On the QCD cusp anomalous dimension

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based on work in progress with

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Cusp anomalous dimension

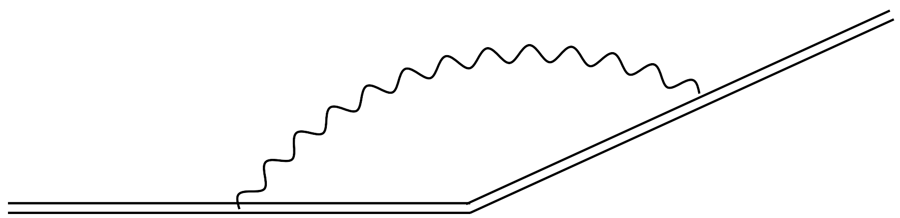
- Cusp anomalous dimension describes infrared divergences

[cf. L. Magnea's talk on Friday]

- $\Gamma_{\rm cusp}(\phi)$ governs UV divergences at cusp

[Polyakov; 1 loop]

[2 loops: Korchemsky, Radyushkin (1987)]



$$\langle W \rangle \sim e^{-|\ln \frac{\mu_{UV}}{\mu_{IR}}|\, \Gamma_{\rm cusp}}$$

- relation to light-like anomalous dimension K

[Korchemsky et al]

$$x = e^{i\phi} \qquad \lim_{x\to 0} \Gamma_{\rm cusp} = -K \log x + \mathcal{O}(x^0)$$

- N=4 SYM susy/non-susy Wilson loop operator

$$\xi = \frac{\cos\theta - \cos\phi}{i \sin\phi} \qquad \theta = \frac{\pi}{2} \quad \longrightarrow \quad \xi = \frac{1+x^2}{1-x^2}$$

Beautiful answers

- Observation: constants in N=4 SYM anomalous dimensions have uniform 'transcendentality' [Kotikov, Lipativ, Velizhanin]

- generalize: pure functions of uniform weight (UT)

- suggests iterative differential structure

- what about QCD?

ref. [JMH, PRL 110 (2013)] suggests QCD integrals can also be chosen UT

do physical results look nice when expressed in a good basis?

Perturbative results in N=4 SYM

- 1 loop $A^{(1)}(\phi) = -\xi \log x$
- 2 loops

[Makeenko, Oleson, Semenoff (2006)]
[Drukker, Forini (2012)]

$$A^{(2)}(\phi) = \frac{1}{3}\xi \left[\pi^2 \log x + \log^3 x\right]$$
$$- \xi^2 \left[\zeta_3 + \zeta_2 \log x + \frac{1}{3}\log^3 x + \log x \mathrm{Li}_2(x^2) - \mathrm{Li}_3(x^2)\right] .$$

- bosonic Wilson loop in N=4 SYM, 2 loops

$$\Gamma^{(2)g}_{\mathrm{cusp}}(\phi) = A^{(2)}(\phi) - A^{(2)}(0) + B^{(2)}(\phi) - B^{(2)}(0) , \qquad \theta = \frac{\pi}{2}$$
$$B^{(2)}(\phi) = \left[\log^2 x + \frac{1}{3}\pi^2\right] - \xi \left[\zeta_2 + \log^2 x + 2\log x \mathrm{Li}_1(x^2) - \mathrm{Li}_2(x^2)\right] .$$

- 3 loops; ξ term at any loop order [Correa, JMH, Maldacena, Sever (2012)]
- 4 loops planar; nonplanar ξ^4 term;
- d-log algorithm for ladder integrals

[JMH, Huber (2013)]

A new look at two loops in QCD

- QCD result

[Korchemsky, Radyushkin (1987)]
nf [Braun, Beneke, 1995]
[Kidonakis (2009)]

$$\Gamma^{(1)} = C_F \left[A^{(1)}(\phi) - A^{(1)}(0) \right]$$

$$\Gamma^{(2)} = C_F C_A \left[A^{(2)}(\phi) - A^{(2)}(0) + B^{(2)}(\phi) - B^{(2)}(0) \right] + \left(-\frac{5}{9} C_F T_F n_f - \frac{67}{36} C_F C_A \right) \left[A^{(1)}(\phi) - A^{(1)}(0) \right] .$$

Only functions from N=4 SYM needed!

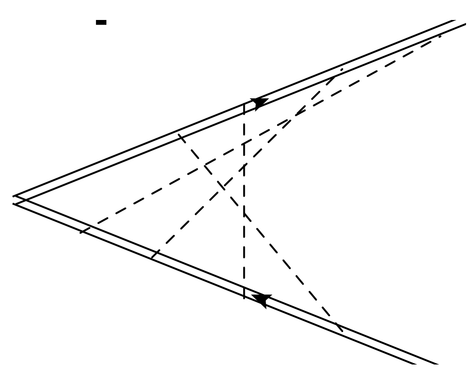
- $A^{(1)}$ uniform weight 1 : from susy WL
- $B^{(2)}$ uniform weight 2 : from bosonic WL
- $A^{(2)}$ uniform weight 3 : from susy WL

- what happens at 3 loops?
- why functions of uniform weight?

Why should we get pure functions?

- For Wilson line integrals, this is easy to see [JMH, Huber, JHEP 1309 (2013) 147]
  - key: 'd-log representations'
  - make it obvious that result is given by pure functions
  - provides algorithm for computing the answer

$$
\begin{aligned}
I_{\mathrm{NP,four-loop}}(x) = &-2\zeta_2(18H_{1,1,1,2}+24H_{1,1,2,1}+18H_{1,2,1,1}+30H_{1,1,1,1,1}) \\
&+48H_{1,1,1,4}+64H_{1,1,2,3}+64H_{1,1,3,2}+48H_{1,2,1,3} \\
&+48H_{1,2,2,2}+80H_{1,1,1,1,3}+80H_{1,1,1,2,2}+24H_{1,1,1,3,1} \\
&+64H_{1,1,2,1,2}+32H_{1,1,2,2,1}+32H_{1,1,3,1,1}+48H_{1,2,1,1,2} \\
&+24H_{1,2,1,2,1}+24H_{1,2,2,1,1}+62H_{1,1,1,1,1,2}+40H_{1,1,1,1,2,1} \\
&+22H_{1,1,1,2,1,1}+8H_{1,1,2,1,1,1}+6H_{1,2,1,1,1,1}+H_{1,1,1,1,1,1,1}
\end{aligned}
$$



  - note: implies that all functions of this family have this property!

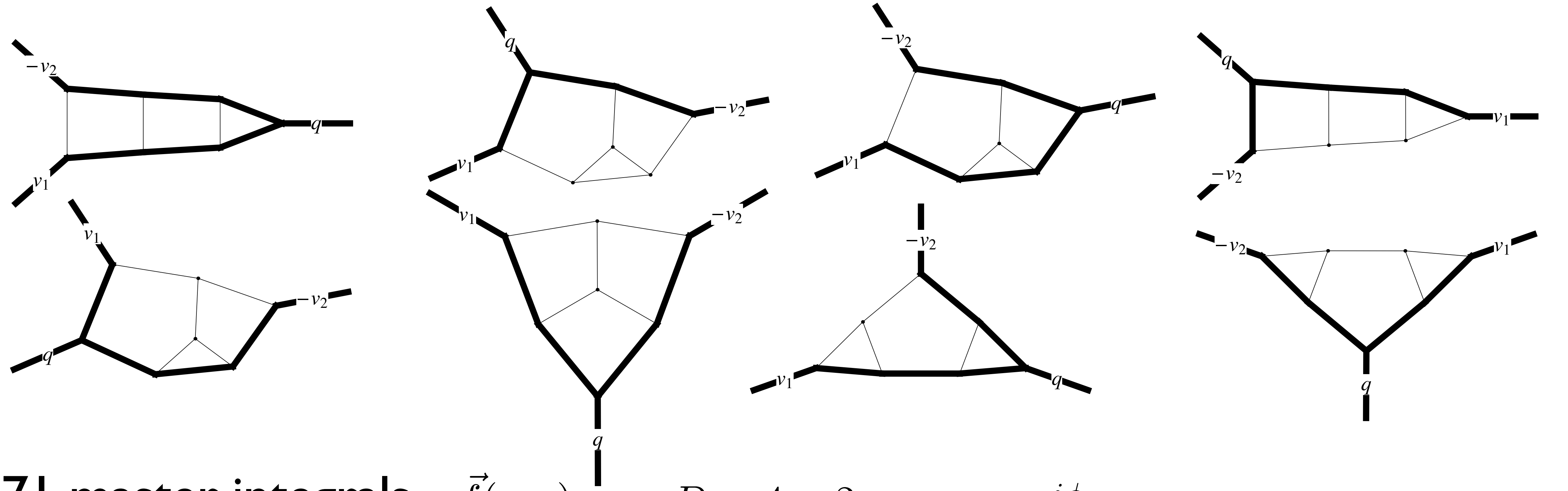
see this more generally: [JMH, PRL 110 (1013) 25]

- algorithm also works for the multi-line case.

other method:
[cf. E. Gardi's talk on Thursday]

Master integrals

- abelian eikonal exponentiation: need only planar integrals



- 71 master integrals $\vec{f}(x;\epsilon)$ $D = 4 - 2\epsilon$ $x = e^{i\phi}$
- differential equations in suitable basis [method: see JMH, PRL 110 (1013) 25]

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1} \right] \vec{f}(x;\epsilon)$$

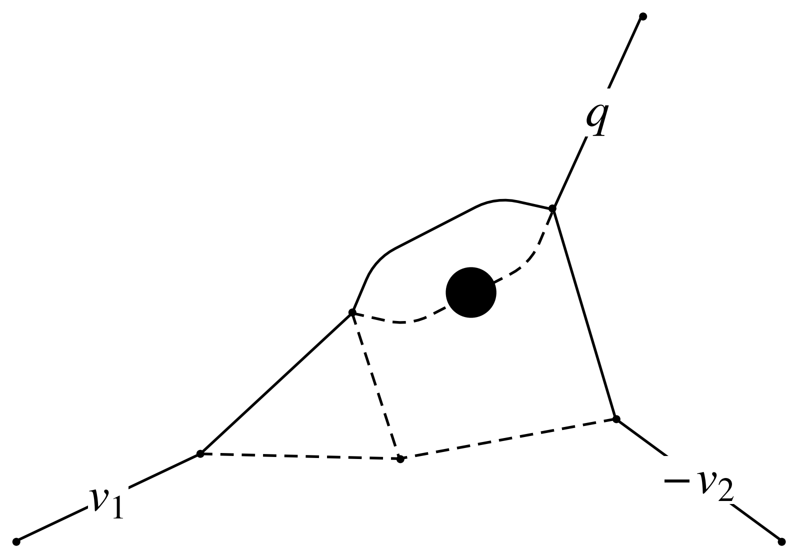
a, b, c constant 71x71 matrices

- boundary conditions trivially from $x = 1$
- solution in terms of harmonic polylogarithms

one integral: [Chetyrkin, Grozin, NP B666 (2003)]

[cf. V. Smirnov's and T. Huber's talks later today for applications to multi-scale cases]

Example



$$f_{44} = \epsilon^5 \frac{1-x^2}{x} G_{1,0,1,0,1,0,1,1,2,0,1,0}$$

$$x = e^{i\phi}$$

$$f_{44} = \epsilon^4 \Big[-\frac{1}{6}\pi^2 H_{0,0}(x) - \frac{2}{3}\pi^2 H_{1,0}(x) - 4H_{0,-1,0,0}(x) + 2H_{0,0,-1,0}(x) + 2H_{0,1,0,0}(x) - 4H_{1,0,0,0}(x) + 4\zeta_3 H_0(x) - \frac{17\pi^4}{360} \Big] + \mathcal{O}(\epsilon^5)$$

- all basis integrals are pure functions of uniform weight
- numerical checks with FIESTA
- confirmed previously known `N=4 SYM` result

Calculation at three loops

(1) compute proper vertex function

(2) take into account renormalization of Lagrangian

(3) compute vertex renormalization

(4) extract Gamma cusp $\Gamma_{\rm cusp} = \frac{\partial}{\partial \log \mu} \log Z$

- color structures $\Gamma^{(3)}_{\rm cusp} : c_1 C_F C_A^2 + c_2 C_F (T_f n_f)^2 + c_3 C_F^2 T_f n_f + c_4 C_F C_A T_F n_f$

$C_F (T_F n_f)^2$ [Braun, Beneke, 1995]

$C_F^2 T_F n_f$

$C_F C_A T_F n_f$ } this talk

$C_F C_A^2$ stay tuned!

Results $\Gamma^{(3)}_{\rm cusp} : c_1 C_F C_A^2 + c_2 C_F (T_f n_f)^2 + c_3 C_F^2 T_f n_f + c_4 C_F C_A T_F n_f$

$$c_2 = -\frac{1}{27} A^{(1)} \qquad c_3 = \left(\zeta_3 - \frac{55}{48}\right) A^{(1)}$$

$$c_4 = -\frac{5}{9}\left(A^{(2)} + B^{(2)}\right) - \frac{1}{6}\left(7\zeta_3 + \frac{209}{36}\right) A^{(1)}$$

$$A = A(\phi) - A(0)$$ Only functions from N=4 SYM needed!

- Checks: expected divergence structure

$$\log Z = -\frac{1}{2\epsilon}\left(\frac{\alpha_s}{\pi}\right)\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{\beta_0}{16\epsilon^2}\Gamma^{(1)} - \frac{1}{4\epsilon}\Gamma^{(2)}\right] + \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right].$$

- Known limit $\lim_{x\to 0}\Gamma_{\rm cusp} = -K\log x + \mathcal{O}(x^0)$

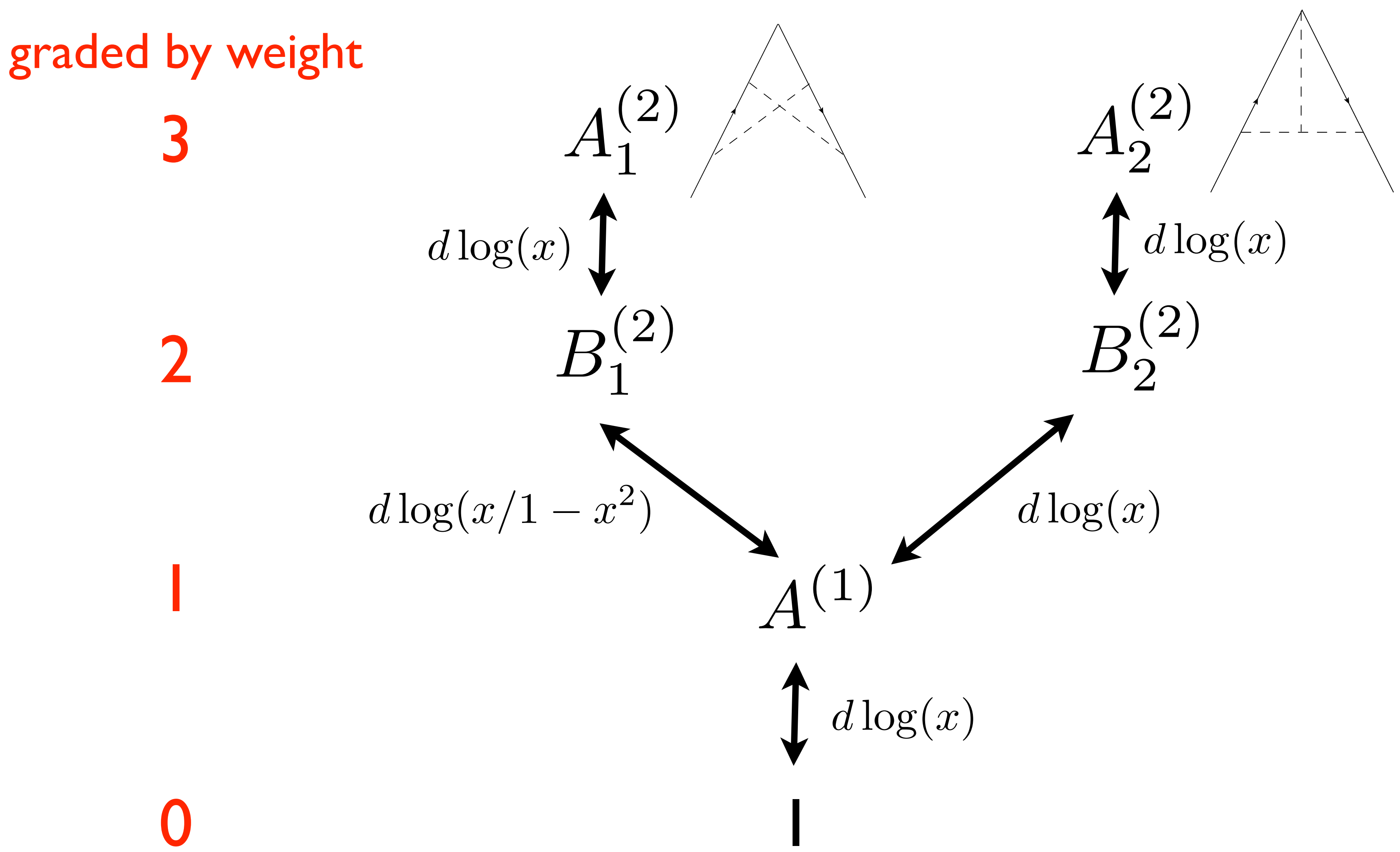
$$K^{(3)} = \frac{1}{4}C_F C_A^2\left(\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2\right) + C_F^2 n_f T_F\left(-\frac{55}{48} + \zeta_3\right)$$
$$+ \frac{1}{2}C_F C_A n_f T_F\left(-\frac{209}{108} + \frac{10}{9}\zeta_2 - \frac{7}{3}\zeta_3\right) + C_F n_f^2 T_F^2\left(-\frac{1}{27}\right)$$

[Vogt (2001)] [Berger (2002)] [Moch, Vermeaseren, Vogt (2004)]

Iterative structure of loop integrals

cf. [Caron-Huot, J.M.H. (2014)

- The physical result is finite as $D \to 4$
- Obtain it from a subset of finite integrals/functions?



- Note: functions appear already in `simpler` N=4 SYM calculations!
- top-down vs. bottom-up approach

Conclusions

- computed nf terms of cusp anomalous dimension at 3 loops
- simple result; closely related to N=4 SYM
- iterative structure of finite loop integrals
- CA^2 CF term in progress

Extra slides

Example: choice of integral basis

three-loop N=4 SYM form factor

$$
\begin{aligned}
F_S^{(3)} = R_\epsilon^3 \Bigg[&+ \frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \\
&- \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} \\
&- \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} \\
&- \frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
&- \frac{16(2D-7)(5D-18)\left(52D^2-485D+1128\right)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} \\
&- \frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
&- \frac{128(2D-7)(3D-8)\left(91D^2-821D+1851\right)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
&- \frac{128(2D-7)\left(1497D^3-20423D^2+92824D-140556\right)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} \\
&+ \frac{4(D-3)}{D-4} B_{8,1} + \frac{64(D-3)^3}{(D-4)^3} B_{6,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} \\
&- \frac{16(3D-10)(3D-8)\left(144D^2-1285D+2866\right)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
&+ \frac{128(2D-7)\left(177D^2-1584D+3542\right)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} \\
&+ \frac{64(2D-5)(3D-8)(D-3)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} \\
&\times \left(2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104\right) B_{4,1} \\
&+ \frac{4(D-3)}{D-4} C_{8,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \Bigg] . \qquad \text{(B.1)}
\end{aligned}
$$

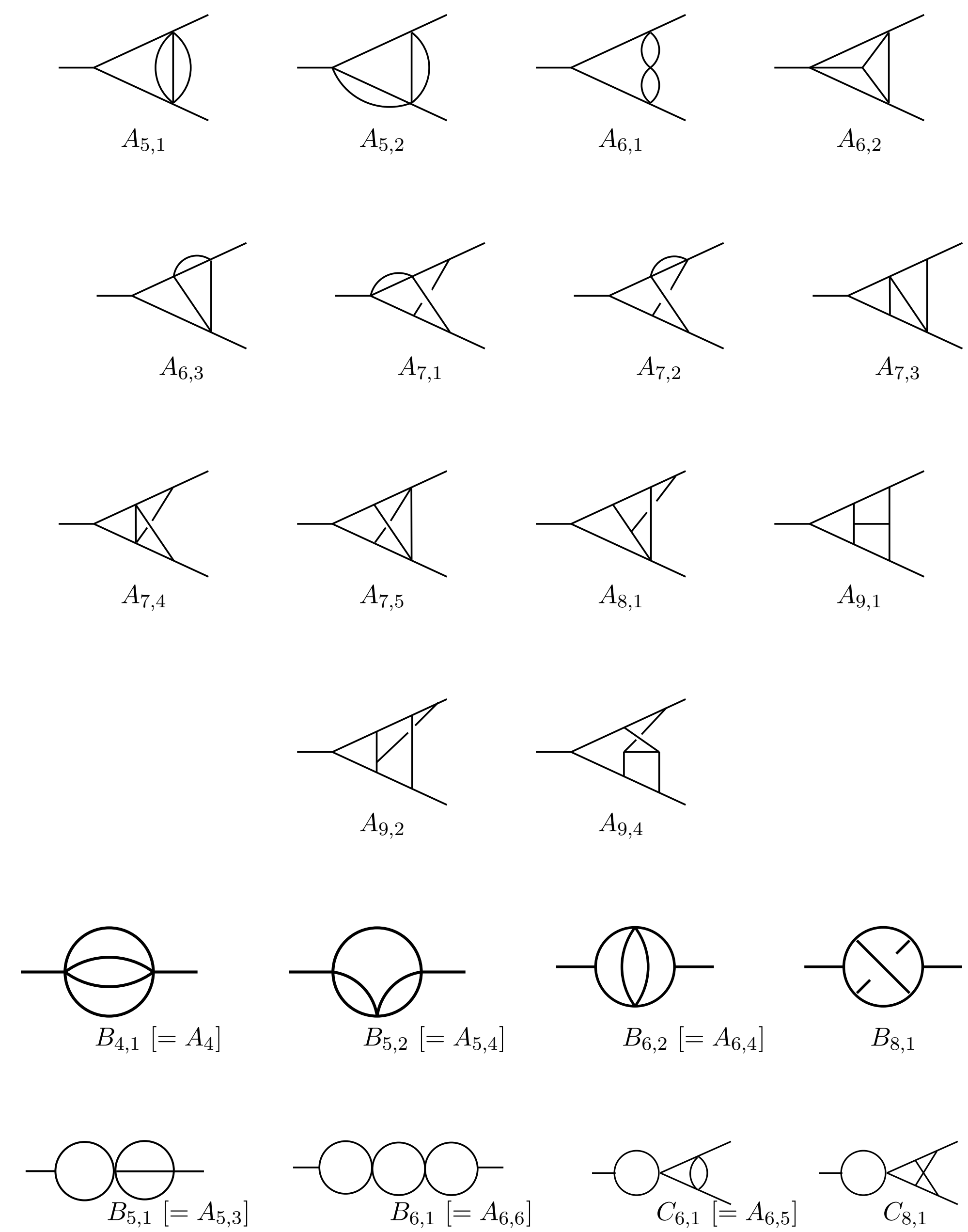


Figure 1: Master integrals for the three-loop form factors. Labels in brackets indicate the naming convention of Ref. [25].

Gehrmann, Glover, Huber, Ikizlerli, Studerus;
Lee, Smirnov & Smirnov

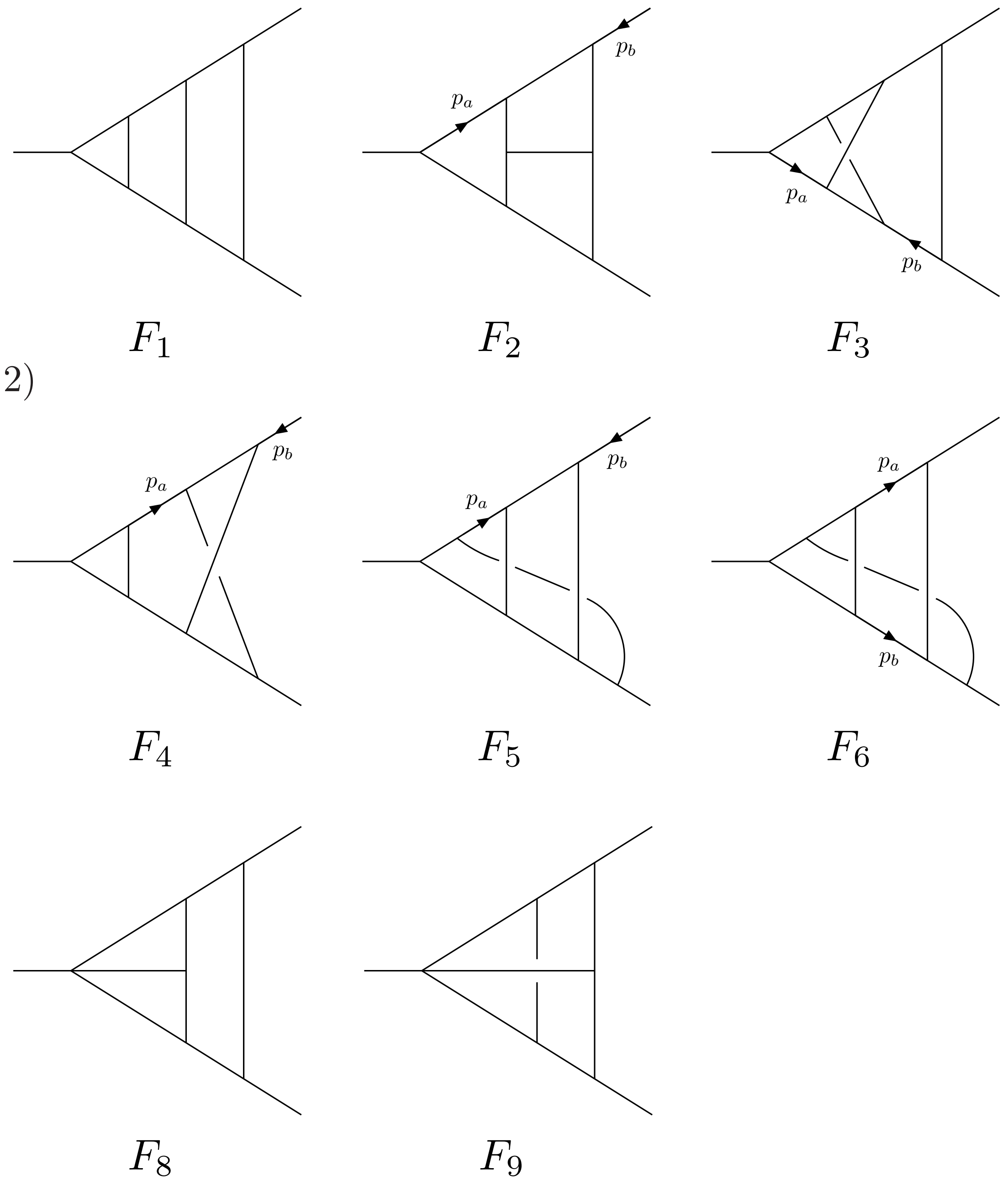
Gehrmann, J.M.H., Huber (2011)

Example: choice of integral basis

three-loop N=4 SYM form factor

$$F_S^{(3)} = R_\epsilon^3 \cdot [8\, F_1^{\rm exp} - 2\, F_2^{\rm exp} + 4\, F_3^{\rm exp} + 4\, F_4^{\rm exp} - 4\, F_5^{\rm exp} - 4\, F_6^{\rm exp} - 4\, F_8^{\rm exp} + 2\, F_9^{\rm exp}]$$

$$\begin{aligned} F_S^{(3)} &= R_\epsilon^3 \cdot [8\, F_1^{\rm exp} - 2\, F_2^{\rm exp} + 4\, F_3^{\rm exp} + 4\, F_4^{\rm exp} - 4\, F_5^{\rm exp} - 4\, F_6^{\rm exp} - 4\, F_8^{\rm exp} + 2\, F_9^{\rm exp}] \\ &= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right) \\ &\quad -\frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right) \\ &\quad +\epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)\,. \qquad (5.2) \end{aligned}$$



- each integral has uniform (and maximal) ``transcendentality''
  T[Zeta[n]] = n
  T[eps^-n] = n
  T[A B] = T[A] + T[B]

- for theories with less susy, other integrals also needed

Gehrmann, J.M.H., Huber (2011)

Iterative structure for finite loop integrals

[Caron-Huot, J.M.H. (2014)]