Bottom Quark Mass from Υ Sum Rules to $\mathcal{O}(\alpha_s^3)$

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[arXiv:1401.7035 [hep-ph]]

Overview

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Motivation

Bottom quark mass $m_b^{\overline{\text{MS}}}(m_b^{\overline{\text{MS}}})$:

- Fundamental parameters of SM
- Precise determination relevant for:
  - Flavour physics
  - GUT physics
  - Collider physics
  - Higgs physics

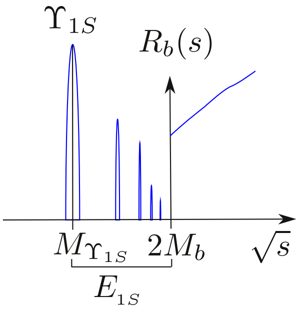
m_b

- Is a free parameter of the SM
- Its value doesn't follow from a first principle, yet
- Currently, one can only extract its value from experimental data

Extraction methods

- Lattice simulations of Υ bound states/spectrum (N.P.)
  [HPQCD'13]
- EFT predictions for Υ_{1S} bound state energy (P.T.)
  [Penin,Steinhauser'02][Beneke,Kiyo,Schuller'05]
- Relativistic Υ sum rules (P.T.)
  [Chetyrkin,Kühn,Maier,Maierhöfer,Marquard,Steinhauser,Sturm('12)]
- Non-relativistic Υ sum rules (P.T.) □
  [Beneke, Bodwin, Braaten, Brambilla, Caswell, Hoang, Kiyo, Kniehl, Kuhn, Lepage, Penin, Pineda, Pivovarov, Ruiz-Femenia, Schuller, Signer, V.A. Smirnov, Soto, Stahlhofen, Steinhauser, Vairo, Voloshin, Yelkhovsky, Yndurain, Zaitsev]

Υ_{1S} bound state energy



1. Match (p)NRQCD perturbatively to QCD $\to \mathcal{H}$

$$\mathcal{H} = \mathcal{H}_C + \dots$$
$$\mathcal{H}_C = -\frac{\partial^2}{M_b} - \alpha_s \frac{C_F}{r}\,.$$

2. Find energy Eigenvalue E_{1S} in dependence of M_b
3. Find M_b such that $2M_b + E_{1S} - M_{\Upsilon_{1S}}(exp) = 0$
4. $M_b \to m_b^{\overline{\text{MS}}}$ transition

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$$
\begin{aligned}
\mathcal{H}\psi_{1S} &= E_{1S}\psi_{1S}\,, \\
\mathcal{H}_C\psi_{1S}^C &= E_{1S}^C\psi_{1S}^C\,, \\
E_{1S} &= E_{1S}^C(1+\dots)\,, \\
E_{nS}^C &= -\frac{M_b C_F^2 \alpha_s^2}{4n^2}\,.
\end{aligned}
$$

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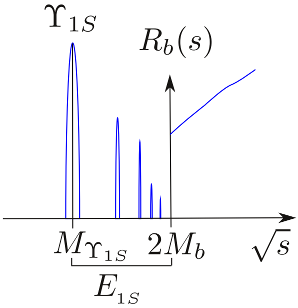

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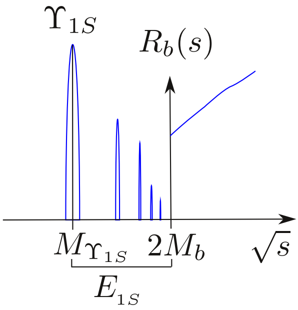

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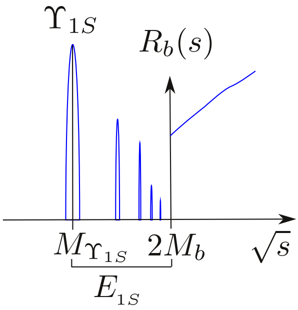

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- (+) least matching effort
- (+) simple extraction method
- (−) misses large N.P. effects $\sim \Lambda^4_{\text{QCD}}/(\alpha_s^4 M_b^4)$

Relativistic Υ Sum Rules (low moments)

Motivation

Consider more "inclusive" observables in order to reduce N.P. effects:

$$\mathcal{M}_n = (4M_b^2)^n \int_0^\infty \frac{R_b(s)ds}{s^{n+1}}$$

- Optical theorem:

$$R_b(s) = 12\pi \,\mathrm{Im}\Pi(s+i\epsilon)$$
$$(p_\mu p_\nu - g_{\mu\nu}p)\Pi(p^2) = i\int d^dx e^{ipx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle$$
$$j_\mu = \bar{b}\gamma_\mu b$$

- Analytic properties of $\Pi(s+i\epsilon)$:

$$\mathcal{M}_n = \frac{12\pi^2}{n!}(4M_b^2)^n \frac{d^n}{ds^n}\Pi(s)\Big|_{s=0}$$

Relativistic Υ Sum Rules (low moments)

1. Calculate $\Pi(s)$ around $z = \frac{s}{4m_b^2} \approx 0$ in pQCD:

$$\Pi(p^2) \sim \sum_{n \geq 0} C_n z^n$$
$$\mathcal{M}_n^{\text{th}} \sim C_n$$

2. Obtain experimental measurement for $R_b(s) \to R_b^{\text{exp}}(s)$
3. Use the analytic id. of the n-th moment to extract m_b for $n = \{1, \ldots, 4\}$

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3. Use the analytic id. of the n-th moment:

$$\underbrace{\mathcal{M}_n(R_b^{\text{ex}}(s))}_{\mathcal{M}_n^{\text{exp}}} \overset{!}{=} \underbrace{\mathcal{M}_n(C_n)}_{\mathcal{M}_n^{\text{th}}},$$

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- (+) Pure pQCD calculation for $\Pi(p^2)$
- (+) Very small N.P. contributions
- (−) $R_b^{\mathrm{exp}}(s)$ cannot be defined unambiguously (Padé approximations!)
- (−) Sensitive to experimental error dominated s regions

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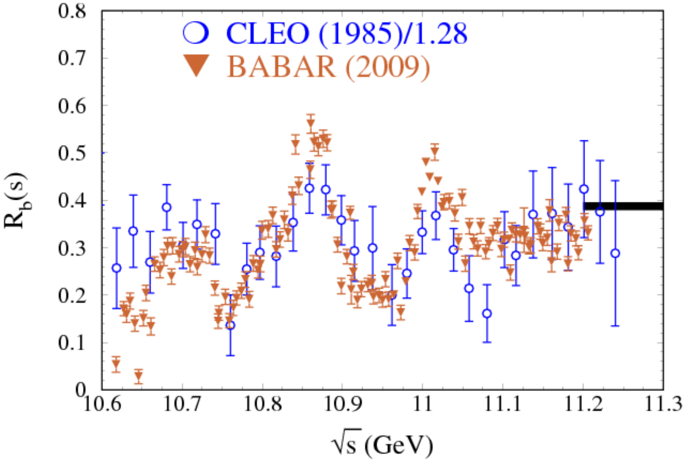

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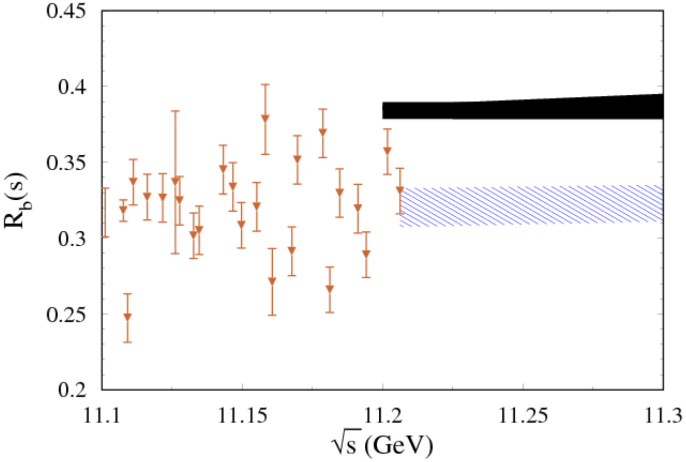

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Non-Relativistic Υ Sum Rules (high moments)

Motivation

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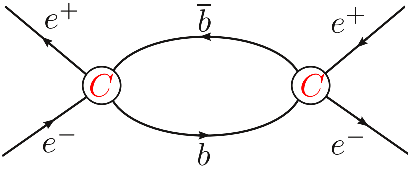
- $\Pi(s)$ expanded in pQCD (α_s) around $z = \frac{s}{4m_b^2} \approx 0$ misses relevant contributions for high n:



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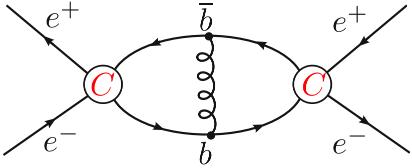

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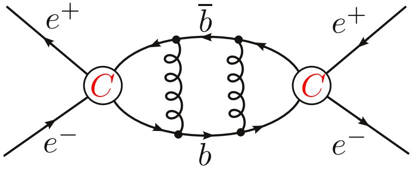

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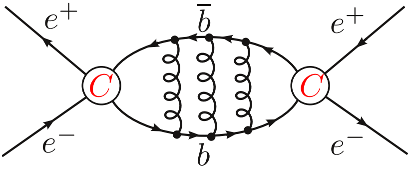

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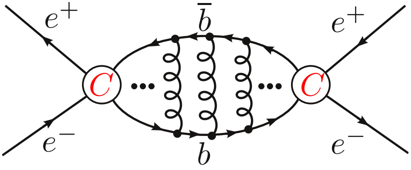

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- Choose "high" n values($6 \lesssim n \lesssim 20$)
- $\Pi(s)$ expanded in pQCD (α_s) around $z = \frac{s}{4m_b^2} \approx 0$ misses relevant contributions for high n
- Naive α_s expansion breaks down close to threshold because of the non-relativistic scaling $\frac{1}{\sqrt{n}} \sim v \sim \alpha_s$:

$$\left(\frac{\alpha_s}{v}\right)^m \sim 1$$

Resummation required. New power counting:

$$LO : \sum_{m=0}^{\infty} \left(\frac{\alpha_s}{v}\right)^m , \qquad NLO : \sum_{m=0}^{\infty} \left(\frac{\alpha_s}{v}\right)^m \{v, \alpha_s\} ,$$

$$NNLO : \sum_{m=0}^{\infty} \left(\frac{\alpha_s}{v}\right)^m \{v^2, \alpha_s v, \alpha_s^2\} , \qquad N^3LO : \sum_{m=0}^{\infty} \left(\frac{\alpha_s}{v}\right)^m \{v^3, \alpha_s v^2, \alpha_s^2 v, \alpha_s^3\}$$

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- Naive α_s expansion breaks down close to threshold because of the non-relativistic scaling $\frac{1}{\sqrt{n}} \sim v \sim \alpha_s$
- Radiative corrections on top of the Coulomb resummation implemented systematically within (p)NRQCD

$$\mathcal{H}_{\text{pNRQCD}} = -\frac{\partial^2}{M_b} - \frac{C_F \alpha_s}{r}[1 + \mathcal{O}(\alpha_s)] + \ldots$$

known including $\sim \alpha_s^3$ (a_3) [Anzai,Kiyo,Sumino'09][2xSmirnov,Steinhauser'09]

Non-Relativistic Υ Sum Rules (high moments)

Construct $\mathcal{M}_n^{\text{th}}$ from Green function

- $\Pi(s)$ ($E = \sqrt{s} - 2M_b \sim M_b v^2$)

$$\Pi(s) = \frac{N_c}{2m_q^2} \underbrace{\left(c_v - \frac{E}{m_q}\frac{d_v}{6} + \dots\right)^2}_{C^2(E)} \left(1 + \frac{E}{2m_q}\right)^{-2} G^s(0,0;E)$$

- Spectral representation of $G^s(0,0;E)$

$$G^s(0,0;E) = \sum_{n=1}^{\infty} \frac{|\psi_n(0)|^2}{E_n - E - i\epsilon} + \dots$$

- $\mathcal{M}_n^{\text{th}}$

$$\mathcal{M}_n^{\text{th}} = (4m_b^2)^n \left(12\pi^2 N_c \sum_{m=1}^{\infty} \frac{C^2(E_m)|\psi_m(0)|^2}{(2m_b + E_m)^{2n+3}} + \int_{4m_b^2}^{\infty} \frac{R(s)ds}{s^{n+1}}\right)$$

Non-Relativistic Υ Sum Rules (high moments)

Basic building blocks for $\mathcal{M}_n^{\text{th}}$

- Vector current MC

$$C(E) = 1 - 2C_F\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2, v^2)\ ☑_{N^3LO}$$ [Marquard,Piclum,Seidel,Steinhauser'14]

- Binding energy

$$E_n = E_n^C \sum_{m=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^m e_n^{(m)}\ ☑_{N^3LO}$$

[Pineda,Yndurain'98]
[Penin,V.A.Smirnov,Steinhauser'05]
[Beneke,Kiyo,Schuller'05]

- Wave function at origin

$$|\psi_n(0)|^2 = |\psi_n^C(0)|^2 \sum_{m=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^m \delta^{(m)}|\psi_n(0)|^2\ ☑_{N^3LO}$$

[Beneke,Kiyo,Schuller'5813]
[Beneke,Kiyo,Penin'7]

- Green function above threshold

$$G^s(0,0;E) = G_C^s(0,0;E) + \sum_{m=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^m \delta^{(m)}G^s\ \square_{N^3LO}$$

Non-Relativistic Υ Sum Rules (high moments)

Basic building blocks for $\mathcal{M}_n^{\text{th}}$

- Vector current MC
- Binding energy
- Wave function at origin
- Green function above threshold

$$G^s(0,0;E) = G^s_C(0,0;E) + \sum_{m=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^m \delta^{(m)} G^s \,\square_{N^3LO}$$

Ultrasoft corrections [Beneke,Kiyo'08]

N^3LO coulomb corrections [Beneke,Kiyo,Schuller'07]

Continuum contribution small (by construction)

Approximation:

$$R^{N^3LO}(s) \approx \rho \left.\frac{R^{N^3LO}}{R^{NNLO}}(s)\right|_{s=M_{\Upsilon(1S)}} R^{NNLO}(s) \qquad \frac{1}{2} \leq \rho \leq 2$$

Non-Relativistic Υ Sum Rules (high moments)

Definition of $\mathcal{M}_n^{\text{exp}}$

$$\mathcal{M}_n^{\text{exp}} = (4M_b^2)^n \frac{9\pi}{Q_b^2 \alpha^2(2M_b)} \left(\sum_m \frac{\Gamma_{\Upsilon(mS)\to l^+l^-}}{M_{\Upsilon(mS)}^{2n+1}} + \dots \right)$$

Very precise data for the first 6 Υ_{nS} resonances [PDG]:

n	1	2	3
$M_{\Upsilon(nS)}$ (GeV)	9.46030(26)	10.02326(31)	10.3552(5)
$\Gamma_{\Upsilon(nS)\to e^+e^-}$ (keV)	1.340(18)	0.612(11)	0.443(8)

n	4	5	6
$M_{\Upsilon(nS)}$ (GeV)	10.5794(12)	10.876(11)	11.019(8)
$\Gamma_{\Upsilon(nS)\to e^+e^-}$ (keV)	0.272(29)	0.31(7)	0.130(30)

Non-Relativistic Υ Sum Rules (high moments)

Extraction of M_b

$$\mathcal{M}_n^{\text{exp}}(M_b) \stackrel{!}{=} \mathcal{M}_n^{\text{th}}(M_b, \mu) \to M_b(n, \mu)$$

Non-Relativistic Υ Sum Rules (high moments)

OS $\rightarrow$ $\overline{\text{MS}}$ conversion

OS mass M_b suffers from Renormalon ambiguity (IR sensitive). Its value does not converge in P.T.

$m_b^{\overline{\text{MS}}} = \overline{m}_b$ is a short distance mass and is infrared insensitive. Better convergence in P.T.

Prescription for OS $\rightarrow$ $\overline{\text{MS}}$ mass conversion: [Hoang,Ligeti,Manohar'1999]

$$M_b^{N^m LO}(\mu) = r^{N^{m+1}LO}(\mu, \overline{m}_b(\overline{m}_b))\overline{m}_b(\overline{m}_b)$$

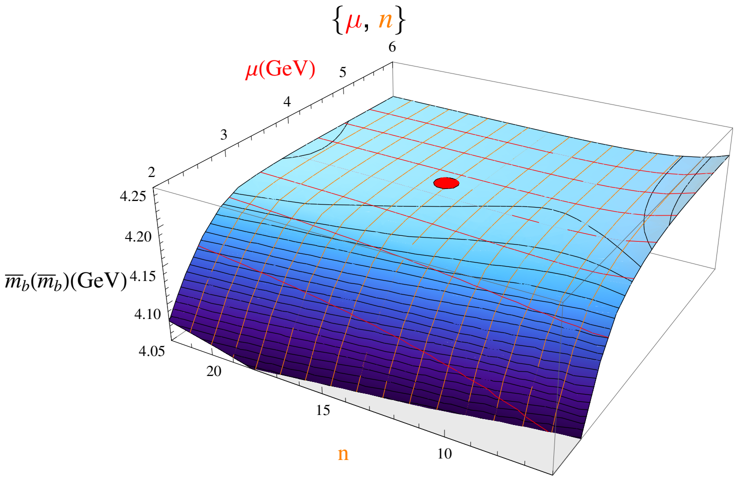
Constant piece of $r^{N^4 LO}$ is currently unknown

- $r_{\beta_0}^{N^4 LO}$: large β_0 approximation
- $r_{\text{ren}}^{N^4 LO}$: renormalon based estimate (default) [Pineda'01]

$$M_b(n, \mu) \rightarrow \overline{m}_b(\overline{m}_b)(n, \mu)$$

Non-Relativistic Υ Sum Rules (high moments)

Results



Non-Relativistic Υ Sum Rules (high moments)
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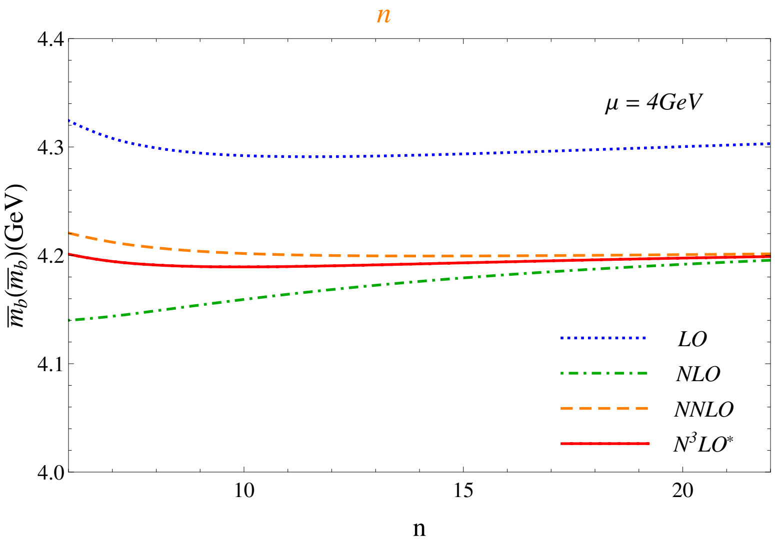

Non-Relativistic Υ Sum Rules (high moments)

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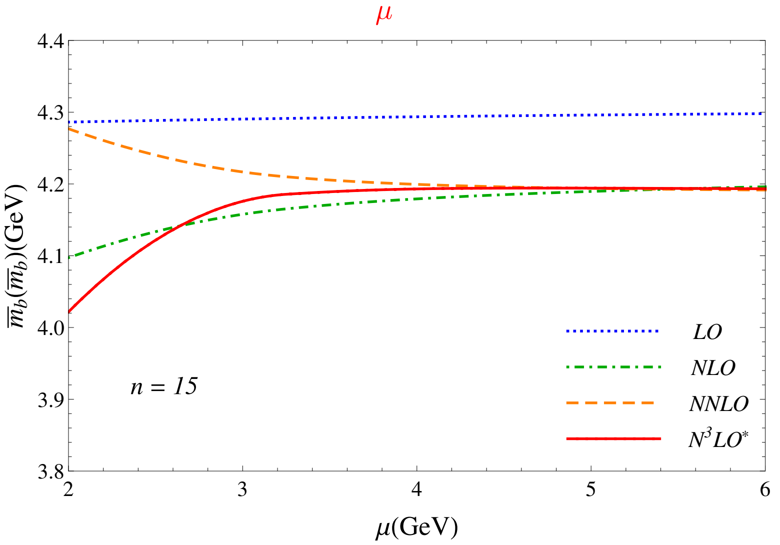


Non-Relativistic Υ Sum Rules (high moments)

P.T. Series for $\overline{m}_b(\overline{m}_b)$

$$\begin{aligned}\overline{m}_b(\overline{m}_b) =& 4.294\,(1_{LO} + 0.0262_{NLO} - 0.0038_{NNLO} + 0.0010_{N^3LO^*} + \ldots)\,GeV \\ =& 4.194 GeV\end{aligned}$$

Charm quark mass effects

⚠ All results have been obtained for $m_c = 0$.
First *NNLO* approximation [Hoang'00] applied to our values:

$$\overline{m}_b(\overline{m}_b)\Big|_{m_c\neq 0} \approx \overline{m}_b(\overline{m}_b)\Big|_{m_c=0} - (25 \pm 5) MeV$$

Non-Relativistic Υ Sum Rules (high moments)

Error Estimation in MeV

Δ_{exp}	Δ_{α_s}	Δ_{ρ}	$\Delta_{r^{(4)}}$	Δ_{n}	$\Delta_{p.t.}$	$\Delta_{n.p.}$	Δ_{m_c}
2.3	1.9	4.2	2.2	3.4	2.1	0.8	5.0

- Δ_{exp}: Coherent variation of experimental data within given errors
- Δ_{α_s}: $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ [PDG]
- Δ_{ρ}: $\frac{1}{2}\left|\overline{m}_b(\overline{m}_b, \rho = \frac{1}{2}) - \overline{m}_b(\overline{m}_b, \rho = 2)\right|$
- $\Delta_{r^{(4)}}$: $\left|\overline{m}_b(\overline{m}_b, r_{\text{ren}}^{N^4LO}) - \overline{m}_b(\overline{m}_b, r_{\beta_0}^{N^4LO})\right|$
- Δ_{n}: $\frac{1}{2}\left|\overline{m}_b(\overline{m}_b, n = 20) - \overline{m}_b(\overline{m}_b, n = 10)\right|$
- $\Delta_{p.t.}$: $\frac{1}{2}\left|\overline{m}_b^{N^3LO}(\overline{m}_b) - \overline{m}_b^{N^2LO}(\overline{m}_b)\right|$
- $\Delta_{n.p.}$: Gluon condensate estimate [Voloshin'95]
- Δ_{m_c}: Unknown contributions from $m_c \neq 0$ effects. [Hoang'00]

Non-Relativistic Υ Sum Rules (high moments)

Final result

$$\overline{m}_b(\overline{m}_b) = 4.169 \pm 0.008_{th} \pm 0.002_{\alpha_s} \pm 0.002_{exp}$$

Comparison with literature

Reference	Method	Approximation	$\overline{m}_b(\overline{m}_b)$ (GeV)
[HPQCD'13]	Lattice	$\mathcal{O}(\alpha_s^2)$	4.166 ± 0.043
[Penin:2002zv]	$\Upsilon(1S)$ mass	$\mathcal{O}(\alpha_s^3)$	4.346 ± 0.070
[Beneke:2005hg]	$\Upsilon(1S)$ mass	$\mathcal{O}(\alpha_s^3)$	4.25 ± 0.08
[Pineda:2006gx]	high moments	partial NNLL	4.190 ± 0.060
[Hoang:2012us]	high moments	partial NNLL	4.235 ± 0.055
[Chetyrkin:2010ic]	low moments	$\mathcal{O}(\alpha_s^3)$	4.163 ± 0.016
This work	high moments	$\mathcal{O}(\alpha_s^3)$	4.169 ± 0.009

Summary and Outlook

Summary

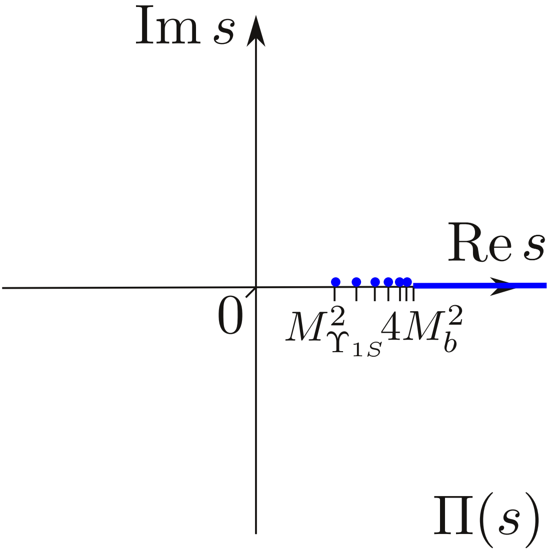
1. N^3LO^* extraction of $\overline{m}_b(\overline{m}_b)$ using n.r. Υ sum rules now available
2. Great stability improvement going from N^2LO to N^3LO^*
3. We find: $\overline{m}_b(\overline{m}_b) = 4.169 \pm 0.008_{th} \pm 0.002_{\alpha_s} \pm 0.002_{exp}$
4. Good agreement with values obtained by complementary methods

Outlook

- N^3LO^* continuum contribution to the moments
- $m_c \neq 0$ effects
- r^{N^4LO}

M. Buffer

Analytic Properties of $\Pi(s)$



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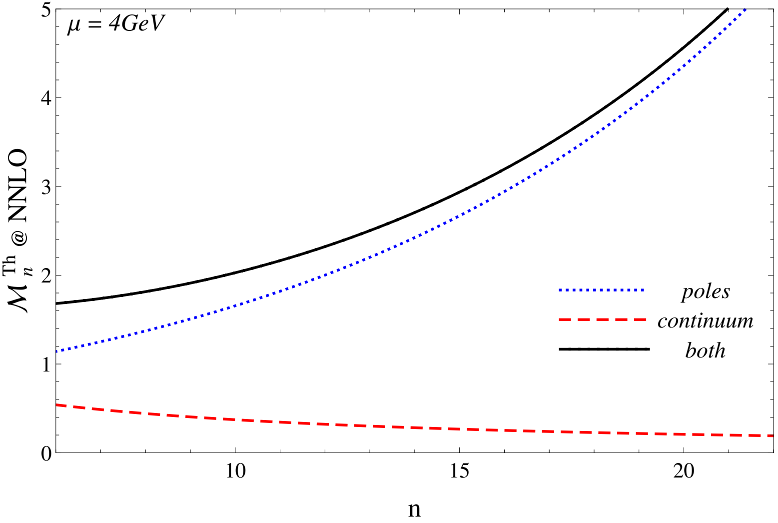


Non-Relativistic Υ Sum Rules (high moments)

N^3LO continuum contribution

Contribution from continuum is strongly suppressed for high n

Use estimate for the N^3LO continuum corrections to G:

$$Z_m = C^2(E)\left(1+\frac{E_m}{2M_b}\right)^{-2}|\psi_m(0)|^2$$

$$\left.\frac{R^{N^3LO}}{R^{NNLO}}(s)\right|_{s=M_{\Upsilon(1S)}} = \frac{Z_1^{N^3LO}}{Z_1^{NNLO}}$$

$$R^{N^3LO}(s) \approx \frac{Z_1^{N^3LO}}{Z_1^{NNLO}} R^{NNLO}(s)$$

Non-Relativistic Υ Sum Rules (high moments)

N^3LO continuum contribution

Contribution from continuum is strongly suppressed for high n

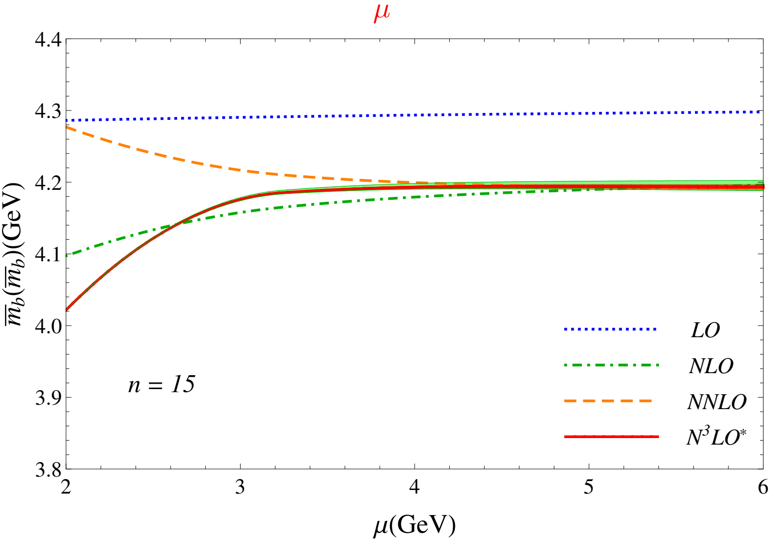
Use estimate for the N^3LO continuum corrections to G:

$$Z_m = C^2(E)\left(1+\frac{E_m}{2M_b}\right)^{-2}|\psi_m(0)|^2$$

$$\left.\frac{R^{N^3LO}}{R^{NNLO}}(s)\right|_{s=M_{\Upsilon(1S)}} = \frac{Z_1^{N^3LO}}{Z_1^{NNLO}}$$

$$R^{N^3LO}(s) \approx \rho\frac{Z_1^{N^3LO}}{Z_1^{NNLO}}R^{NNLO}(s) \qquad \frac{1}{2} \leq \rho \leq 2$$

ρ-Variation: $\frac{1}{2} \leq \rho \leq 2$



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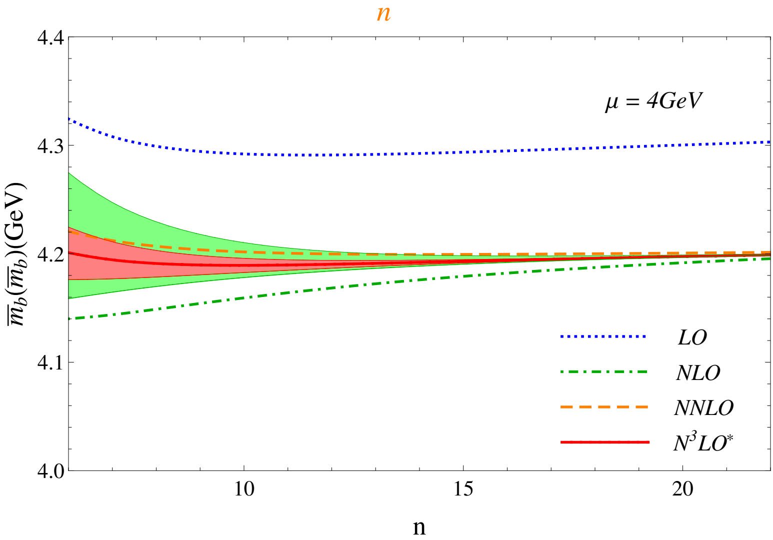
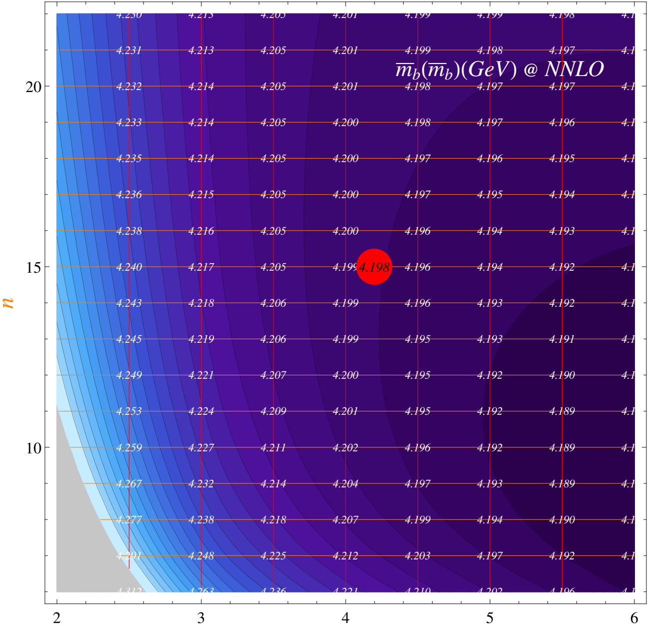


Extraction @ *NNLO*



Extraction @ N^3LO^*