DIS sum rules in four loops: news and update

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in collaboration with

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Loops & Legs, Monday 28 April 2014

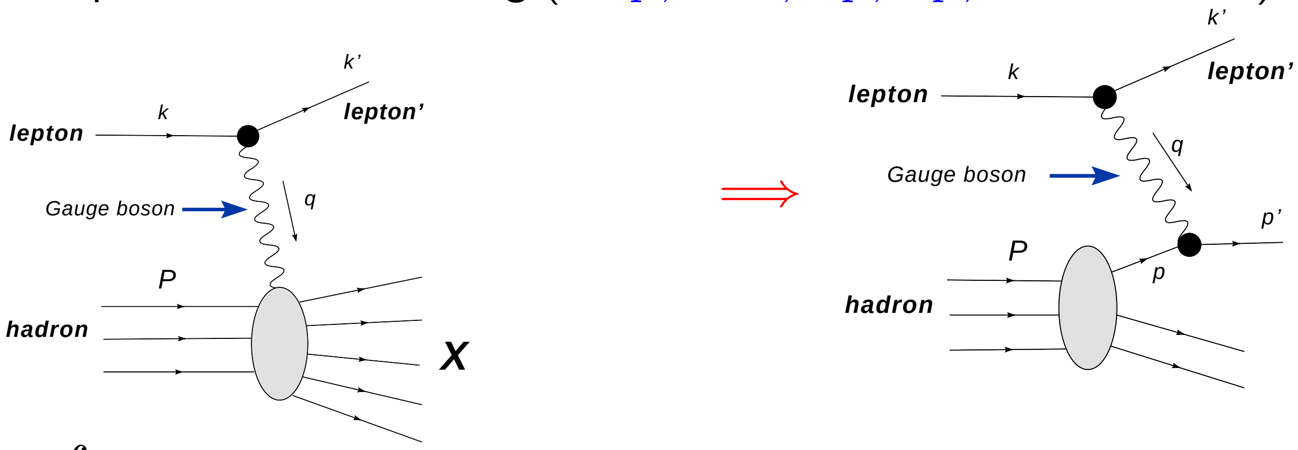
Outline

- Bjorken SR for unpolarized scattering at $\mathcal{O}({\alpha_s}^4)$ (4 loops) /new result!/

- update of the Ellis-Jaffe Sum Rule at Four-Loop (a two years-old result has been corrected, but numerics has changed only insignificantly)

- some news on interplay between higher order PT corrections to the Bjorken SR for polarized scattering and higher twist contributions

- Deep-inelastic lepton-hadron scattering ($e^{\pm}p,\ e^{\pm}n,\ \nu p,\ \bar{\nu}p,\ \ldots$ - collisions)



$$
\begin{aligned}
W_{\mu\nu} &= \frac{1}{4\pi}\int d^4z e^{iqz}\langle p,s|J_\mu(z)J_\nu^+(0)|p,s\rangle \\
&= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\boxed{F_1(x,Q^2)}+\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)\frac{1}{p\cdot q}F_2(x,Q^2) \\
&\quad +i\epsilon_{\mu\nu\rho\sigma}q_\rho\left(\frac{s_\sigma}{p\cdot q}\boxed{g_1(x,Q^2)}+\frac{s_\sigma p\cdot q-p_\sigma q\cdot s}{(p\cdot q)^2}g_2(x,Q^2)\right)
\end{aligned}
$$

where $x = Q^2/(2p\cdot q)$ and $Q^2 = -q^2$,

J_μ is either EM $J_\mu = \sum_{i=1}^{n_f} e_i\overline{\psi_i}\gamma_\mu\psi_i \equiv \overline{\psi}E\gamma_\mu\psi$ or the weak charged current

Structure functions F_1 and g_1 appear in three DIS sum rules: the Bjorken SR's for unpolarized (F_1) and polarized DIS (g_1) and the Ellis-Jaffe sum rule (g_1); all three will be discussed in the talk

Parton model prediction (Bjorken sum rule for F_1) reads:

$$\int_0^1 dx\, F_1^{\bar{\nu}p-\nu p} = 1$$

in QCD the SR receives higher order PT correctons, which can be related to thec oefficient function (CF) C^{Bj}_{unp} of the corresponding Operator Product Expansion (OPE)

$$i\int dz e^{iqz} T\{J_\mu(z) J^+_\nu(0)\} \stackrel{Q^2\to\infty}{=\!=\!=} \tag{1}$$

$$(-g_{\mu\nu} + q_\mu q_\nu/q^2)\ C^{Bj}_{unp}\ q_\nu\, J^V_\nu + \cdots \text{(other operators)},$$

where J^V_ν is a *vector* quark current with a proper flavour structure.

The OPE above is one of the simplest ones (no epsilon tensors, no γ_5). Experimentally, this SR is presumably very difficult to deal with. On the other hand, it it has a reach history of calculations.

Bjorken SR for unpolarized scattering

$$\int_0^1 dx F_1^{\bar{\nu}p-\nu p} = 1 \quad \leftarrow \quad \text{/Bjorken (1967)/}$$

$$- \quad \frac{2}{3} a_s \quad \leftarrow \quad \text{/Bardin, Buras, Duke, Muta (1978); Altarelli, R. K. Ellis, Martinelli (1978)/}$$

$$+ \quad \underbrace{a_s^2 \left(-\frac{23}{6} + \frac{8}{27} n_f \right) \quad \leftarrow \quad \text{K.Ch., Gorishny, Larin, Tkachov (1984).}}_{\Uparrow} \qquad \boxed{\mathbf{a_s \equiv \alpha_s/\pi}}$$

First real application of the first (SCHOONSHIP) version of the legendary MINCER program for the first two loop calculation in DIS! It was also first "real life (QCD)" application of the powerfull method of projectors / Gorishny, Larin, Tkachov (1983);Gorishny, Larin (1987)/ to deal with OPE; it is now routinely being used in virtually every calculation of CF's of OPE

$$+ \underbrace{a_s^3 \left(-\frac{4075}{108} + \frac{622}{27}\zeta_3 - \frac{680}{27}\zeta_5 + n_f \left[\frac{3565}{648} - \frac{59}{27}\zeta_3 + \frac{10}{3}\zeta_5 \right] - \frac{155}{972} n_f^2 \right)}_{\Uparrow} \quad \leftarrow \quad \boxed{\text{Larin, Tkachov, Vermaseren (1991)}}$$

First real application of the second (FORM 2) version of the MINCER program for the first THREE loop calculation in DIS!

We extended the above results to the four-loop level:

$$\int_0^1 F_1^{\bar{\nu}p-\nu p} \equiv C^{Bj}_{unpol} = 1 + \ldots$$
$$+ \; a_s^4 \left(-\frac{12053285}{31104} + \frac{70315}{162}\zeta_3 + \frac{67}{2}\zeta_3^2 - \frac{939995}{1296}\zeta_5 + \frac{341075}{2592}\zeta_7 \right.$$
$$+ \; n_f \left[\frac{2756269}{31104} - \frac{256543}{3888}\zeta_3 - \frac{743}{81}\zeta_3^2 + \frac{36835}{324}\zeta_5 - \frac{49}{24}\zeta_7 \right]$$
$$\left. + \; n_f^2 \left[-\frac{548725}{93312} + \frac{29}{24}\zeta_3 + \frac{1}{3}\zeta_3^2 - \frac{55}{18}\zeta_5 \right] + \frac{445}{4374} n_f^3 \right)$$

Transcendentally structure : at order α_s^4 ζ_7 does appear BUT not ζ_4, ζ_6 and $\zeta_3 \zeta_4$! /while they do abound in separate contributions!/

This is now well understood /Broadhurst (1999); Baikov, K.Ch.,(2010) / as a consequence of 2 facts:

- **peculiar structure of the four-loop masters**
- **the rationality (somewhat mysterious, as separate diagrams do contain ζ_3) of three-loop QCD β-function**

four-loop result for C^{Bj}_{unp}; con-ed

Numerically, the result is

$$
\begin{aligned}
C^{Bj}_{unp} &= 1. - 0.6667 a_s + a_s^2(-3.833 + 0.2962\, n_f) \\
&+ a_s^3\,(-36.155 + 6.33135\, n_f - 0.1595\, n_f^2) \\
&+ a_s^4\,(-436.768 + 111.873\, n_f - 7.115\, n_f^2 + 0.10174\, n_f^3)
\end{aligned}
$$

$$
C^{Bj}_{unp}(n_f = 3) = 1 - 0.6667\, a_s - 2.9444\, a_s^2 - 18.5963\, a_s^3 - 162.436\, a_s^4
$$

We observe two typical patterns:

(i) signifcant cancellations between n_f^0 and n_f^1 terms

(ii) almost geometrical sign-non-alternating growth of the coefficients of α_s series

It is very amusing to compare unpolarized case with the polarized one:

$$\begin{aligned} C^{Bj}_{unp} &= 1. - 0.6667 a_s + a_s^2(-3.833 + 0.2962\, n_f) \\ &+ a_s^3\,(-36.155 + 6.33135\, n_f - 0.1595\, n_f^2) \\ &+ a_s^4\,(-436.768 + 111.873\, n_f - 7.115\, n_f^2 + 0.10174\, n_f^3) \end{aligned}$$

$$\begin{aligned} C^{Bjp}_{pol} &= 1 - a_s + a_s^2\,(-4.583 + 0.3333\, n_f) \\ &+ a_s^3\,\left(-41.44 + 7.607\, n_f - 0.1775\, n_f^2\right) \\ &+ a_s^4\,\left(-479.4 + 123.4\, n_f - 7.697\, n_f^2 + 0.1037\, n_f^3\right) \end{aligned}$$

$$\begin{aligned} C^{Bj}_{unp}(n_f = 3) &= 1 - \frac{2}{3}\, a_s - 2.9444\, a_s^2 - 18.5963\, a_s^3 - 162.436\, a_s^4 \\ C^{Bj}_{pol}(n_f = 3) &= 1 - a_s \quad - 3.583 a_s^2 \quad - 20.22 a_s^3 \quad - 175.7\, a_s^4 \end{aligned}$$

Ellis-Jaffe sum rule

OPE of 2 EM currents:

$$i\int dz e^{iqz} T\{J_\mu(z)J_\nu(0)\} \overset{Q^2\to\infty}{=} \epsilon_{\mu\nu\rho\sigma}\frac{q_\rho}{q^2}\Big[\quad C^{NS}\left(L, a_s(\mu)\right)\sum_a C^a J^{5,a}_\sigma(0)$$

$$+ C^S\left(L, a_s(\mu)\right) J^5_\sigma(0)\Big] + \cdots \text{(higher twists)}$$

$a_s = \alpha_s/\pi,\ L = \log(\frac{\mu^2}{Q^2})$, flavour CF: $C^a = \mathrm{Tr}\left(E^2 \cdot T^a\right)$

non-singlet axial current: $J^{5,a}_\sigma(x) = \overline{\psi}\gamma_\sigma\gamma_5 t^a\psi(x)$ $\Longrightarrow$ conserved

singlet axial current: $J^5_\sigma(x) = \sum_{i=1}^{n_f}\overline{\psi_i}\gamma_\sigma\gamma_5\psi_i(x)$ $\Longrightarrow$ not conserved due to the (non-abelean!) anomaly.

As a result, operator J^5_σ develops non-zero anomalous dimension (starting from 2 loops). This compicates RG-improvement, which now requires the evaluation of the anom. dimension with (L+1)- loop accuracy in addition to L-loop coef. function C^S

OPE of 2 EM currents:

$$i\int dz e^{iqz} T\{J_\mu(z)J_\nu(0)\} \quad \overset{Q^2\to\infty}{=} \epsilon_{\mu\nu\rho\sigma}\frac{q\rho}{q^2}\left[\quad C^{NS}\sum_a C^a J_\sigma^{5,a}(0) + C^s\, J_\sigma^5(0)\right] + \cdots \text{(higher twists)}$$

The OPE results to Ellis-Jaffe sum-rule:

$$\int_0^1 dx g_1^{p(n)}(x,Q^2) \quad = \quad C^{\rm ns}(L,a_s(\mu))\,(\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8) + C^{\rm s}(L,a_s(\mu))\,\frac{1}{9}a_0(\mu^2)$$

$$\begin{aligned}
|g_A|s_\sigma &= 2\langle p,s|J_\sigma^{5,3}|p,s\rangle &&= (\Delta u - \Delta d)s_\sigma, \quad /\Delta u \equiv \overline{\psi}_u\gamma_\sigma\gamma_5\psi_u(x),\ \text{etc.}/ \\
a_8 s_\sigma &= 2\sqrt{3}\langle p,s|J_\sigma^{5,8}|p,s\rangle &&= (\Delta u + \Delta d - 2\Delta s)s_\sigma, \\
a_0(\mu^2)s_\sigma &= \langle p,s|J_\sigma^5|p,s\rangle &&= (\Delta u + \Delta d + \Delta s)s_\sigma = \Delta\Sigma(\mu^2)s_\sigma.
\end{aligned}$$

$g_A = 1.270 \pm 0.003$ from neutron beta decays

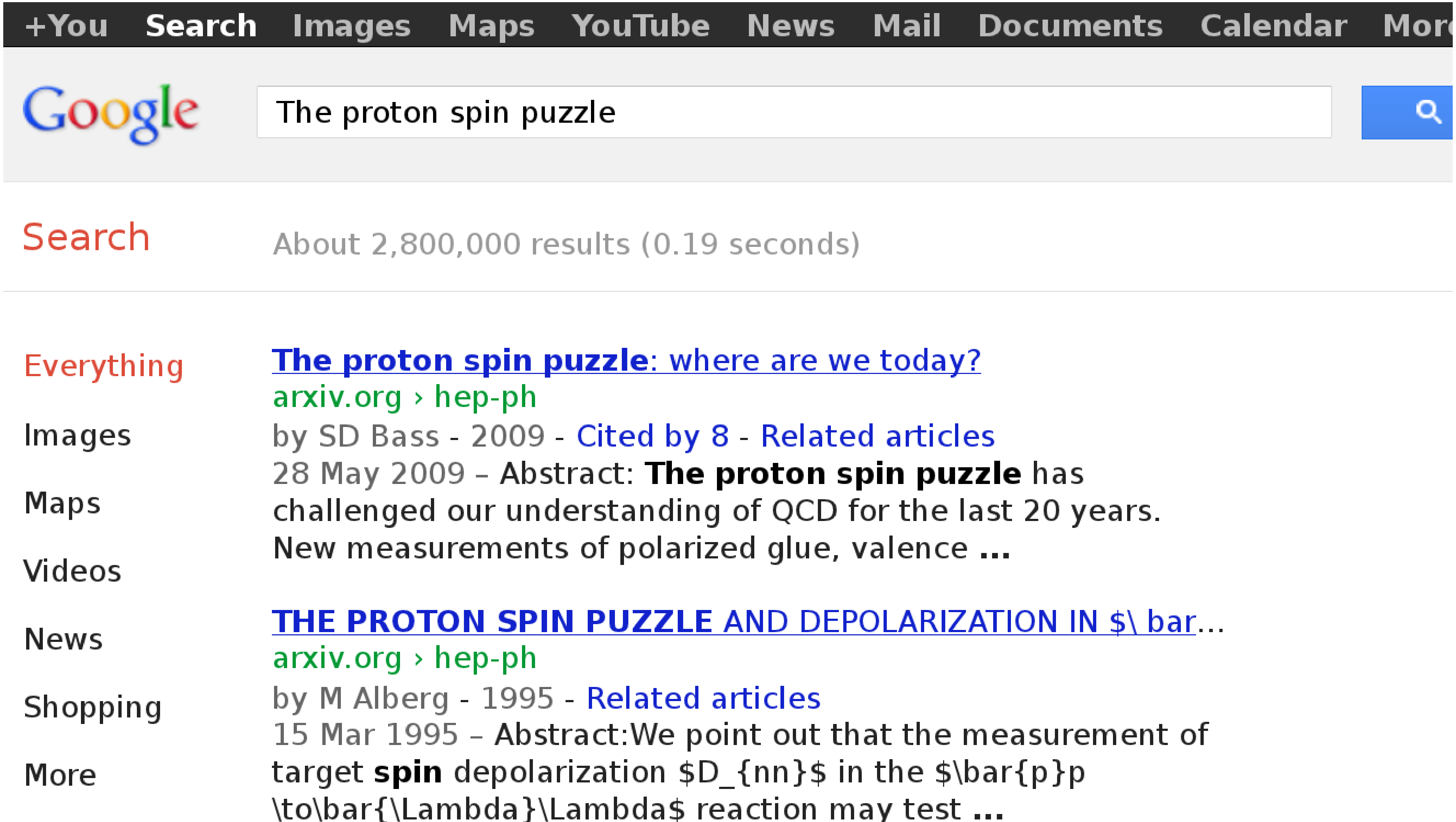
$a_8 = 0.58 \pm 0.03$ from hyperon beta decays (assuming SU(3))

$a_0 = 0.33 \pm 0.03$ from EJ sum-rule (COMPASS Collab., Phys. Lett. B647 (2007) 8)

(in the naive parton model a_0 interpred as the fraction of the proton's spin carried by its quarks (and antiquarks) and was assumed to be between one and a half!

$\Longrightarrow$ "proton spin crisis"!

Around a quarter of century ago the European Muon Collaboration (EMC) published their polarized deep inelastic measurement of the proton's g_1 spin dependent structure function and the flavour-singlet axial-charge $g_A^{(0)}$. They established that the quark's intrinsic spin contributes little of the proton's spin. The challenge was christened as "the proton spin puzzle" and inspired a vast programme of theoretical activity and new experiments at CERN, DESY, JLab, RHIC and SLAC.



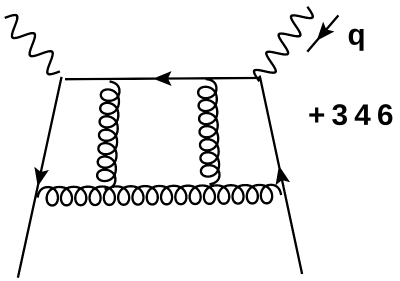
$$\int_0^1 dx g_1^{p(n)}(x,Q^2) \quad = \quad C^{\rm ns}(L,a_s(\mu))\,(\pm\frac{1}{12}|g_A| + \frac{1}{36}a_8) + C^{\rm s}(L,a_s(\mu))\,\frac{1}{9}a_0(\mu^2)$$

If one consider a difference $g_1^p - g_1^n$ then only first term survives:
$\Longrightarrow$ Bjorken sum-rule

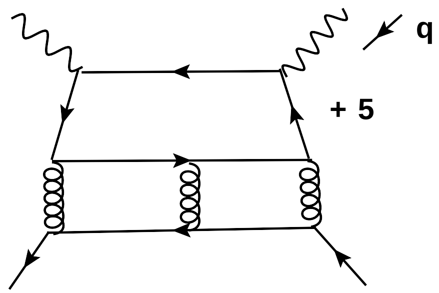
$$\int_0^1 dx g_1^{(p-n)}(x,Q^2) \quad = \quad C^{\rm ns}(L,a_s(\mu))\,\frac{1}{6}|g_A|$$

Typical diagrams at ${\alpha_s}^3$ (known from /Larin, van Ritbergen, Vermaseren /1997/),

C^{NS}



C^{NS} *and* C^S



C^{NS} contributes to the (polarized) Bjorken sum rule and has been computed at order α_s^4 (P. Baikov, K.Ch. J.Kühn, PLR 104 (2010) 132004).

After the RG improvement: the choice $\mu = Q$ (the *non-singlet* axial vector current $J_\sigma^{5,3}$ is scale invariant[1])

$$C^{\rm ns}(0, a_s(Q))|_{n_f=3} = 1 - a_s(Q) - 3.583\, a_s^2(Q) - 20.215\, a_s^3(Q)$$
$$-175.75\, a_s^4(Q)$$

The Bjorken sum rule

$$\int_0^1 dx g_1^{(p-n)}(x, Q^2) \quad = \quad C^{\rm ns}(L, a_s(\mu))\, \frac{1}{6} |g_A|$$

can be employed to fit $\alpha_s(Q)$. Unfortunately, the maximal value of Q^2 being used is only 3 GeV^2, which is too small and leads to a significant residual μ-dependence, which does not decrease with adding more PT terms. Still, one could try at least to see a direction in which higher orders drive theory prediction

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[1] Better to say it could (and, in fact, must) renormalized in such a way to be scale-invariant

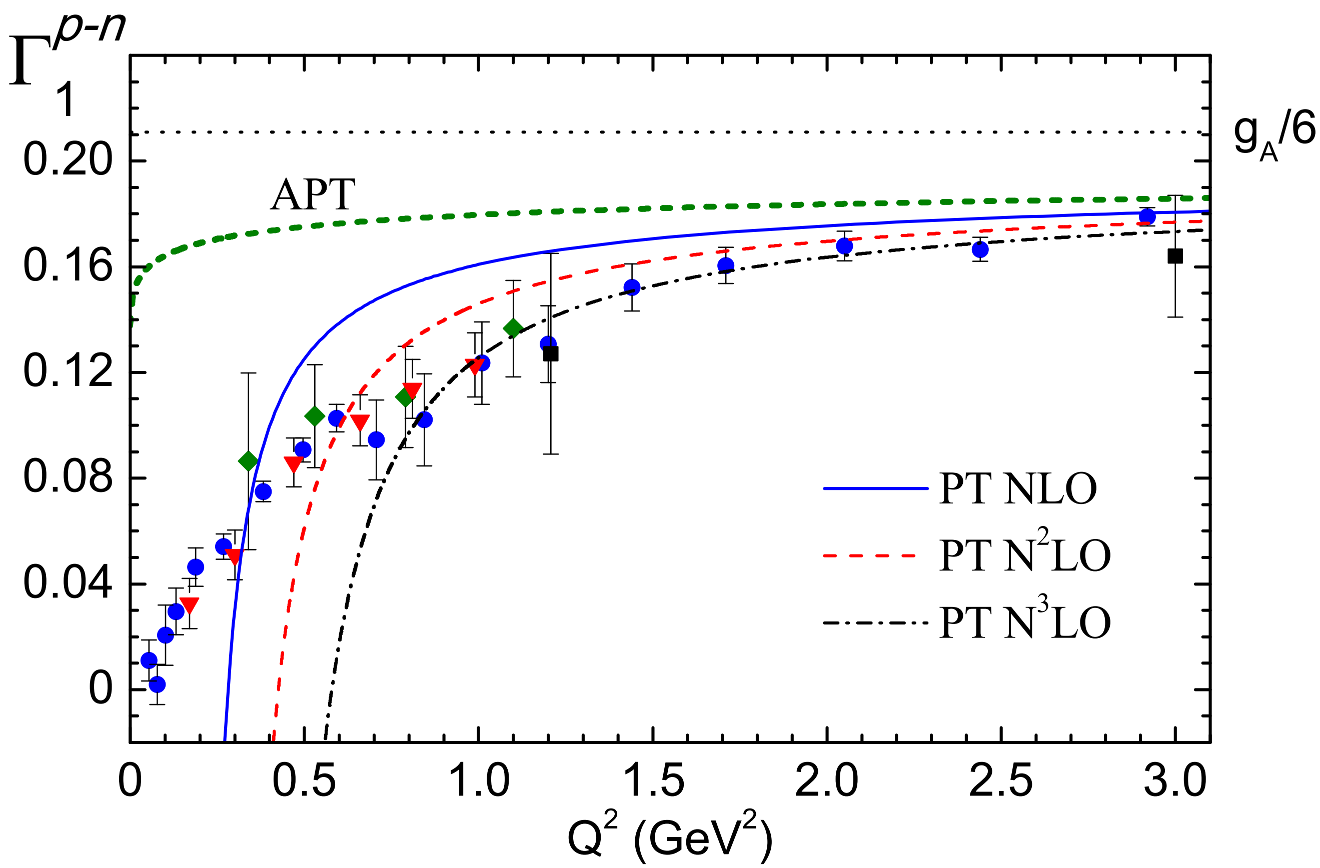

Figure 1: Perturbative part of the BSR as a function of the momentum transfer squared Q^2 in different orders in both the APT and standard PT approaches against the combined set of the Jefferson Lab (taken from V.L. Khandramai, R.S. Pasechnik, D.V. Shirkov, O.P. Solovtsova, O.V. Teryaev, *Four-loop QCD analysis of the Bjorken sum rule vs data*, Phys.Lett.B706:340-344,2012).

Higher twist contribution to the Borken (polarized) SR

$$\Gamma_1^{p-n}(Q^2) = \frac{|g_A|}{6}\left[1 - \Delta_{\rm Bj}^{\rm PT}(Q^2)\right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}} \,,$$

Recent analysis of μ_4 from exp. data + PT
/Khandramai, Solovtsova, Teryaev, arXiv:1302.3952v1 (2013)/ has demonstrated a huge sensitivity to the higher order corrections:

B. Higher twist contribution

We expand our consideration, including the HT part which is presented in expression (1). In Table I we show our results for values of the coefficient μ_4 (the errors are statistical only) fitted to the low Q^2 data [10, 11] in different PT orders. One can see that μ_4-values extracted changes rather strongly between different PT orders. The absolute value of μ_4 decreases with the order of PT and just at N^{3}LO becomes compatible to zero. It can be interpreted as a manifestation of duality between higher orders and HT (see Ref. [21]). Note that a value extracted in the

TABLE I: Results of μ_4-extraction from the data on the Bjorken sum rule in different PT orders.

PT order	LO	NLO	N^{2}LO	N^{3}LO
μ_4, GeV^2	-0.037 ± 0.003	-0.025 ± 0.004	-0.012 ± 0.006	0.005 ± 0.008

leading-order (LO) is consistent with a value $\mu_4 = -0.047 \pm 0.025$ GeV^2 presented in Ref. [22] as well as with a value $\mu_4 = -0.028 \pm 0.019$ GeV^2 obtained from the next-to-leading-order (NLO) fit based on the x-dependent structure functions data [23] .

The Ellis-Jaffe sum-rule after RG-improvement:

$$\frac{1}{2}\int_0^1 dx g_1^{p+n}(x,Q^2) \quad = \quad C^{\rm NS}(0,a_s(Q^2))\frac{1}{36}a_8 + \hat{C}^{\rm S}(0,a_s(Q^2))\frac{1}{9}\hat{a}_0$$

$$\hat{a}_0 = \exp\left(-\int^{a_s(\mu^2)} da'_s \frac{\gamma^s(a'_s)}{\beta(a'_s)}\right) a_0(\mu^2)$$

We have analytically computed (with a dedicated FORM program – BAICER (see below)) $C^S_{\overline{MS}}$ at $\boldsymbol{\mathcal{O}(\alpha_s{}^4)}$ (4 loops) (about 10^4 diagrams)

γ^s at $\boldsymbol{\mathcal{O}(\alpha_s{}^5)}$ (5 loops!) / about 10^5 daigrams; more complicated: an involved IR tricks are necessary to express the pole part of the 5-loop diagrams in terms of 4-loop massless propagators doable with BAICER/

The results were reported at previous LL 2 years ago. Unfortunately, later we have found that in one of auxiliary files the ϵ-expansion had not been done sufficiently deep. The problem has affected only the anomalous dimension. Next slide presents the corrected result.

Our final results (in numerical form and for $n_f = 3$)

$$C^S_{\overline{MS}}(0, a_s(Q)) = 1. - 2.333\, a_s + 0.14023\, a_s^2 + 4.79185\, a_s^3 - 29.791\, a_s^4$$

$$\gamma^S = -4.5\, a_s^2 - 16.896\, a_s^3 + 1.375\, a_s^4 - \frac{\cancel{49.01}}{42.54}\, a_s^5$$

And, finally,

$$\hat{C}^S(0, a_s(Q)) = 1. - 0.3333\, a_s - 0.5496\, a_s^2 - 4.4473\, a_s^3 - \frac{\cancel{36.814}}{35.840}\, a_s^4$$

cmp. to the non-singlet case:

$$C^{\rm ns}(0, a_s(Q))|_{n_f=3} = 1 - a_s - 3.583\, a_s^2 - 20.215\, a_s^3 - 175.75\, a_s^4$$

The 3% shift of the coefficient of a_s^4 is not numericaly relevant for phenomenological analysis

Phenomenological Implications

COMPASS Collaboration (2007) have measured the *deuteron* structure function g_1:

$$\Gamma_1^N \equiv \int_0^1 \frac{dx}{2}(g_1^p + g_1^n)(x, Q^2 = 3\,\mathrm{GeV}^2) = 0.050 + \pm\, 0.007$$

Theory gives (without higher twists!):

$$\Gamma_1^N = \frac{1}{36} C^{NS}(Q^2 = 3\,\mathrm{GeV}^2)\, a_8 + \frac{1}{9}\hat{C}^S(Q^2 = 3\,\mathrm{GeV}^2)\, \hat{a}_0$$

Assuming $a_8 = 0.585 \pm 0.025$ /from hyperon β decay/, we find:

$$\mathrm{LO} : \hat{a}_0 = 0.332 \pm\ 0.03(\mathrm{stat.}) \pm\ 0.02(\mathrm{syst})$$

$$\mathrm{NLO} : \hat{a}_0 = 0.340 \pm\ 0.03(\mathrm{stat.}) \pm\ 0.02(\mathrm{syst})$$

$$\mathrm{NNLO} : \hat{a}_0 = 0.345 \pm\ 0.03(\mathrm{stat.}) \pm\ 0.02(\mathrm{syst})$$

$$\mathrm{N^3LO} : \hat{a}_0 = 0.350 \pm\ 0.03(\mathrm{stat.}) \pm\ 0.02(\mathrm{syst})$$

which could be compared with the recent lattice result
(QCDSF collaboration, arXiv:1112.3354 , December 2011):

$$\hat{a}_0 = 0.376 + 0.08$$

Our tool-box

- we compute CF's of OPE with the method of projectors (cancellation of IR singularities of on-shell scattering amplitudes against UV singularities of the relevant composite operators) $\Longrightarrow$ 4-loop CF's in terms of 4-loop massless propagators
- the Baikov's way of doing reduction of resulting millions of 4-loop massless propagators with the help of $1/D$ expansion of the corresponding coefficient functions in front of masters (analytically known from **two!** independent calculations /K.Ch, P.Baikov (2010), R. Lee, V. Smirnov (2012)/).
- automatic generation of Feynman diagrams with QGRAF /Nogueira (1993)/
- the FORM program BAICER which implements $1/D$ expansion
- last but not the least: **the very FORM in two versions: ParFORM and T-FORM:**

  M. Tentyukov et al. "ParFORM: Parallel Version of the Symbolic Manipulation Program", PoS ACAT2010 (2010) 072

  M. Tentyukov, H. M. Staudenmaier, and J. A. M. Vermaseren. "ParFORM: Recent development". *Nucl. Instrum. Meth.*, A559:224–228, 2006.

  M. Tentyukov and J. A. M. Vermaseren. "The multithreaded version of FORM", hep-ph/0702279"

Conclusions

- The result for Bjorken SR for unpolarized scattering is available at 4 loops ($\mathcal{O}({\alpha_s}^4)$).

- The 4-loop result for the Ellis-Jaffe sum rule has been corrected. Numerically, the changes are quite small.

- the available higher order (up to and including α_s^4) results for various 1-scale QCD quantites display striking comon features (geometrical growth of coeffiients of sign-non-alternating α_s series)

- taken at face value higher orders tend to improve agreement *pure* PT predictions with experimental data even at as small Q^2 as 2–3 GeV^2. As a result the estimations of higher twist contributions to DIS sum rules depend strongly on the number of PT terms taken into account