Helicity amplitudes with off-shell partons

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Narodowe Centrum Badań i Rozwoju

LIDER

Outline

- high-energy factorization
- amplitudes with off-shell initial-state gluons
- amplitudes with an arbitrary number of off-shell gluons
- BCFW recursion for off-shell gluons

High–energy factorization

Collins, Ellis 1991
Catani, Ciafaloni, Hautmann 1991

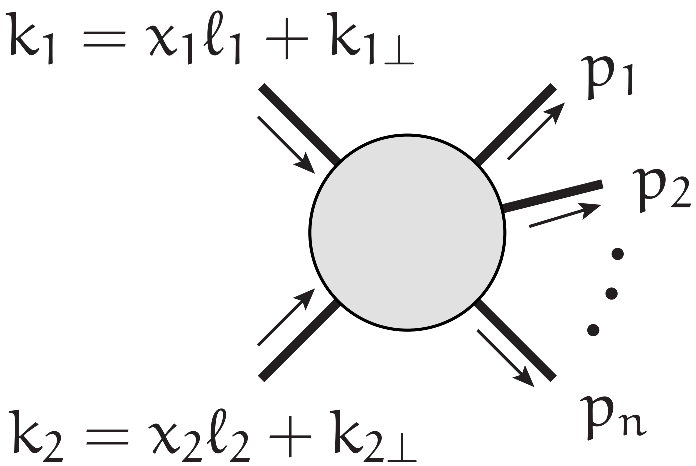
$$\sigma_{h_1,h_2\to QQ} = \int d^2k_{1\perp}\,\frac{dx_1}{x_1}\,\mathcal{F}(x_1,k_{1\perp})\,d^2k_{2\perp}\,\frac{dx_2}{x_2}\,\mathcal{F}(x_2,k_{1\perp})\,\hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s},\frac{k_{1\perp}}{m},\frac{k_{2\perp}}{m}\right)$$

- to be applied in the 3-scale regime $s \gg m^2 \gg \Lambda^2_{QCD}$
- reduces to collinear factorization for $s \gg m^2 \gg k_\perp^2$,
  but holds also for $s \gg m^2 \sim k_\perp^2$
- *unintegrated pdf* $\mathcal{F}$ may satisfy BFKL-eqn, CCFM-eqn, BK-eqn, KGBJS-eqn, ...
- typically associated with small-x physics
- relevant for forward physics, saturation physics, heavy-ion physics...
- $k_\perp$ gives a handle on the size of the proton
- allows for higher-order kinematical effects at leading order

Matrix elements

The issue:

High-energy factorization requires matrix elements for parton-level scattering process with off-shell initial states



where ℓ_1, ℓ_2 are light-like momenta associated with the scattering hadrons, and $k_{1\perp}, k_{2\perp}$ are perpendicular to both ℓ_1 and ℓ_2.

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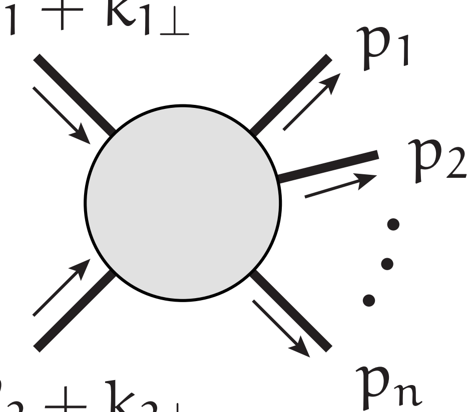
$$k_1 = x_1 \ell_1 + k_{1\perp}$$



$$k_2 = x_2 \ell_2 + k_{2\perp}$$

where ℓ_1, ℓ_2 are light-like momenta associated with the scattering hadrons, and $k_{1\perp}, k_{2\perp}$ are perpendicular to both ℓ_1 and ℓ_2.

Matrix elements, squared and summed over final-state spins, may be calculated using spin amplitudes.

Amplitudes must be gauge invariant

- must be calculable in any gauge
- must satisfy Ward identities.
- must preferably be practical.

We cannot just take a prescription to calculate on-shell matrix elements and keep initial-state momenta off-shell, because we won't have gauge invariance.

Using some kind of projectors on off-shell external gluons won't be enough.

Lipatov's effective action

Lipatov 1995, Antonov, Lipatov, Kuraev, Cherednikov 2005

Effective action in terms of quarks $\psi, \bar{\psi}$ gluons v_μ and reggeized gluons $A_\pm$.

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ind}}$$

$$\mathcal{L}_{\mathrm{QCD}} = i\bar{\psi}\not{D}\psi + \frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}^2 \qquad D_\mu = \partial_\mu + g v_\mu \qquad G_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu]$$

$$\mathcal{L}_{\mathrm{ind}} = -\mathrm{Tr}\Big\{ \frac{1}{g}\partial_+ \left[\mathcal{P}\exp\left(-\frac{g}{2}\int_{-\infty}^{x^+} v_+(y)\, dy^+ \right) \right] \cdot \partial_\sigma^2 A_-(x)$$
$$+ \frac{1}{g}\partial_- \left[\mathcal{P}\exp\left(-\frac{g}{2}\int_{-\infty}^{x^-} v_-(y)\, dy^- \right) \right] \cdot \partial_\sigma^2 A_+(x) \Big\}$$

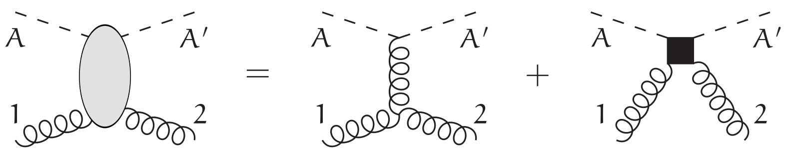
$$k_\pm = \frac{1}{E}(\ell_\mu^\pm)k^\mu \qquad (\ell^-)^2 = (\ell^+)^2 = 0 \qquad \ell^+ \cdot \ell^- = 2E^2$$

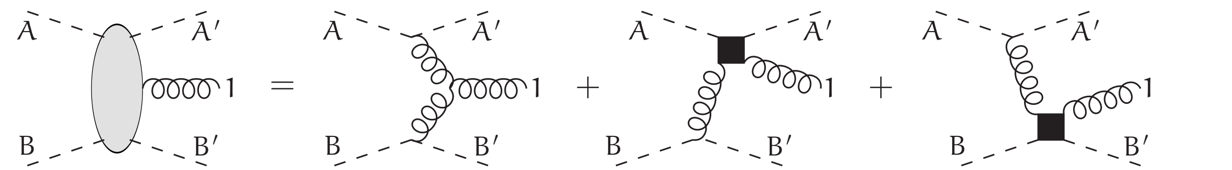
Reggeized gluon $\Longrightarrow$ gluon with momentum $x_\pm \ell^\pm + k_\perp$.

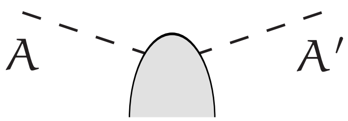
Effective action $\Longrightarrow$ vertices of arbitrary order.

Reggeon–gluon vertices

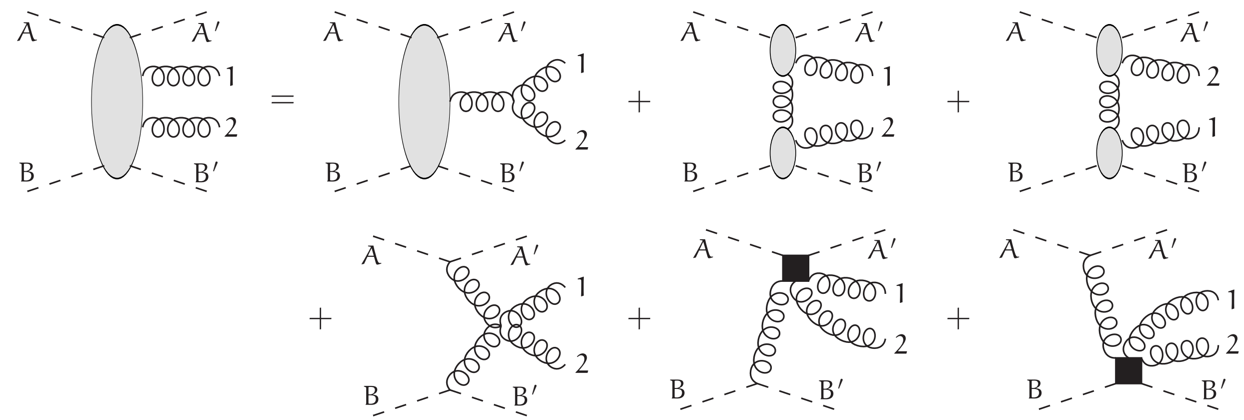
Antonov, Lipatov, Kuraev, Cherednikov
2005

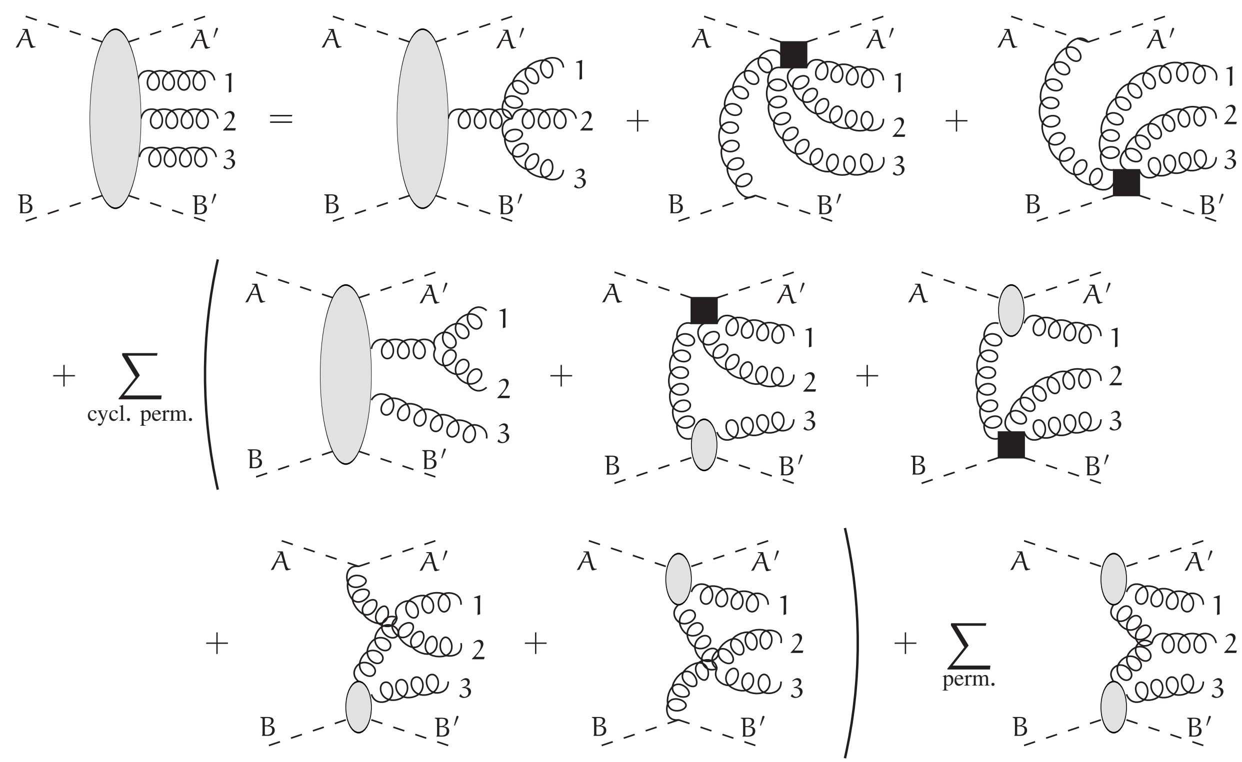






= reggeon A





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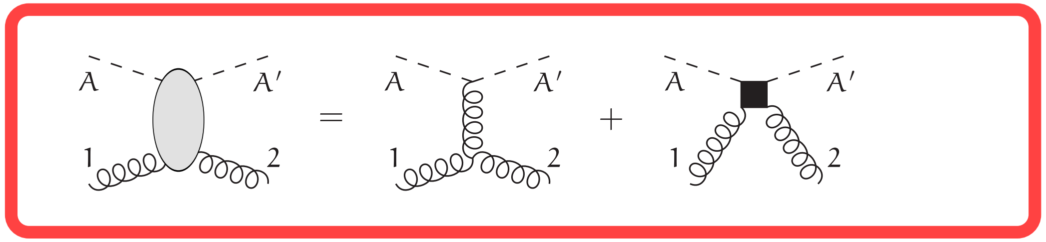
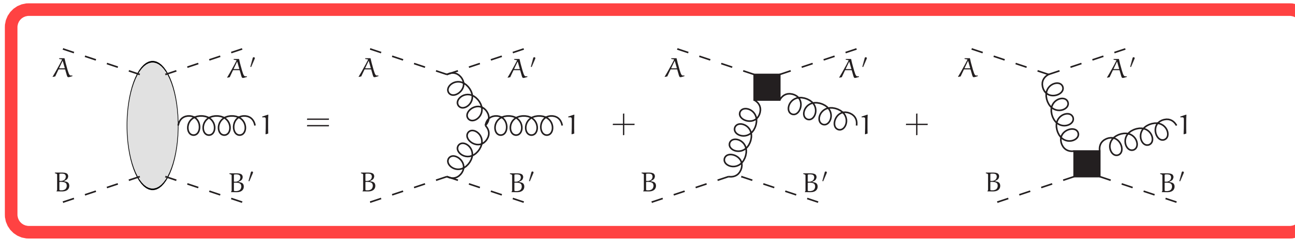
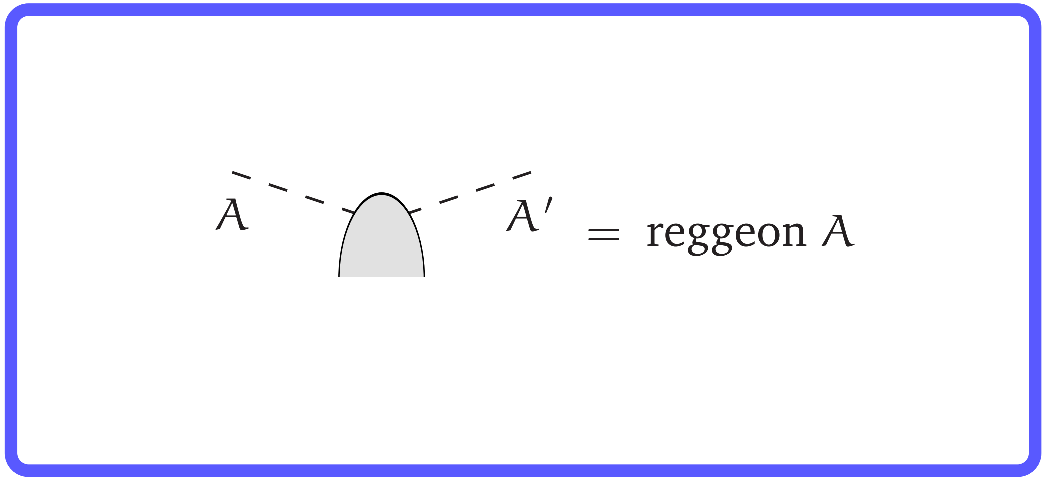
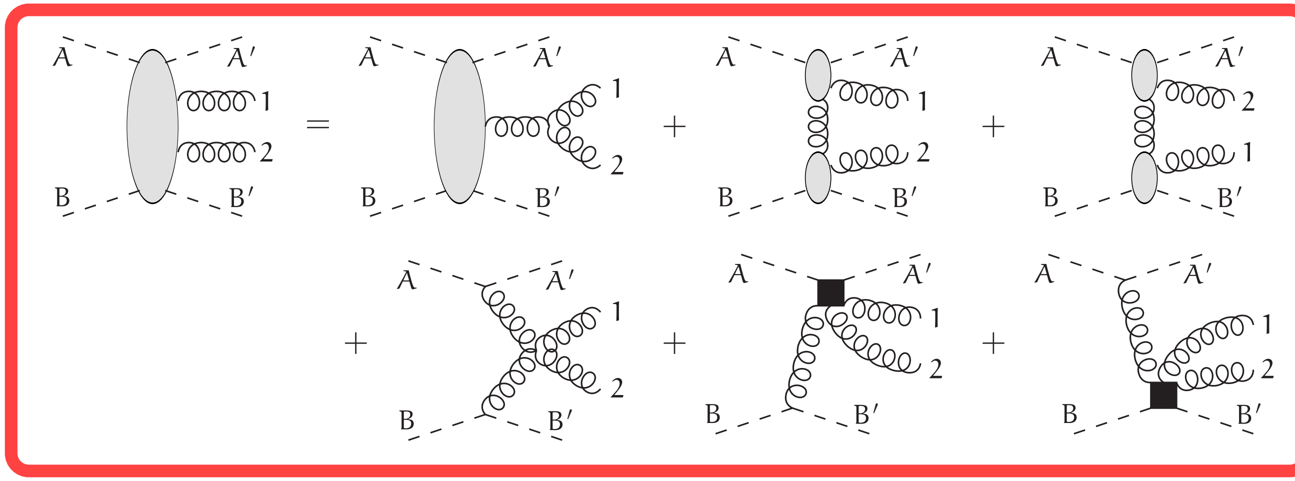
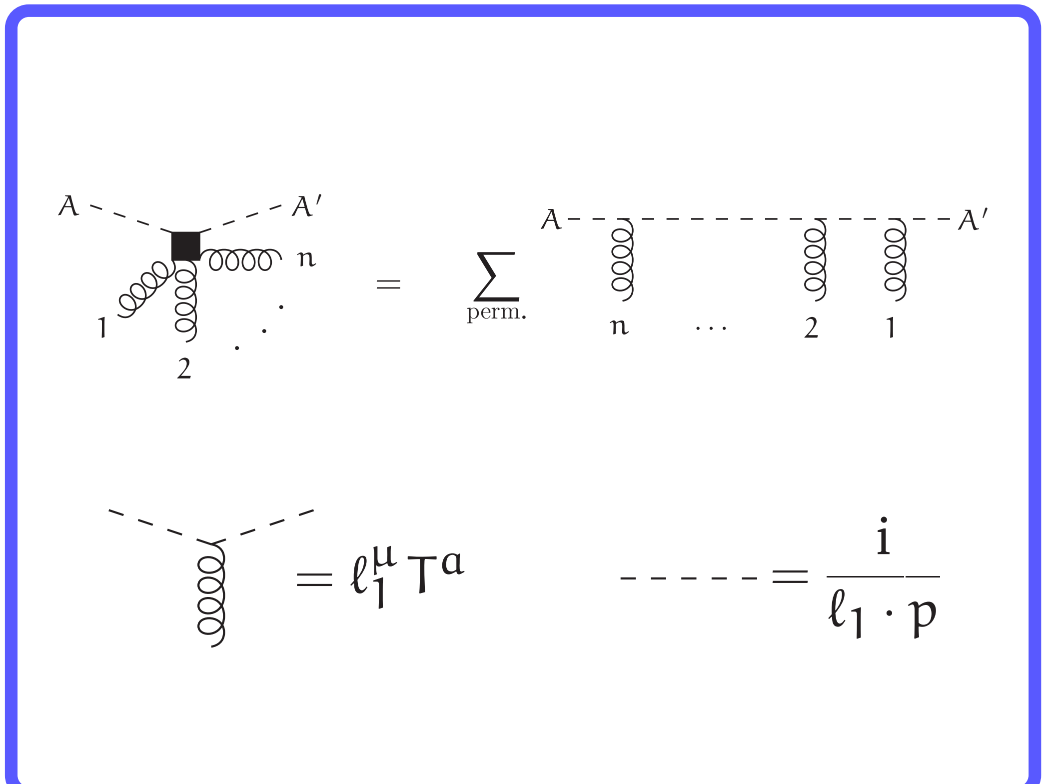
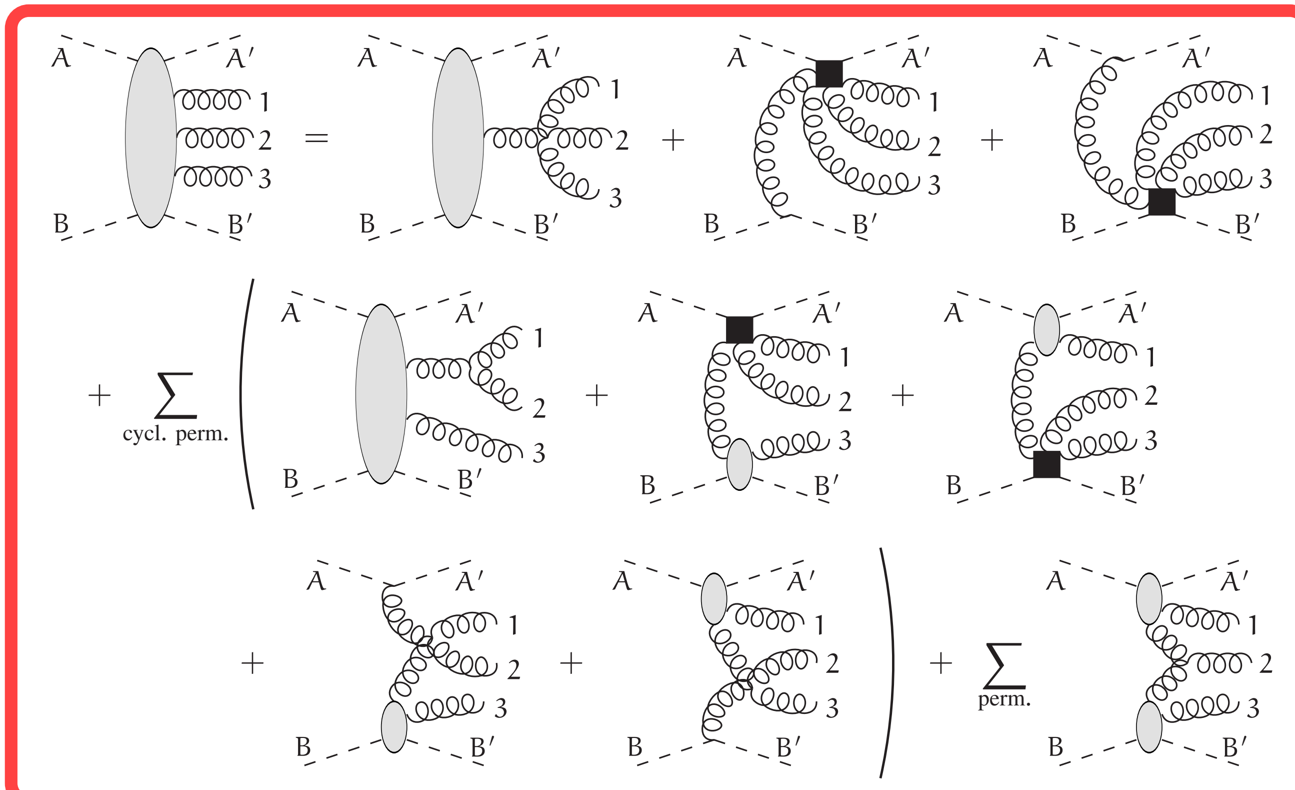








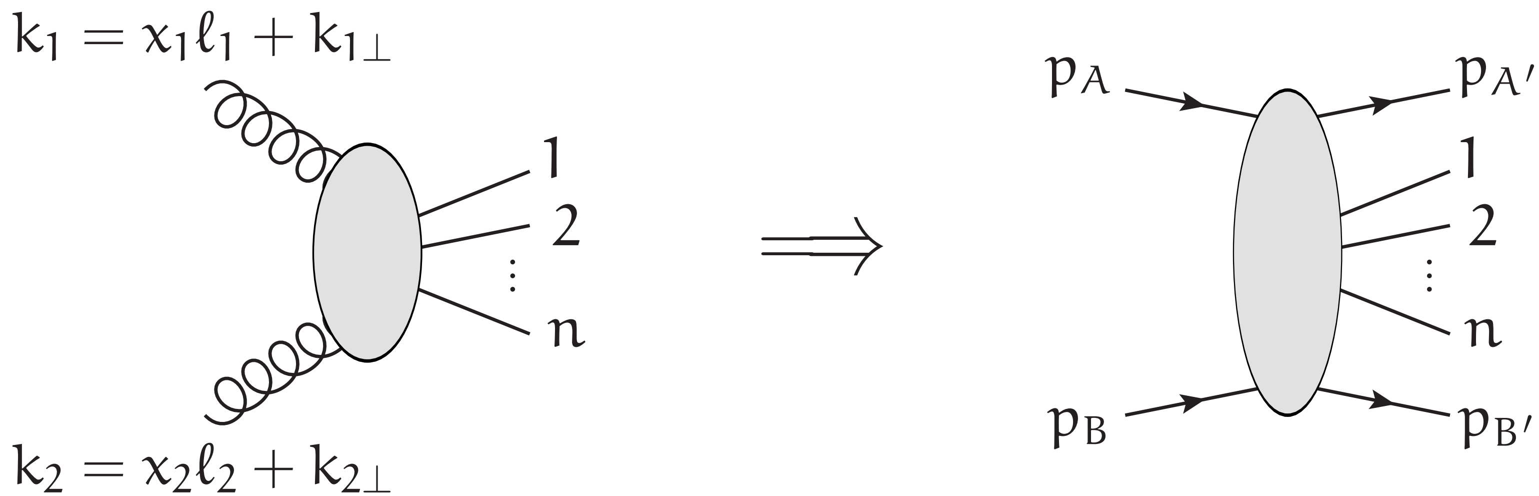




Prescription for g* g* –> X

AvH, Kotko, Kutak 2013

1. Consider the embedding $q_A q_B \to q_A q_B X$



with momentum flow as if the momenta p_A, p_B of the initial-state quarks and $p_{A'}, p_{B'}$ of the final-state quarks are given by

$$p_A^\mu = k_1^\mu \quad , \quad p_B^\mu = k_2^\mu \quad , \quad p_{A'}^\mu = p_{B'}^\mu = 0 \, .$$

2. Assign the spinors $|\ell_1], \langle\ell_1|$ to the external A-quarks, and assign $i\,\not{\ell}_1/(2\ell_1\cdot p)$ instead of $i\,\not{p}/p^2$ to the propagators on the A-quark line.
3. Do the same with the B quark line, using ℓ_2 instead of ℓ_1.
4. Multiply the amplitude with $g_s^{-1}x_1\sqrt{-k_{1\perp}^2/2} \times g_s^{-1}x_2\sqrt{-k_{2\perp}^2/2}$.
5. For the rest, normal Feynman rules apply.

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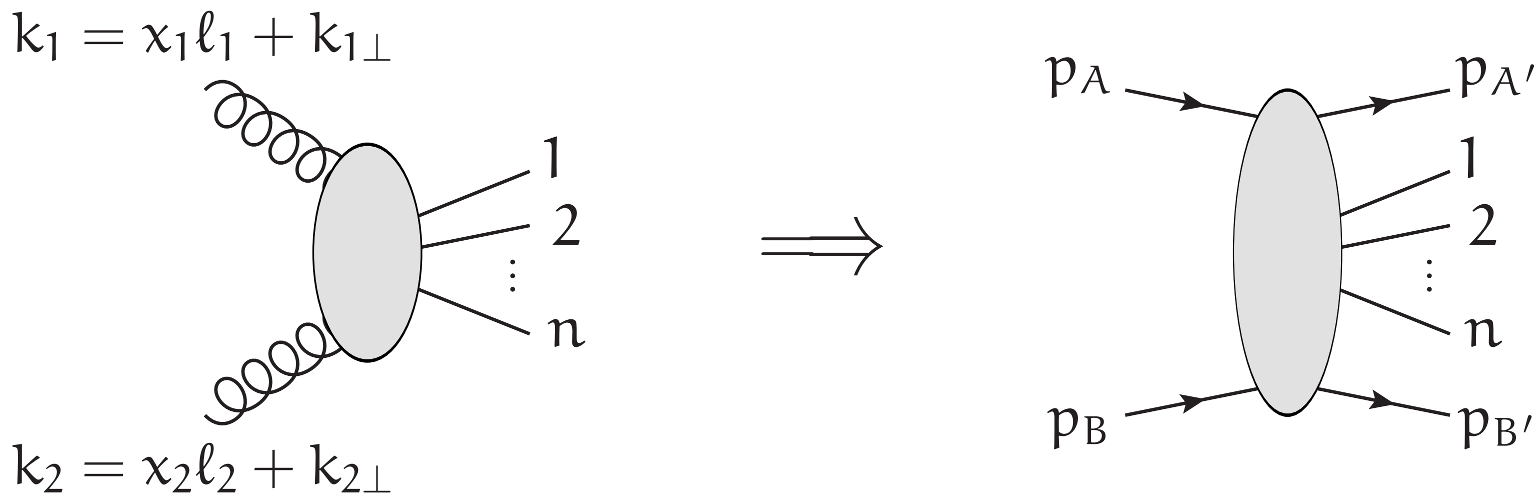

with … s and $p_{A'}$, p …

In agreement with Lipatov's effective action!

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Analytic result for g* g –> g g

AvH, Kotko, Kutak 2012

$$0 \to g^*(p_1 + k_T)\, g(p_2)\, g(p_3)\, g(p_4)$$

$$\mathcal{M}^{a_1 a_2 a_3 a_4}(1,2,3,4) = \frac{4g_s^2}{\sqrt{2}} \sum_{\text{non-cyclic}} \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,\mathcal{A}(1,2,3,4)$$

$$\mathcal{A}(2^-,3^-,4^-) = 0 \qquad \mathcal{A}(2^+,3^+,4^+) = 0$$

$$\mathcal{A}(2^-,3^-,4^+) = \frac{[3|\not{k}_T|1\rangle}{|k_T|[31]}\,\frac{[41]^4}{[12][23][34][41]} \qquad \mathcal{A}(2^+,3^+,4^-) = \frac{\langle 1|\not{k}_T|3]}{|k_T|\langle 13\rangle}\,\frac{\langle 41\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

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$$\left|\frac{[i|\not{k}_T|1\rangle}{|k_T|[i1]}\right| = \left|\frac{\langle 1|\not{k}_T|i]}{|k_T|\langle 1i\rangle}\right| = 1$$

Off–shell gluons from Wilson lines

Kotko 2014

A Wilson line along path C, defined as

$$[x,y]_C = \mathcal{P}\exp\left\{ig\int_C dz_\mu\, A_b^\mu(z)T^b\right\}\ ,$$

transforms under local gauge transformations as $\quad [x,y]_C \mapsto U(x)[x,y]_C U^\dagger(y)\ .$

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Use an infinite Wilson line with “direction” p^μ

$$[y]_p = \mathcal{P}\exp\left\{ig\int_{-\infty}^{\infty} ds\, p\cdot A_b(y + sp)T^b\right\}$$

to define the operator

$$\mathcal{R}^a(p, k) = \int d^4y\, e^{iy\cdot k}\,\mathrm{Tr}\left\{\frac{1}{\pi g}\,T^a\,[y]_p\right\} \ .$$

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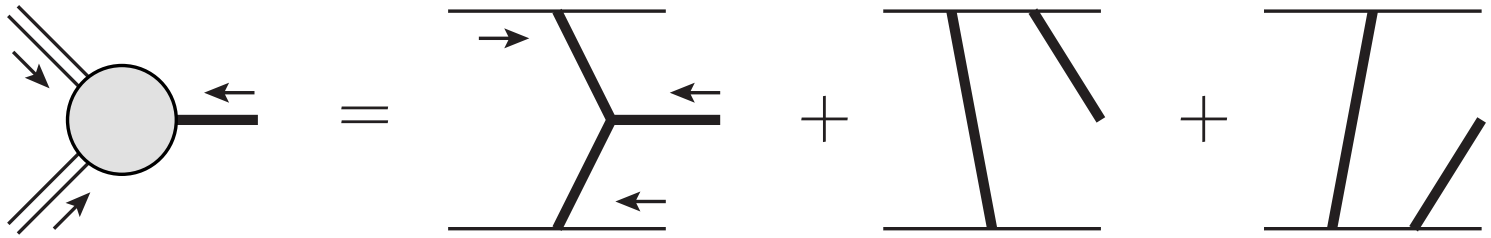
Amplitudes with n on-shell gluons and m off-shell gluons defined by

$$\begin{aligned}
&\left\langle k_1,k_2,\ldots,k_n \right| \mathcal{R}^{a_{n+1}}(p_{n+1},k_{n+1})\,\mathcal{R}^{a_{n+2}}(p_{n+2},k_{n+2})\cdots\mathcal{R}^{a_{n+m}}(p_{n+m},k_{n+m})\left|0\right\rangle \\
&\qquad = \delta(p_{n+1}\cdot k_{n+1})\,\delta(p_{n+2}\cdot k_{n+2})\cdots\delta(p_{n+m}\cdot k_{n+m})\,\delta^4(k_1+k_2+\cdots+k_{n+m}) \\
&\qquad \times \mathcal{A}(k_1,k_2,\ldots,k_{n+m}\,;\,p_{n+1},p_{n+2},\ldots,p_{n+m})
\end{aligned}$$

Feynman rules

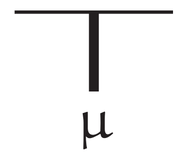
Planar graphs for the process $\emptyset \to g^* g^* g$:



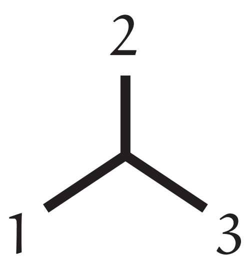
The Feynman rules in the Feynman gauge:

$$= \frac{-\eta^{\mu\nu}}{K^2}$$

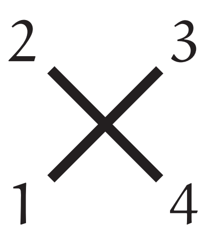
$$= \frac{1}{2p\cdot K}$$



$$= \sqrt{2}\, p^{\mu}$$



$$= \frac{1}{\sqrt{2}} \left[(K_1 - K_2)^{\mu_3} \eta^{\mu_1\mu_2} + (K_2 - K_3)^{\mu_1} \eta^{\mu_2\mu_3} + (K_3 - K_1)^{\mu_2} \eta^{\mu_3\mu_1} \right]$$



$$= \frac{-1}{2} \left[2\,\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right]$$

where p^{μ} is the direction associated with the eikonal line.

BCFW recursion

Britto, Cachazo, Feng 2004
Britto, Cachazo, Feng, Witten 2005

For a rational function f of a complex variable z which vanishes at infinity, we have

$$\lim_{z\to\infty} f(z) = 0 \quad \Rightarrow \quad \oint \frac{dz}{2\pi i}\,\frac{f(z)}{z} = 0\,,$$

where the integration contour expands to infinity and necessarily encloses all poles of f. This directly leads to the relation

$$f(0) = \sum_i \frac{\lim_{z\to z_i} f(z)(z - z_i)}{-z_i}\,,$$

where the sum is over all poles of f, and z_i is the position of pole number i.
For color-ordered tree-level multi-gluon amplitudes, this can be translated to

$$\mathcal{A}(1^+,2,\ldots,n{-}1,n^-) = \sum_{i=2}^{n-1}\sum_{h=+,-}\mathcal{A}(\hat{1}^+,2,\ldots,i,-\hat{K}^h_{1,i})\,\frac{1}{K^2_{1,i}}\,\mathcal{A}(\hat{K}^h_{1,i},i{+}1,\ldots,n{-}1,\hat{n}^-)$$

where the lower-point on-shell amplitudes have "shifted" momenta.

BCFW for off–shell gluons

n-gluon amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$ and n directions $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$\begin{aligned} k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 && \text{momentum conservation} \\ p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 && \text{light-likeness} \\ p_1\cdot k_1 = p_2\cdot k_2 = \cdots = p_n\cdot k_n &= 0 && \text{eikonal condition} \end{aligned}$$

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With the help of an auxiliary four-vector q^μ with $q^2 = 0$, we define

$$
k_T^\mu(q) = k^\mu - x(q) p^\mu \quad \text{with} \quad x(q) \equiv \frac{q \cdot k}{q \cdot p}
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Construct k_T^μ explicitly in terms of p^μ and q^μ:

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \ , \quad \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually

κ and κ^* are independent of q^μ.

BCFW: shifted momenta

$$f(0) = \sum_i \frac{\lim_{z\to z_i} f(z)(z - z_i)}{-z_i}$$

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We choose two external gluons i and j, and use the direction of one as the auxiliary momentum to define the transverse momentum of the other, and vice versa, so

$$k_i^\mu = x_i(p_j)p_i^\mu - \frac{\kappa_i}{2}\frac{\langle i|\gamma^\mu|j]}{[ij]} - \frac{\kappa_i^*}{2}\frac{\langle j|\gamma^\mu|i]}{\langle ji\rangle} \quad , \quad k_j^\mu = x_j(p_i)p_j^\mu - \frac{\kappa_j}{2}\frac{\langle j|\gamma^\mu|i]}{[ji]} - \frac{\kappa_j^*}{2}\frac{\langle i|\gamma^\mu|j]}{\langle ij\rangle}$$

We adopt the notation $|i\rangle \equiv |p_i\rangle$, $|j\rangle \equiv |p_j\rangle$ etc..

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$$e^\mu \equiv \tfrac{1}{2}\langle i|\gamma^\mu|j] \quad \Rightarrow \quad p_i\cdot e = p_j\cdot e = e\cdot e = 0$$

Using this vector, we then define the shifted momenta

$$\hat{k}_i^\mu(z) \equiv k_i^\mu + ze^\mu = x_i(p_j)p_i^\mu - \frac{\kappa_i - [ij]z}{2}\frac{\langle i|\gamma^\mu|j]}{[ij]} - \frac{\kappa_i^*}{2}\frac{\langle j|\gamma^\mu|i]}{\langle ji\rangle}$$

$$\hat{k}_j^\mu(z) \equiv k_j^\mu - ze^\mu = x_j(p_i)p_j^\mu - \frac{\kappa_j}{2}\frac{\langle j|\gamma^\mu|i]}{[ji]} - \frac{\kappa_j^* + \langle ij\rangle z}{2}\frac{\langle i|\gamma^\mu|j]}{\langle ij\rangle}$$

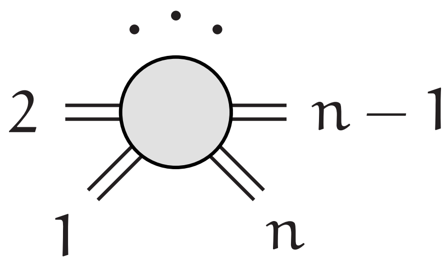
Total momentum is conserved, and also

$$p_i\cdot\hat{k}_i = 0 \quad , \quad p_j\cdot\hat{k}_j = 0$$

BCFW recursion

$$f(0) = \sum_i \frac{\lim_{z\to z_i} f(z)(z - z_i)}{-z_i}$$

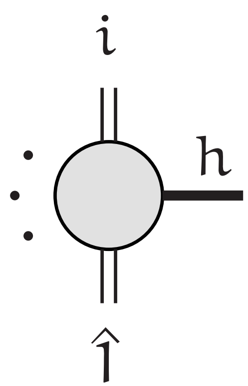
The BCFW recursion formula becomes



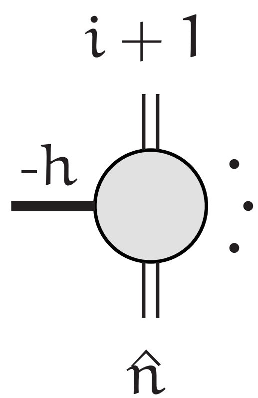
$$= \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D\,,$$

where

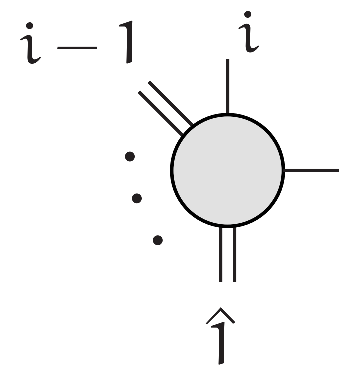
$$A_{i,h} =$$



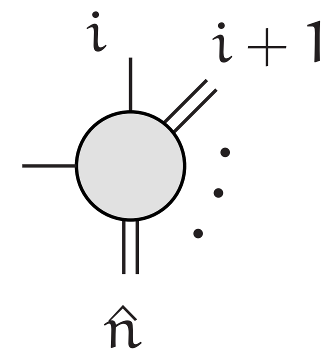
$$\frac{1}{K_{1,i}^2}$$



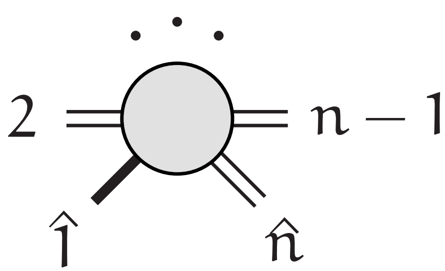
$$B_i =$$



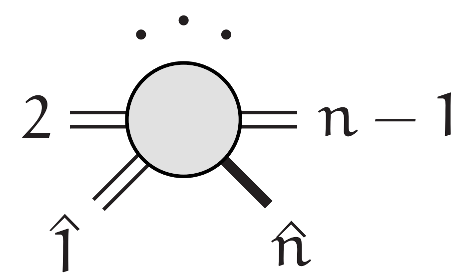
$$\frac{1}{2p_i \cdot K_{i,n}}$$



$$C = \frac{1}{\kappa_1}$$



$$D = \frac{1}{\kappa_n^*}$$



The hatted numbers label the shifted external gluons.

On–shell limit

For each off-shell gluon j, we can identify the following terms in the amplitude

$$\mathcal{A}(k_j) = \frac{1}{\kappa_j^*} U(k_j) + \frac{1}{\kappa_j} V(k_j) + W(k_j)$$

The actual amplitude needs a factor proportional to $\sqrt{-k_j^2}$, we choose κ_j^*:

$$\kappa_j^* \mathcal{A}(k_j) = U(k_j) + \frac{\kappa_j^*}{\kappa_j} V(k_j) + \kappa_j^* W(k_j)$$

The ratio κ_j^*/κ_j does not vanish in the on-shell limit, and an angle dependence remains.

$$\left|\kappa_j^* \mathcal{A}(k_j)\right|^2 \xrightarrow{k_j^2 \to 0} \left|U(p_j)\right|^2 + \left|V(p_j)\right|^2 + e^{2i\varphi_j} U(p_j) V(p_j)^* + e^{-2i\varphi_j} U(p_j)^* V(p_j)$$

Interference terms vanish upon integration over φ.

- the $-$ helicity can be associated with U, or $1/\kappa_j^*$
- the $+$ helicity can be associated with V, or $1/\kappa_j$

3–gluon amplitudes

$\emptyset \to ggg$, at least two shifted momenta

$$\mathcal{A}(1^-,2^+,3^+) = \frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle} \quad , \quad \mathcal{A}(1^+,2^-,3^-) = \frac{[32]^3}{[21][13]}$$

$\emptyset \to g^*gg$

$$\mathcal{A}(1^*,2^+,3^-) = \frac{1}{\kappa_1^*}\frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle} = \frac{1}{\kappa_1}\frac{[21]^3}{[13][32]}$$

$\emptyset \to gg^*g^*$

$$\mathcal{A}(1^+,2^*,3^*) = \frac{1}{\kappa_2^*\kappa_3^*}\frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle} \quad , \quad \mathcal{A}(1^-,2^*,3^*) = \frac{1}{\kappa_2\kappa_3}\frac{[32]^3}{[21][13]}$$

$\emptyset \to g^*g^*g^*$

$$\mathcal{A}(1^*,2^*,3^*) = \frac{\langle 12\rangle^3[32]^3}{\kappa_3\kappa_1^*\langle 1|\not{k}_3|2]\langle 2|\not{k}_1|3]\langle 2|\not{k}_1|2]} + (231) + (312)\,,$$

where the second and third term are obtained by applying the cyclic permutations on the arguments and indices of the first term.

4–gluon amplitudes, 2 off–shell

$$\mathcal{A}(1^*,2^+,3^+,4^*) = \frac{1}{\kappa_4^*\kappa_1^*}\,\frac{\langle 41\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle} \quad , \quad \mathcal{A}(1^*,2^+,3^*,4^+) = \frac{1}{\kappa_1^*\kappa_3^*}\,\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$\mathcal{A}(1^*,2^+,3^-,4^*) = \frac{1}{\kappa_1^*\kappa_4}\,\frac{-\langle 1|\not{p}_3+\not{k}_4|4]^4}{\langle 2|\not{k}_1|4]\langle 1|\not{k}_4|3]\langle 12\rangle[43](p_3+k_4)^2}$$
$$+\frac{1}{\kappa_1}\,\frac{\langle 34\rangle^3[14]^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 2|\not{k}_1|4]\langle 4|\not{k}_1|4]\langle 23\rangle} + \frac{1}{\kappa_4^*}\,\frac{[21]^3\langle 14\rangle^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 1|\not{k}_4|3]\langle 1|\not{k}_4|1][32]}$$

$$\mathcal{A}(1^*,2^-,3^*,4^+) = \frac{\langle 13\rangle^3[13]^3}{\langle 34\rangle\langle 41\rangle\langle 1|\not{k}_3+\not{p}_4|3]\langle 3|\not{k}_1+\not{p}_4|1][32][21]}$$
$$+\frac{1}{\kappa_1^*\kappa_3}\,\frac{\langle 12\rangle^3[43]^3}{\langle 2|\not{k}_3|4]\langle 1|\not{k}_3+\not{p}_4|3](k_3+p_4)^2} + \frac{1}{\kappa_1\kappa_3^*}\,\frac{\langle 23\rangle^3[14]^3}{\langle 2|\not{k}_1|4]\langle 3|\not{k}_1+\not{p}_4|1](k_1+p_4)^2}$$

MHV amplitudes

$$\mathcal{A}(1^-, i^-, (\text{the rest})^+) = \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^+, i^+, (\text{the rest})^-) = \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$

$$\mathcal{A}(1^*, i^-, (\text{the rest})^+) = \frac{1}{\kappa_1^*} \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^*, i^+, (\text{the rest})^-) = \frac{1}{\kappa_1} \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$

$$\mathcal{A}(1^*, i^*, (\text{the rest})^+) = \frac{1}{\kappa_1^* \kappa_i^*} \frac{\langle p_1 p_i \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

$$\mathcal{A}(1^*, i^*, (\text{the rest})^-) = \frac{1}{\kappa_1 \kappa_i} \frac{[p_i p_1]^4}{[p_1 p_n][p_n p_{n-1}][p_{n-1} p_{n-2}] \cdots [p_3 p_2][p_2 p_1]}$$