scattering

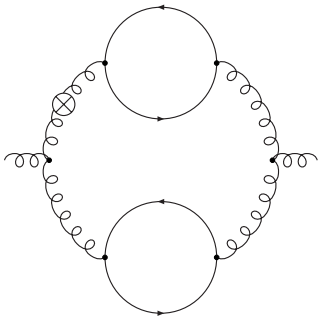
Heavy flavor Wilson coefficients

3-loop contributions in the asymptotic region by massive operator matrix elements at general N

2-loop: [Buza, Matiounine, Smith, Migneron, Neerven hep-ph/9601302]



[Ablinger, Blümlein, CGR, Schneider, Wißbrock 1403.1137]



[Ablinger, Blümlein, De Freitas, Hasselhuhn, Klein, Schneider, Wißbrock 1212.5950]

Main steps

1. Dimensional regularization
2. ε-expansion via symbolic summation tools
3. Analytic continuation to complex N

[Ablinger, Blümlein, Round, Schneider 1210.1685]

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Harmonic sums: [Vermaseren hep-ph/9806280]

$$a(i) = \frac{(\pm 1)^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized harmonic sums (S-sums): [Moch, Uwer, Weinzierl hep-ph/0110083]

$$a(i) = \frac{c^i}{i^m}$$

Expressions in the ε-expansion

Nested sums

$$\sum_{i_1=1}^{N} a_1(i_1) \sum_{i_2=1}^{i_1} a_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k)$$

Summand structure

Generalized (inverse) binomial sums: [Ablinger, Blümlein, CGR, Schneider DESY 14-021]

$$a(i) = \frac{c^i}{i^m} \binom{2i}{i}^{\pm 1}, \qquad a(i) = \frac{c^i}{i^m}, \qquad a(i) = \frac{c^i}{i^m(2i+1)} \binom{2i}{i}^{-1}$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} =$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} =$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} =$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} = 2 \int_0^1 \frac{x^N - 1}{x - 1} \operatorname{arctanh}\left(\sqrt{1-x}\right) dx$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = -\frac{1}{\pi} \int_0^1 \frac{(4x)^N - 1}{x - \frac{1}{4}} \left(\frac{\arccos(2x-1)^3}{6} - \zeta_2 \arccos(2x-1) \right) dx$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = \frac{1}{2} \int_0^1 \frac{(-x)^N - 1}{x+1} \frac{x\, \mathrm{H}^*_{(0,\{-\frac{1}{4}\}),0,0}(x)}{\sqrt{x + \frac{1}{4}}} dx - {}_5F_4(const) \int_0^1 \frac{\left(\frac{x}{4}\right)^N - 1}{x - 4} \frac{x}{\sqrt{1-x}} dx$$

Nested sums and their analytic continuation

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} = 2 \int_0^1 \frac{x^N - 1}{x-1} \operatorname{arctanh}\left(\sqrt{1-x}\right) dx$$

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Nested sums and their asymptotic expansion

Examples

$$\sum_{i=1}^{N} \frac{4^i}{i^2 \binom{2i}{i}} = 3\zeta_2 + \sqrt{\frac{\pi}{N}} \left(-2 + \frac{5}{12N} - \frac{21}{320N^2} + \dots \right)$$

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \frac{4^N}{\sqrt{\pi N}} \left(\frac{4\zeta_2}{3N} + \frac{3\zeta_2 - 8}{6N^2} + \frac{363\zeta_2 - 80}{288N^3} + \dots \right) - \frac{2}{3}\zeta_3$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{(-1)^j \binom{2j}{j}}{j^3}}{(2i+1)\binom{2i}{i}} = const + (-1)^N \left(\frac{1}{5N^4} - \frac{16}{25N^5} + \frac{67}{125N^6} + \dots \right) - \frac{{}_5F_4(const)}{4^N} \sqrt{\frac{\pi}{N}} \left(-\frac{1}{3} + \frac{25}{72N} - \frac{683}{1152N^2} + \dots \right)$$

Integral representations

Integral representations

General form

$$\sum_{i=1}^{N} a(i) = const + \sum_{j=1}^{k} c_j^N \int_0^1 x^N f_j(x)\, dx$$

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General form

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Mellin transform

$$\mathbf{M}\left(f(x)\right)(N) := \int_0^1 x^N f(x)\, dx$$

Mellin transform

Basic properties

$$\begin{aligned}
\mathbf{M}\left(c_1 f_1(x) + c_2 f_2(x)\right)(N) &= c_1 \mathbf{M}\left(f_1(x)\right)(N) + c_2 \mathbf{M}\left(f_2(x)\right)(N) \\
\mathbf{M}\left(f(x)\right)(N+k) &= \mathbf{M}\left(x^k f(x)\right)(N) \\
\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) &= \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t}) f_2(t)}{t}\, dt\right)(N)
\end{aligned}$$

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\end{aligned}$$

Summation

$$\sum_{k=1}^{N} c^k \mathbf{M}\left(f(x)\right)(k) = c^N \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(N) - \mathbf{M}\left(\frac{x}{x-\frac{1}{c}} f(x)\right)(0)$$

Mellin transform (cont.)

Basic transforms

$$\frac{1}{N} = \mathbf{M}\left(\frac{1}{x}\right)(N)$$

$$\binom{2N}{N} = \frac{4^N}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(N)$$

$$\frac{1}{N\binom{2N}{N}} = \frac{1}{4^N}\mathbf{M}\left(\frac{1}{x\sqrt{1-x}}\right)(N)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)$$

Example

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(-\frac{\ln(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{1}{x}\right)(i) \left(\zeta_2 + \mathbf{M}\left(\frac{\ln(x)}{1-x}\right)(i)\right)
\end{aligned}$$

Example

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\begin{aligned}
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&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)
\end{aligned}
$$

Example

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&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i) \\
&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)
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Example

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&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\zeta_2 \frac{\arccos(2x-1)}{x} - \frac{\arccos(2x-1)^3}{6x}\right)(i)
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$$

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&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\zeta_2 \frac{\arccos(2x-1)}{x} - \frac{\arccos(2x-1)^3}{6x}\right)(i) \\
&= \frac{4^N}{\pi} \mathbf{M}\left(\zeta_2 \frac{\arccos(2x-1)}{x-1/4} - \frac{\arccos(2x-1)^3}{6(x-1/4)}\right)(N) - \frac{2}{3}\zeta_3
\end{aligned}
$$

Iterated integrals

General form

$$\mathrm{H}_{a_1,\ldots,a_k}(x) := \int_0^x f_{a_1}(t_1) \int_0^{t_1} f_{a_2}(t_2) \ldots \int_0^{t_{k-1}} f_{a_k}(t_k)\, dt_k \ldots dt_1$$

Harmonic polylogarithms

[Remiddi, Vermaseren hep-ph/9905237]

$$f_1(x) := \frac{1}{1-x}, \qquad f_0(x) := \frac{1}{x}, \qquad f_{-1}(x) := \frac{1}{1+x}$$

Iterated integrals

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Harmonic polylogarithms

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$$f_1(x) := \frac{1}{1-x}, \qquad f_0(x) := \frac{1}{x}, \qquad f_{-1}(x) := \frac{1}{1+x}$$

Multiple polylogarithms

[Kummer 1840, Lappo-Danilevsky 1953, Goncharov 1998]

$$f_a(x) := \frac{1}{|a| - \mathrm{sign}(a)x}, \qquad f_0(x) := \frac{1}{x}$$

Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \ldots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \ldots dt_1$$

Integrands with square roots

[Hermite 1883, Aglietti, Bonciani hep-ph/0401193, Ablinger, Blümlein, CGR, Schneider DESY 14-021]

$$f_a(x) \quad := \quad \frac{\mathrm{sign}(1-a-0)}{x-a},$$

Iterated integrals

General form

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$$\begin{aligned} f_a(x) &:= \frac{\mathrm{sign}(1-a-0)}{x-a}, \\ f_{\{a_1,\ldots,a_k\}}(x) &:= f_{a_1}(x)^{1/2} \ldots f_{a_k}(x)^{1/2} \qquad k \geq 2, \end{aligned}$$

Iterated integrals

General form

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Iterated integrals

General form

$$\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x) := \int_x^1 f_{\mathbf{a_1}}(t_1) \int_{t_1}^1 f_{\mathbf{a_2}}(t_2) \ldots \int_{t_{k-1}}^1 f_{\mathbf{a_k}}(t_k)\, dt_k \ldots dt_1$$

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f_{(a_0,\{a_1,\ldots,a_k\})}(x) &:= f_{a_0}(x) f_{a_1}(x)^{1/2} \ldots f_{a_k}(x)^{1/2} \qquad k \geq 1, \\
f_{(\{a_1,\ldots,a_k\},j)}(x) &:= x^j f_{\{a_1,\ldots,a_k\}}(x) \qquad j \in \{1,\ldots,k-2\}
\end{aligned}$$

Examples

Depth 1

- $\mathrm{H}_0^*(x) = \int_x^1 \frac{1}{t}\, dt = -\ln(x)$
- $\mathrm{H}_{\{0,1\}}^*(x) = \int_x^1 \frac{1}{\sqrt{t(1-t)}}\, dt = \arccos(2x-1)$
- $\mathrm{H}_{(0,\{1\})}^*(x) = \int_x^1 \frac{1}{t\sqrt{1-t}}\, dt = 2\,\mathrm{arctanh}(\sqrt{1-x})$

Depth 2

- $\mathrm{H}_{1,0}^*(x) = \int_x^1 \frac{1}{1-t}\, \mathrm{H}_0^*(t)\, dt = \mathrm{Li}_2(1-x)$
- $\mathrm{H}_{(2,\{0,8\}),1}^*(x) = \int_x^1 \frac{\ln(1-t)}{(2-t)\sqrt{t(8-t)}}\, dt$

Algebraic relations

Shuffle relations

$$
\begin{aligned}
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1}}(x) \\
\mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x) \\
&\vdots
\end{aligned}
$$

Algebraic relations

Shuffle relations

$$\begin{aligned} \mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1}}(x) \\ \mathrm{H}^*_{\mathbf{a_1}}(x)\,\mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2}}(x) &= \mathrm{H}^*_{\mathbf{a_1},\mathbf{b_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{a_1},\mathbf{b_2}}(x) + \mathrm{H}^*_{\mathbf{b_1},\mathbf{b_2},\mathbf{a_1}}(x) \\ &\vdots \end{aligned}$$

Theorem

The functions $\mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)$ do not satisfy any algebraic relations apart from the shuffle relations.

Mellin transforms

Theorem

Let $h(x) = \frac{rat(x)}{\sqrt{poly(x)}}$ and let $\mathbf{a_1}, \ldots, \mathbf{a_k}$ be some generalized letters as above. If for all $i \in \{0, \ldots, k\}$ the product

$$h(x) f_{\mathbf{a_1}}(x) \ldots f_{\mathbf{a_i}}(x)$$

has at most one square-root singularity different from $x = 0$, then

$$\mathbf{M}\left(h(x)\, \mathrm{H}^*_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)\right)(N)$$

is expressible in terms of generalized (inverse) binomial sums.

Examples

Basic examples

- $$\mathbf{M}\left(\frac{1}{\sqrt{x-a}}\right)(N) = \frac{(4a)^N}{(N+1/2)\binom{2N}{N}}\left(\sqrt{1-a}-\sqrt{-a}+\sqrt{1-a}\sum_{i=1}^{N}\frac{\binom{2i}{i}}{(4a)^i}\right)$$
- $$\mathbf{M}\left(\frac{1}{\sqrt{x(x-a)}}\right)(N) = \left(\frac{a}{4}\right)^N\binom{2N}{N}\left(2\,\mathrm{arcsinh}\left(\frac{1}{\sqrt{-a}}\right)+\sqrt{1-a}\sum_{i=1}^{N}\frac{(4/a)^i}{i\binom{2i}{i}}\right)$$

Mellin convolution via differential equations

Mellin convolution

Recall

$$\mathbf{M}\left(f_1(x)\right)(N)\,\mathbf{M}\left(f_2(x)\right)(N) = \mathbf{M}\left(\int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt\right)(N)$$

Approach

Compute

$$F(x) = \int_x^1 \frac{f_1(\frac{x}{t})f_2(t)}{t}\,dt$$

as solution of a differential equation

$$c_m(x)F^{(m)}(x) + \cdots + c_0(x)F(x) = r(x)$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

1. Find $g(x,t)$ and $c_0(x), \ldots, c_m(x)$ s.t.

$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

Construction of the ODEs

Parametric integration

Consider $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$

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2. Transfer this to a relation of corresponding integrals

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3. Rewrite as ODE for $F(x)$

$$c_m(x) F^{(m)}(x) + \cdots + c_0(x) F(x) = r(x)$$

Computer algebra

Problem

Given: C a field, V a C-vector space, $f(x,t), \ldots, D_x^m f(x,t) \in V$

Find: $g(x,t) \in V$ and $c_0(x), \ldots, c_m(x) \in C$ s.t.

$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

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Find: $g(x,t) \in V$ and $c_0(x), \ldots, c_m(x) \in C$ s.t.

$$c_m(x) D_x^m f(x,t) + \cdots + c_0(x) f(x,t) = D_t g(x,t)$$

Differential fields [Risch 1969, Singer, Saunders, Caviness 1985, Bronstein 1990/97, CGR 2012]

- (V, D_t) a differential field containing $f(x,t), \ldots, D_x^m f(x,t)$
- $C = \mathrm{Const}_{D_t}(V)$ its constant field

Ore algebras [Almkvist, Zeilberger 1990, Chyzak 2000, Chyzak, Kauers, Salvy 2009, Koutschan 2009]

- V a $C(t)[D_x, D_t]$-module containing $f(x,t)$
- C the field of rational functions in x

Example

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{1}{\sqrt{x(1-x)}}\right)(i)\, \mathbf{M}\left(\frac{\zeta_2 - \mathrm{Li}_2(1-x)}{x}\right)(i)$$

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$$F(x) = \int_x^1 \frac{\zeta_2 - \mathrm{Li}_2(1-t)}{t\sqrt{x(t-x)}}\, dt$$

$$D_x^3 f - \frac{3(4x-3)}{2x(1-x)} D_x^2 f - \frac{7x-3}{x^2(1-x)} D_x f - \frac{1}{x^2(1-x)} f =$$
$$D_t \frac{4(t-x)^2 + 2t(t-x)\ln(t) - (1-t)(t+2x)(\zeta_2 - \mathrm{Li}_2(1-t))}{4(1-x)x^2(t-x)^2\sqrt{x(t-x)}}$$

Example

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$$F^{(3)}(x) - \frac{3(4x-3)}{2x(1-x)} F''(x) - \frac{7x-3}{x^2(1-x)} F'(x) - \frac{1}{x^2(1-x)} F(x) =$$
$$\frac{1}{x^2(1-x)\sqrt{x(1-x)}}$$

Solution of the ODEs

Observations

- ODEs obtained factor completely into first-order ODEs

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Consequence

Solutions can be written as linear combinations of expressions of the form

$$\frac{r(x)}{\sqrt{p(x)}} \mathrm{H}^{*}_{\mathbf{a_1},\ldots,\mathbf{a_k}}(x)$$

where r is a rational function and p is a squarefree polynomial

Example

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k^2}}{j2^j}}{i\binom{2i}{i}} = \sum_{i=1}^{N} \frac{1}{i\binom{2i}{i}} \left(\frac{1}{2^i} \mathbf{M} \left(\frac{\mathrm{H}^*_{1,0}(x) - \zeta_2}{2-x} \right)(i) + \frac{5}{8}\zeta_3 \right)$$

involves convolution $F(x) = \int_x^1 \frac{\mathrm{H}^*_{1,0}(t) - \zeta_2}{x(2-t)\sqrt{1-\frac{x}{t}}}\, dt$ satisfying

$$F^{(5)}(x) + \cdots + \frac{15}{4x^3(1-x)(2-x)} F(x) = -\frac{3}{4x^4(1-x)^{7/2}(2-x)}$$

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Solution is given by

$$F(x) = \frac{1}{x}\left(\mathrm{H}^*_{0,0,(0,\{1\})}(x) - \mathrm{H}^*_{(0,\{1\}),(0,\{1\}),(0,\{1\})}(x) + \zeta_2\, \mathrm{H}^*_{(0,\{1\})}(x)\right) - \frac{2}{x\sqrt{2-x}}\left(\mathrm{H}^*_{(0,\{2\}),0,(0,\{1\})}(x) - \mathrm{H}^*_{(0,\{1,2\}),(0,\{1\}),(0,\{1\})}(x) + \zeta_2\, \mathrm{H}^*_{(0,\{1,2\})}(x)\right)$$

Rewrite Rules

Alternative method

Absorption of simple prefactors

$$\frac{1}{N}\mathbf{M}\left(f(x)\right)(N) \quad = \quad \mathbf{M}\left(\frac{1}{x}\int_x^1 f(t)\,dt\right)(N)$$

Alternative method

Absorption of simple prefactors

$$\begin{aligned}
\frac{1}{N}\mathbf{M}\left(f(x)\right)(N) &= \mathbf{M}\left(\frac{1}{x}\int_x^1 f(t)\,dt\right)(N)\\
\binom{2N}{N}\mathbf{M}\left(f(x)\right)(N) &= \frac{4^N}{\pi}\mathbf{M}\left(\frac{1}{\sqrt{x}}\int_x^1 \frac{f(t)}{\sqrt{t-x}}\,dt\right)(N)\\
\frac{1}{N\binom{2N}{N}}\mathbf{M}\left(f(x)\right)(N) &= \frac{1}{4^N}\mathbf{M}\left(\frac{1}{x}\int_x^1 \frac{\sqrt{t}f(t)}{\sqrt{t-x}}\,dt\right)(N)\\
&\vdots
\end{aligned}$$

Assume $f(x) = h(x)\,\mathrm{H}^*_{\vec{a}}(x)$

- Construct rewrite rules exploiting the structure of integrands
- Apply rules recursively
- Sufficient set of rules avoids construction of ODEs

Rules for Mellin convolution integrals

- $$\int_x^1 \frac{f(t)}{(t-c)\sqrt{t-x}}\,dt = \\ = \frac{1}{\sqrt{x-c}}\int_x^1 \frac{1}{\sqrt{t-c}}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{f'(u)}{\sqrt{u-t}}\,du\right)dt$$
- $$\int_x^1 \frac{f(t)}{\sqrt{t-a}\sqrt{t-x}}\,dt = \\ = \int_x^1 \frac{1}{t-a}\left(\frac{\sqrt{1-a}f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u-a}f'(u)}{\sqrt{u-t}}\,du\right)dt$$
- $$\int_x^1 \frac{\sqrt{t}f(t)}{(t-c)\sqrt{t-x}}\,dt = \\ = \int_x^1 \frac{1}{t}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u}f'(u)}{\sqrt{u-t}}\,du\right)dt + \\ + \frac{c}{\sqrt{x-c}}\int_x^1 \frac{1}{t\sqrt{t-c}}\left(\frac{f(1)}{\sqrt{1-t}} - \int_t^1 \frac{\sqrt{u}f'(u)}{\sqrt{u-t}}\,du\right)dt$$

Patterns

Theorem

$$\binom{2N}{N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_1,\ldots,a_k}(x)}{x-a_0}\right)(N) = \frac{4^N}{\pi}\mathbf{M}\left(\frac{\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{\sqrt{x(x-a_0)}}\right)(N)$$

where $\mathbf{b_i} = \{a_i, a_{i+1}\}$ for $i \in \{0,\ldots,k-1\}$ and $\mathbf{b_k} = \{1, a_k\}$

Theorem

$$\frac{1}{N\binom{2N}{N}}\mathbf{M}\left(\frac{x\,\mathrm{H}^*_{\mathbf{b_0},\ldots,\mathbf{b_k}}(x)}{(x-c)\sqrt{x(x-a_0)}}\right)(N) = \frac{\pi}{4^N}\mathbf{M}\left(\frac{\mathrm{H}^*_{a_0,\ldots,a_k}(x)}{x} + c\frac{\mathrm{H}^*_{\mathbf{w},a_1,\ldots,a_k}(x)}{x\sqrt{x-c}}\right)(N)$$

where $\mathbf{w} = (a_0, \{c\})$ and $\mathbf{b_0},\ldots,\mathbf{b_k}$ are as above

Example revisited

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

Example revisited

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)$$

Example revisited

$$\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} = \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j)$$

$$= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right)$$

$$= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{H}_{1,0}^*(x)}{x}\right)(i)$$

Example revisited

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{H}_{1,0}^*(x)}{x}\right)(i) \\
&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{\zeta_2 \mathrm{H}_{\{0,1\}}^*(x) - \mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x}\right)(i)
\end{aligned}$$

Example revisited

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \frac{1}{j^2} &= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \sum_{j=1}^{i} \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x}\right)(j) \\
&= \sum_{i=1}^{N} \frac{\binom{2i}{i}}{i} \left(\zeta_2 + \mathbf{M}\left(\frac{\mathrm{H}_0^*(x)}{x-1}\right)(i)\right) \\
&= \sum_{i=1}^{N} \binom{2i}{i} \mathbf{M}\left(\frac{\zeta_2 - \mathrm{H}_{1,0}^*(x)}{x}\right)(i) \\
&= \sum_{i=1}^{N} \frac{4^i}{\pi} \mathbf{M}\left(\frac{\zeta_2 \mathrm{H}_{\{0,1\}}^*(x) - \mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x}\right)(i) \\
&= \frac{4^N}{\pi} \mathbf{M}\left(\frac{\zeta_2 \mathrm{H}_{\{0,1\}}^*(x) - \mathrm{H}_{\{0,1\},\{0,1\},\{0,1\}}^*(x)}{x-1/4}\right)(N) \\
&\quad -\frac{2}{3}\zeta_3
\end{aligned}
$$

Recall

Mellin transform

$$\mathbf{M}\left(f(x)\right)(n) := \int_0^1 x^n f(x)\, dx$$

Properties

$$\begin{aligned}
\mathbf{M}\left(x^k f(x)\right)(n) &= \mathbf{M}\left(f(x)\right)(n+k) \\
\mathbf{M}\left(\frac{f(x)}{x-c}\right)(n) &= c^n \left(\int_0^1 \frac{f(x)}{x-c}\, dx + \sum_{i=1}^{n} \frac{1}{c^i} \mathbf{M}\left(f(x)\right)(i-1)\right) \\
\mathbf{M}\left(f(x)\right)(n) &= \frac{1}{n+1}\left(f(1) - \mathbf{M}\left(f'(x)\right)(n+1)\right)
\end{aligned}$$

Rules for Mellin transforms involving square roots

- $$\mathbf{M}\left(\frac{f(x)}{\sqrt{x-a}}\right)(n) = \\ = \frac{(4a)^n}{(2n+1)\binom{2n}{n}}\left(\int_0^1 \frac{f(x)}{\sqrt{x-a}}\,dx + 2\sum_{i=1}^{n}\frac{\binom{2i}{i}}{(4a)^i}\left(\sqrt{1-a}f(1) - \mathbf{M}\left(\sqrt{x-a}f'(x)\right)(i)\right)\right)$$

- $$\mathbf{M}\left(\frac{f(x)}{\sqrt{x}\sqrt{x-a}}\right)(n) = \\ = \frac{a^n}{4^n}\binom{2n}{n}\left(\int_0^1 \frac{f(x)}{\sqrt{x}\sqrt{x-a}}\,dx + \sum_{i=1}^{n}\frac{(4/a)^i}{i\binom{2i}{i}}\left(\sqrt{1-a}f(1) - \mathbf{M}\left(\frac{\sqrt{x-a}}{\sqrt{x}}f'(x)\right)(i)\right)\right)$$

Generating functions

Ordinary generating function

$$\sum_{n=0}^{\infty} f(n)x^n$$

Properties

$$\sum_{n=0}^{\infty} \frac{f(n)}{n} x^n = \int_0^x \frac{1}{t} \sum_{n=0}^{\infty} f(n)t^n \, dt$$

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} f(i)x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} f(n)x^n$$

$\rightarrow$ see also the talk by Janusz Gluza at this conference

Rules for generating functions containing $\binom{2n}{n}$

- $$\sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{i=0}^{n} f(i) = \\ = \frac{1}{\sqrt{\frac{1}{4}-x}} \left(\frac{f(0)}{2} + \frac{1}{4} \int_0^x \frac{1}{t\sqrt{\frac{1}{4}-t}} \sum_{n=0}^{\infty} t^n n \binom{2n}{n} f(n)\, dt \right)$$
- $$\sum_{n=0}^{\infty} x^n \frac{1}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{n} f(i) = \\ = \frac{2}{\sqrt{x}\sqrt{4-x}} \int_0^x \frac{1}{\sqrt{t}\sqrt{4-t}} \sum_{n=0}^{\infty} t^n \frac{f(n)}{\binom{2n}{n}}\, dt$$
- $$\sum_{n=1}^{\infty} x^n \frac{1}{n\binom{2n}{n}} \sum_{i=0}^{n} f(i) = \\ = \sum_{n=1}^{\infty} x^n \frac{f(n)}{n\binom{2n}{n}} + \frac{\sqrt{x}}{\sqrt{4-x}} \int_0^x \frac{1}{\sqrt{t}\sqrt{4-t}} \sum_{n=0}^{\infty} t^n \frac{f(n)}{\binom{2n}{n}}\, dt$$

Rules for generating functions of moments

$$\sum_{n=k}^{\infty} c_n x^n = \int_0^1 \frac{(tx)^k}{1-tx} f(t)\,dt \quad \text{where} \quad c_n := \int_0^1 t^n f(t)\,dt$$

- $$\int_0^1 \frac{(tx)^k}{1-tx}\frac{t}{t-c} f(t)\,dt = \\ = \frac{1}{1-cx}\left(c_k x^k + \frac{1}{x}\int_0^x \left(f(1)\frac{t^{k+1}}{1-t} - \int_0^1 \frac{(tu)^{k+1}}{1-tu} u f'(u)\,du\right)dt\right)$$

- $$\int_0^1 \frac{(tx)^k}{1-tx}\frac{1}{\sqrt{t}\sqrt{t-a}} f(t)\,dt = \\ = \frac{1}{\sqrt{1-ax}}\left(kc_k\int_0^x \frac{t^{k-1}}{\sqrt{1-at}}\,dt + \int_0^x \frac{1}{\sqrt{1-at}}\left(\sqrt{1-a}f(1)\frac{t^k}{1-t} - \int_0^t \frac{(tu)^k}{1-tu}\sqrt{u}\sqrt{u-a}f'(u)\,du\right)dt\right)$$

Conclusion

Nested (inverse) binomial sums and their associated integrals

- Conversion between sum and integral representations
- Asymptotic expansion of the sums
- Basis of new iterated integrals involving square roots
- Conditions for Mellin transform as nested binomial sums

Parametric integration

Compute ODEs for parameter integrals (e.g. Mellin convolutions)

Rewrite rules

- Patterns in Mellin convolutions
- Reduce computational effort by available rules and patterns
- Also for Mellin transforms and generating functions