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Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations

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joint work with A. Behring, J. Blümlein, A. De Freitas (DESY, Zeuthen)

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Recent symbolic summation methods

A challenging diagram

A new method for coupled systems

A general tactic

Feynman integrals

A general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

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↓ non-trivial transformations (DESY)

multiple sums

↓ symbolic summation

compact expression in terms of special functions

A general tactic

Feynman integrals

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compact expression in terms of special functions

Tactic 1: Expand the summand and simplify

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$$\text{GIVEN } F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\times \\ \underbrace{\times B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{f(N,k)}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Tactic 1: Expand and simplify

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FIND the first coefficients of the ϵ-expansion

$$F(N)\stackrel{?}{=}\varepsilon^{-3}F_{-3}(N)+\varepsilon^{-2}F_{-2}(N)+\varepsilon^{-1}F_{-1}(N)+\dots$$

Tactic 1: Expand and simplify

$$\text{GIVEN } F(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \Big(-2 - \frac{3\varepsilon}{2} \Big)! \times \underbrace{\times B\Big(2+k, \frac{\varepsilon}{2}\Big) B(-\varepsilon+k, -\varepsilon) B\Big(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\Big) \binom{N}{k}}_{f(N,k)}$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(N,k) = \quad f_{-3}(N,k)\ \varepsilon^{-3} + \quad f_{-2}(N,k)\ \varepsilon^{-2} + \quad f_{-1}(N,k)\ \varepsilon^{-1} +$$

Tactic 1: Expand and simplify

$$\text{GIVEN } F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!\times$$
$$\times\underbrace{B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}}_{f(N,k)}$$

Step 2: Simplify the sums in

$$\sum_{k=1}^{N}f(N,k)=\Big(\sum_{k=1}^{\infty}f_{-3}(N,k)\Big)\varepsilon^{-3}+\Big(\sum_{k=1}^{\infty}f_{-2}(N,k)\Big)\varepsilon^{-2}+\Big(\sum_{k=1}^{\infty}f_{-1}(N,k)\Big)\varepsilon^{-1}+$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

where

$$S_a(N)=\sum_{i=1}^{N}\frac{\text{sign}(a)^i}{i^a}\text{ and }\zeta_a=\sum_{i=1}^{\infty}\frac{1}{i^a}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$\downarrow$ (summation package `Sigma.m`)

$$\begin{aligned}&\big(16N^3+144N^2+413N+384\big)(N+1)^2F_{-1}(N)\\ &\quad-(N+2)(2N+5)\big(16N^3+112N^2+221N+113\big)F_{-1}(N+1)\\ &\quad+(N+3)^2\big(16N^3+96N^2+173N+99\big)F_{-1}(N+2)\\ &\quad=\frac{1}{2}\big(4N^2+21N+29\big)\zeta_2+\frac{-64N^5-500N^4-1133N^3+203N^2+3516N+3090}{3(N+2)(N+3)}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

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$$\begin{aligned}&\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}\\&+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}\\&+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\,|\,c_1,c_2\in\mathbb{Q}\}\end{aligned}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$\bigcap$$

$$\Big\{c_1\frac{1-4N}{N+1}+c_2\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}\Big|c_1,c_2\in\mathbb{Q}\Big\}$$

Simplify

$$F_{-1}(N)=\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\Big(\frac{(2+3k)\big(-2+3k+7k^2+3k^3\big)}{3k^2(1+k)^3}+\frac{2S_2(k)}{1+k}+\frac{\zeta_2}{2(1+k)}\Big)$$

$$||$$

$$\Big(\frac{1}{12}-\frac{1}{8}\zeta_2\Big)\frac{1-4N}{N+1}+1\,\frac{-14N-13}{(N+1)^2}$$
$$+\frac{(4N-1)S_1(N)}{N+1}+\frac{(1-4N)S_1(N)^2}{6(N+1)}+\frac{(14N+13)S_1(N)}{3(N+1)^2}$$
$$+\frac{175N^2+334N+155}{12(N+1)^3}+\frac{(1-4N)S_2(N)}{6(N+1)}+\frac{\zeta_2}{8(N+1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{n} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums

(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND all solutions expressible by indefinite nested products/sums
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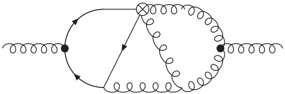
3. Find a "closed form"

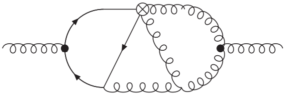
F(N)=combined solutions in terms of indefinite nested sums.

Consider a massive 3–loop ladder graph (Ablinger, Blümlein,Hasselhuhn,Klein, CS,Wissbrock, 2012)



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Consider a massive 3–loop ladder graph (Ablinger, Blümlein,Hasselhuhn,Klein, CS,Wissbrock, 2012)



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$
$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$\boxed{F_0(N)}=$ (using Sigma.m, EvaluateMultiSums.m and J. Ablinger's HarmonicSums.m package)

$$\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)S_2(N)+\Big(\frac{29}{3}-(-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N)+\Big(\frac{3}{4}+(-1)^N\Big)S_2(N)^2$$
$$-2(-1)^NS_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)S_2(N)+S_{-2}(N)\Big(10S_1(N)^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)S_3(N)+\Big(\frac{19}{2}-2(-1)^N\Big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^NS_{-2}(N)\right)\zeta_2$$

A general tactic

Feynman integrals

$\downarrow$ non-trivial transformations (DESY)

multiple sums

$\downarrow$ symbolic summation

compact expression in terms of special functions

Tactic 2: Expand a recurrence in ε

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\left(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\right)F(N+1)$$
$$-(2\varepsilon-N-1)(\varepsilon+2N+6)F(N+2)=0\varepsilon^{-3}-\frac{16}{3}\varepsilon^{-2}+\frac{40}{3}\varepsilon^{-1}-\left(2\zeta_2-\frac{68}{3}\right)\varepsilon^0+\dots$$

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k}$$

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$$F(1)=\frac{2}{3}\varepsilon^{-3}-\frac{11}{6}\varepsilon^{-2}+\frac{\zeta_2}{4}+\frac{79}{24}\varepsilon^{-1}+\dots$$
$$F(2)=\frac{8}{9}\varepsilon^{-3}-\frac{73}{27}\varepsilon^{-2}+\frac{\zeta_2}{3}+\frac{1415}{324}\varepsilon^{-1}+\dots$$

Ansatz (for power series)

$$a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]$$
$$+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]$$
$$+$$
$$\vdots$$
$$+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]$$
$$=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

REC solver: Using the initial values $F_0(1),F_0(2),\dots$ determine $F_0(N)$ in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
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$$

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$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}$$

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&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
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+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_0(N+d)+F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_d(0,N)F_0(N+d)=h_0(N)}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad =h'_0(N)+h'_1(N)\varepsilon+h'_2(N)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,N)\Big[F_1(N+d)\varepsilon+F_2(N+d)\varepsilon^2+\dots\Big]\\
&\qquad=\underbrace{h'_0(N)}_{=0}+h'_1(N)\varepsilon+h'_2(N)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)+F_2(N)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_1(N+1)+F_2(N+1)\varepsilon+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[F_1(N+d)+F_2(N+d)\varepsilon+\dots\Big]=h_1'(N)+h_2'(N)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

[Blümlein, Klein, CS, Stan, 2012; arXiv:1011.2656v2]

$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\Big(-2-\frac{3\varepsilon}{2}\Big)!B\Big(2+k,\frac{\varepsilon}{2}\Big)B(-\varepsilon+k,-\varepsilon)B\Big(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\Big)\binom{N}{k}$$

$\downarrow$ (summation package `Sigma.m`)

$$2(N+1)^2F(N)+\big(3\varepsilon^2+3\varepsilon N+9\varepsilon-4N^2-12N-8\big)F(N+1)$$
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$$F(N)=\sum_{k=1}^{N}(-1)^k e^{-\frac{3\varepsilon\gamma}{2}}\left(-2-\frac{3\varepsilon}{2}\right)!B\left(2+k,\frac{\varepsilon}{2}\right)B(-\varepsilon+k,-\varepsilon)B\left(1-\frac{\varepsilon}{2}+k,1+\frac{\varepsilon}{2}\right)\binom{N}{k}$$

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$$F(N)=\frac{4N}{3(N+1)}\varepsilon^{-3}-\left(\frac{2(2N+1)}{3(N+1)}S_1(N)+\frac{2N(2N+3)}{3(N+1)^2}\right)\varepsilon^{-2}$$
$$\left(\frac{(1-4N)}{6(N+1)}S_1(N)^2-\frac{N\left(N^2-2\right)}{3(N+1)^3}+\frac{(3N+2)(4N+5)}{3(N+1)^2}S_1(N)+\frac{(1-4N)}{6(N+1)}S_2(N)+\frac{N\zeta_2}{2(N+1)}\right)\varepsilon^{-1}+\dots$$

Find a recurrence for the integral/sum

$$F(N)=\int_0^1\dots\int_0^1\Phi(\varepsilon,N,x_1,x_2,\dots,x_7)dx_1dx_2\dots dx_7$$
$$\stackrel{?}{=}F_{-3}(N)\varepsilon^{-3}+F_{-2}(N)\varepsilon^{-2}+F_{-1}(N)\varepsilon^{-1}+\dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon,N)F(N)+\dots+a_d(\varepsilon,N)F(N+d)=h(\varepsilon,N)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

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Holonomic/difference field Approach
(Mark Round)

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ε-recurrence solver

multivariate
Almquist/Zeilberger
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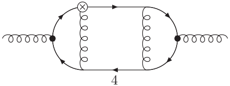
A challenging diagram and

an algorithm for coupled systems

A challenging diagram (ladder graph with 6 massive fermion lines)

$D_4(N) =$ 

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Strategies:

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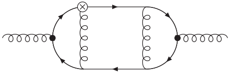
$D_4(N) =$ 

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[Ablinger, Blümlein, Raab, Schneider, Wissbrock, 2014; arXiv:1403.1137 [hep-ph]]

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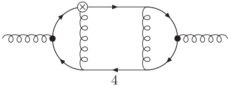
$D_4(N) =$ 

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- New approach: for the complete diagram

Consider the power series of $D_4(N)$:

$$D_4(N) \longrightarrow \hat{D}_4(x) = \sum_{N=0}^{\infty} D_4(N) x^N$$

(holonomic closure properties)

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IBP (extension of REDUZE_2, A.v. Manteuffel) gives

$$\underbrace{\sum_{N=0}^{\infty} D_4(N) x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5 x^5 - 14325922\varepsilon^5 x^4 \cdots + 1524096x^2 - 653184x\right)}{23328\varepsilon^2(x-1)x^5}} \hat{B}_1(x)$$

$$+ \square \hat{B}_2(x) + \cdots + \square \hat{B}_{52}(x) \ldots$$

$\hat{B}_1(x), \ldots, \hat{B}_{52}(x)$ can be handled with sophisticated Mellin-Barnes techniques (DESY colleagues) and symbolic summation (see first slides).

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$\hat{B}_1(x), \dots, \hat{B}_{52}(x)$ can be handled with sophisticated Mellin-Barnes techniques (DESY colleagues) and symbolic summation (see first slides).

E.g.,

$$\hat{B}_1(x) = \sum_{N=0}^{\infty} B_1(N) x^N$$

with

$$B_1(N) = \sum_{k=1}^{N} (-1)^k e^{-\frac{3\varepsilon\gamma}{2}} \big(-2 - \frac{3\varepsilon}{2}\big)! B\big(2+k, \frac{\varepsilon}{2}\big) B(-\varepsilon+k, -\varepsilon) B\big(1 - \frac{\varepsilon}{2} + k, 1 + \frac{\varepsilon}{2}\big) \binom{N}{k}$$

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$$= \frac{4N}{3(N+1)}\varepsilon^{-3} - \left(\frac{2(2N+1)}{3(N+1)}S_1(N) + \frac{2N(2N+3)}{3(N+1)^2}\right)\varepsilon^{-2} + \dots$$

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$$+ \square \hat{B}_2(x) + \cdots + \square \hat{B}_{52}(x)$$

$$+ \boxed{-\frac{\big(122\varepsilon^4 x^3 - 2647\varepsilon^4 x^2 + \cdots - 304\varepsilon + 24x^3 - 24x\big)}{4\varepsilon x^4}} \hat{I}_1(x)$$

$$+ \boxed{\frac{\big(589\varepsilon^5 x^3 - 20123\varepsilon^5 x^2 + \cdots - 896\varepsilon + 96x^3 - 96x\big)}{16\varepsilon^2 x^4}} \hat{I}_2(x)$$

$$+ \boxed{\frac{\big(589\varepsilon^5 x^3 - 21509\varepsilon^5 x^2 + \cdots - 1152\varepsilon + 96x^3 - 96x\big)}{16\varepsilon^2 x^4}} \hat{I}_3(x)$$

$$+ \square \hat{I}_4(x) + \cdots + \square \hat{I}_{15}(x)$$

However, $\hat{I}_1(x), \ldots, \hat{I}_{15}(x)$ are hard to handle. Luckily...

... there are differential relations among the integrals. E.g.,

$$D_x\hat{I}_1(x) = -\frac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \frac{2}{(x-1)x}\hat{I}_2(x) + \frac{1}{(x-1)x}\hat{B}_1(x) + \dots$$

$$D_x\hat{I}_2(x) = \frac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x) + \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) + \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots$$

$$D_x\hat{I}_3(x) = -\frac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x) + \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x) - \frac{1}{4}\frac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots$$

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$$\begin{aligned} D_x \hat{I}_2(x) = {} & \frac{\big(3(\varepsilon+4)^2-22(\varepsilon+4)+40\big)}{4(x-1)}\hat{I}_1(x) \\ & + \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) \\ & + \frac{1}{4}\frac{\big((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\big)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

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Step 1: From a DE system to a REC system

$$
\begin{aligned}
D_x \hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \hat{I}_1(x) \\
&- \tfrac{2}{(x-1)x} \hat{I}_2(x) \\
&+ \tfrac{1}{(x-1)x} \hat{B}_1(x) + \dots
\end{aligned}
$$

Step 1: From a DE system to a REC system

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\begin{aligned}
D_x \sum_{N=0}^{\infty} I_1(N)x^N = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x} \sum_{N=0}^{\infty} I_1(N)x^N \\
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Step 1: From a DE system to a REC system

$$\sum_{N=1}^{\infty} I_1(N) N x^{N-1} = -\frac{(-\varepsilon+x-1)}{(x-1)x} \sum_{N=0}^{\infty} I_1(N) x^N$$

$$-\frac{2}{(x-1)x} \sum_{N=0}^{\infty} I_2(N) x^N$$

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$\downarrow$ Nth coefficient

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... there are differential relations among the integrals. E.g.,

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A coupled system of difference equations

$$
\begin{aligned}
NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)&\\
=&B_1(N)+\dots
\end{aligned}
$$

$$
\begin{aligned}
2(\varepsilon+2N+2)I_2(N)-2(3\varepsilon+2N+1)I_2(N-1)&\\
+\varepsilon(3\varepsilon+2)I_1(N-1)-2(\varepsilon+1)I_3(N-1)&\\
=&(5\varepsilon+4)B_1(N)-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+\dots
\end{aligned}
$$

$$
\begin{aligned}
4(\varepsilon-N)I_3(N)-2\varepsilon(3\varepsilon+2)I_1(N)+\varepsilon(3\varepsilon+2)I_1(N-1)&\\
-2(3\varepsilon+1)I_2(N-1)+2(5\varepsilon+2)I_2(N)&\\
-2(\varepsilon-2N+1)I_3(N-1)&\\
=&-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+(5\varepsilon+4)B_1(N)+\dots
\end{aligned}
$$

A coupled system of difference equations

$$\begin{aligned} &N I_1(N-1) - (\varepsilon+N+1) I_1(N) + 2 I_2(N) \\ &\quad = + \tfrac{4(N+2)}{3(N+1)}\varepsilon^{-3} + \Big(\tfrac{2(2N+1)}{3(N+1)} S_1(N) - \tfrac{2\left(6N^2+13N+8\right)}{3(N+1)^2}\Big)\varepsilon^{-2} + \dots \end{aligned}$$

$$\begin{aligned} &2(\varepsilon+2N+2) I_2(N) - 2(3\varepsilon+2N+1) I_2(N-1) \\ &\quad + \varepsilon(3\varepsilon+2) I_1(N-1) - 2(\varepsilon+1) I_3(N-1) \\ &\quad = \tfrac{8}{3}\varepsilon^{-3} + \Big(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6\Big)\varepsilon^{-1} + \dots \end{aligned}$$

$$\begin{aligned} &4(\varepsilon-N) I_3(N) - 2\varepsilon(3\varepsilon+2) I_1(N) + \varepsilon(3\varepsilon+2) I_1(N-1) \\ &\quad - 2(3\varepsilon+1) I_2(N-1) + 2(5\varepsilon+2) I_2(N) \\ &\quad - 2(\varepsilon-2N+1) I_3(N-1) \\ &\quad = - \tfrac{8}{3}\varepsilon^{-3} - \Big(\tfrac{8}{3} S_1(N) - 4\Big)\varepsilon^{-2} \\ &\quad - \Big(\tfrac{4}{3} S_1(N)^2 - \tfrac{4(N+1)}{N} S_1(N) + \tfrac{4}{3} S_2(N) + \zeta_2 + 6\Big)\varepsilon^{-1} + \dots \end{aligned}$$

Step 2: Uncouple the system

$$
\begin{aligned}
\Box I_1(N-1) + \Box I_1(N) + \Box I_2(N)&\\
&= \Box\varepsilon^{-3} + \Box\varepsilon^{-2} + \Box\varepsilon^{-1} + \dots\\
\Box I_2(N) + \Box I_2(N-1) + \Box I_1(N-1) + \Box I_3(N-1)&\\
&= \Box\varepsilon^{-3} + \Box\varepsilon^{-1} + \dots\\
\Box I_3(N) + \Box I_1(N) + \Box I_1(N-1) + \Box I_2(N) + \Box I_3(N-1)&\\
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Step 2: Uncouple the system

$$\Box I_1(N-1)+\Box I_1(N)+\Box I_2(N) = \Box\varepsilon^{-3}+\Box\varepsilon^{-2}+\Box\varepsilon^{-1}+\dots$$

$$\Box I_2(N)+\Box I_2(N-1)+\Box I_1(N-1)+\Box I_3(N-1) = \Box\varepsilon^{-3}+\Box\varepsilon^{-1}+\dots$$

$$\Box I_3(N)+\Box I_1(N)+\Box I_1(N-1)+\Box I_2(N)+\Box I_3(N-1) = \Box\varepsilon^{-3}+\Box\varepsilon^{-2}+\Box\varepsilon^{-1}+\dots$$

$\downarrow$ (uncoupling algorithms[a], S. Gerhold's `OrseSys.m`)

$$\Box I_1(N)+\Box I_1(N+1)+\Box I_1(N+2)+\Box I_1(N+3) = \Box\varepsilon^{-3}+\Box\varepsilon^{-2}+\Box\varepsilon^{-1}+\dots$$

$I_2(N) =$ expression in $I_1(N)$

$I_3(N) =$ expression in $I_1(N)$

[a] We use Zürcher's uncoupling algorithm (1994)

More precisely, we get:

$$
\begin{aligned}
&-2(N+1)(N+2)(\varepsilon+N+2)I_1(N)-(N+2)\big(2\varepsilon^2-5\varepsilon N-7\varepsilon-6N^2-28N-32\big)\\
&+\big(\varepsilon^3+4\varepsilon^2N+14\varepsilon^2-4\varepsilon N^2-13\varepsilon N-3\varepsilon-6N^3-50N^2-136N-120\big)I_1(N+2)\\
-&(\varepsilon-N-2)(\varepsilon+N+4)(\varepsilon+2N+8)I_1(N+3)\\
&=-\frac{4(N+2)}{3(N+3)}\varepsilon^{-3}+\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\varepsilon^{-2}+\dots
\end{aligned}
$$

Step 3: Solve the scalar recurrence

$$
\begin{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
I_1(1) &= \tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+\big(\tfrac{15\zeta_2}{8}+\tfrac{1223}{48}\big)\varepsilon^{-1}+\dots\\
I_1(2) &= \tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+\big(\tfrac{65\zeta_2}{36}+\tfrac{46379}{1944}\big)\varepsilon^{-1}+\dots\\
I_1(3) &= \tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+\big(\tfrac{169\zeta_2}{96}+\tfrac{470071}{20736}\big)\varepsilon^{-1}+\dots
\end{aligned}
$$

using, e.g., an extension of `MATAD` (M. Steinhauser); see also A. De Freitas' talk

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using, e.g., an extension of MATAD (M. Steinhauser); see also A. De Freitas' talk

↓ (Sigma.m's recurrence solver, see first slides)

$$
\begin{aligned}
I_1(N)=&\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1(N)}{3(N+1)}\Big)\varepsilon^{-3}\\
&-\Big(\frac{2\big(20N^3+58N^2+57N+22\big)}{3(N+1)^3}+\frac{S_1(N)^2}{N+1}+\frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2}-\frac{S_2(N)}{N+1}\Big)\varepsilon^{-2}+\dots
\end{aligned}
$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$

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Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$, i.e.,

$$I_2(N) = \square I_1(N) + \square I_1(N+1) + \square I_1(N+2)$$
$$- \frac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\frac{6N^3+25N^2+33N+15}{3(N+1)^2(N+2)} + \frac{(-2N-1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots$$

$$I_3(N) = \square I_1(N) + \square I_1(N+1) + \square I_1(N+2)$$
$$+ \frac{2(N+2)}{3(N+1)}\varepsilon^{-3} + \left(\frac{-2N^3-3N^2+3N+3}{3(N+1)^2(N+2)} + \frac{(2N+1)}{3(N+1)}S_1(N)\right)\varepsilon^{-2} + \dots$$

Step 4: Compute $I_2(N)$ and $I_3(N)$:

Recall: by uncoupling we expressed $I_2(N)$ and $I_3(N)$ by $I_1(N)$, i.e.,

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This yields

$$I_2(N) = \frac{4}{3\varepsilon^3} - \frac{2}{\varepsilon^2} + \left(-\tfrac{1}{3}S_1(N)^2 + \tfrac{2}{3}S_1(N) - \tfrac{1}{3}S_2(N) + \tfrac{5N+7}{3(N+1)} + \tfrac{\zeta_2}{2}\right)\varepsilon^{-1} + \dots$$

$$I_3(N) = \tfrac{8}{3\varepsilon^3} + \left(\tfrac{4(N+2)}{3(N+1)}S_1(N) - \tfrac{4\left(4N^2+7N+2\right)}{3(N+1)^2}\right)\varepsilon^{-2}$$
$$+ \left(-\tfrac{2\left(4N^2+11N+10\right)}{3(N+1)^2}S_1(N) + \tfrac{2\left(12N^3+32N^2+25N+2\right)}{3(N+1)^3}\right.$$
$$\left. + \tfrac{(N-2)}{3(N+1)}S_1(N)^2 + \tfrac{(N-2)}{3(N+1)}S_2(N) + \zeta_2\right)\varepsilon^{-1} + \dots$$

Compute the remaining integrals

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5x^5-14325922\varepsilon^5x^4\cdots+1524096x^2-653184x\right)}{23328\varepsilon^2(x-1)x^5}}\hat{B}_1(x)$$

$$+\Box\hat{B}_2(x)+\cdots+\Box\hat{B}_{52}(x)$$

$$+\boxed{-\frac{\left(122\varepsilon^4x^3-2647\varepsilon^4x^2+\cdots-304\varepsilon+24x^3-24x\right)}{4\varepsilon x^4}}\hat{I}_1(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-20123\varepsilon^5x^2+\cdots-896\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_2(x)$$

$$+\boxed{\frac{\left(589\varepsilon^5x^3-21509\varepsilon^5x^2+\cdots-1152\varepsilon+96x^3-96x\right)}{16\varepsilon^2x^4}}\hat{I}_3(x)$$

$$+\Box\hat{I}_4(x)+\cdots+\Box\hat{I}_{15}(x)$$

Analogously, all $\hat{I}_j(x)=\sum_{N=0}^{\infty} I_j(N)x^N$, $j=1,\dots,15$ can be computed.

Final step: Insert all subresults

$$\underbrace{\sum_{N=0}^{\infty} D_4(N)x^N}_{\hat{D}_4(x)} = \boxed{\frac{\left(1545842\varepsilon^5 x^5 - 14325922\varepsilon^5 x^4 \cdots + 1524096x^2 - 653184x\right)}{23328\varepsilon^2(x-1)x^5}} \hat{B}_1(x)$$

$$+ \square \hat{B}_2(x) + \cdots + \square \hat{B}_{52}(x)$$

$$+ \boxed{-\frac{\left(122\varepsilon^4 x^3 - 2647\varepsilon^4 x^2 + \cdots - 304\varepsilon + 24x^3 - 24x\right)}{4\varepsilon x^4}} \hat{I}_1(x)$$

$$+ \boxed{\frac{\left(589\varepsilon^5 x^3 - 20123\varepsilon^5 x^2 + \cdots - 896\varepsilon + 96x^3 - 96x\right)}{16\varepsilon^2 x^4}} \hat{I}_2(x)$$

$$+ \boxed{\frac{\left(589\varepsilon^5 x^3 - 21509\varepsilon^5 x^2 + \cdots - 1152\varepsilon + 96x^3 - 96x\right)}{16\varepsilon^2 x^4}} \hat{I}_3(x)$$

$$+ \square \hat{I}_4(x) + \cdots + \square \hat{I}_{15}(x)$$

Plugging in all expansion and extracting the N-th coefficient (using `HarmonicSums.m`, `Sigma.m`, `EvaluateMultiSum.m`, `SumProduction.m`) yield

$$I_4(N) = \left(\frac{64\left(N^2+N-1\right)}{3(N+1)(N+2)(N+3)(N+4)} - \frac{64S_1(N)}{3(N+3)(N+4)}\right)\varepsilon^{-3}$$

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$$+ \Big(\frac{4(5N+27)}{3(N+2)(N+3)(N+4)}S_1(N)^2 - \frac{4\left(3N^5+68N^4+379N^3+648N^2-98N-696\right)}{3(N+1)(N+2)^2(N+3)^2(N+4)^2}S_1(N)$$

$$+ \frac{4\left(14N^6+214N^5+1179N^4+3050N^3+4097N^2+3094N+1200\right)}{3(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{4(N+1)(4N+17)S_2(N)}{3(N+2)(N+3)(N+4)}\Big)\varepsilon^{-2}$$

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$$+\Big(\frac{-23N^2-35N-176}{9(N+2)(N+3)(N+4)(N+5)}S_1(N)^3 - \frac{2(10N^2+53N+106)}{3(N+2)(N+3)(N+4)}S_{2,1}(N) + \zeta_2\Big(\frac{8(N^2+N-1)}{(N+1)(N+2)(N+3)(N+4)}$$

$$-\frac{8}{(N+3)(N+4)}S_1(N)\Big) + \frac{-8N^3-95N^2-171N-56}{3(N+2)(N+3)(N+4)(N+5)}S_2(N)S_1(N) + \frac{2(30N^3+469N^2+1873N+2542)}{9(N+2)(N+3)(N+4)(N+5)}S_3(N)$$

$$+\frac{25N^6+213N^5+491N^4-1007N^3-7942N^2-15988N-10340}{3(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)}S_1(N)^2$$

$$+\frac{-85N^6-1469N^5-8965N^4-23889N^3-25644N^2-3724N+5780}{3(N+1)(N+2)^2(N+3)^2(N+4)^2(N+5)}S_2(N)$$

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Arising objects:

$\zeta_2, \zeta_3, (-1)^N, 2^N, S_{-3}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-2,1}(N),$
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[J.A.M. Vermaseren, 1998; J. Blümlein/S. Kurth, 1998]

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$$S_{1,1,1,1}\left(2,\tfrac{1}{2},1,1,N\right) = \sum_{k=1}^{N} \frac{2^k \sum_{i=1}^{k} \frac{2^{-i}\sum_{j=1}^{i}\frac{\sum_{r=1}^{j}\frac{1}{r}}{j}}{i}}{k}$$

[S. Moch, P. Uwer, S. Weinzierl, 2002; J. Ablinger, J. Blümlein, CS, 2013] see J. Ablinger's talk.

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- More involved massive 3-loop topologies are currently investigated.