The three-loop splitting functions in QCD: the helicity-dependent case

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with Sven Moch (Hamburg Univ.) and Jos Vermaseren (NIKHEF)

- Polarized PDFs, their evolution, α_s^2 calculations (1990s), large-x limit
- α_s^3 via $g_1^{\mathrm{e.m.}}$ (2008, all-N) & graviton-exch. DIS (new, extreme Mincer)
- All-N expressions, via end-point knowledge and number theory tools

Loops & Legs 2014, Weimar, 30-04-14

Parton distributions and their evolution

Long. polarized proton: q/g distributions $f_i^{\rightarrow}, f_i^{\leftarrow}$ for same, opposite helicity

Unpolarized and polarized parton distribution functions (PDFs)

$$
\begin{aligned}
f_i(x,\mu^2) &= f_i^{\rightarrow}(x,\mu^2) + f_i^{\leftarrow}(x,\mu^2) \\
\Delta f_i(x,\mu^2) &= f_i^{\rightarrow}(x,\mu^2) - f_i^{\leftarrow}(x,\mu^2)
\end{aligned}
$$

x : momentum fraction, μ : factorization scale (= renorm. scale, w.l.o.g.)

Scale dependence: renormalization-group evolution equations

$$
\frac{d}{d\ln\mu^2}(\Delta)f_i(x,\mu^2) = \left[(\Delta)P_{ik}(\alpha_S(\mu^2)) \otimes (\Delta)f_k(\mu^2)\right](x)
$$

Pert. expansion of the splitting functions ($\Leftrightarrow$ twist-2 anomalous dimensions)

$$
(\Delta)P_{ik}(x,\mu^2) = \sum_{n=0} a_S^{n+1}(\Delta)P_{ik}^{(n)}(x) \quad , \quad a_S = \frac{\alpha_S(\mu^2)}{4\pi}
$$

Here: 3^{rd}-order (NNLO) contributions $\Delta P_{ik}^{(2)}$ for the polarized case

Second-order calculations of the 1990s

Splitting functions $\Delta P_{ik}^{(1)}$, coefficient functions for g_1 in polarized e.m. DIS

- Structure function g_1 analogous to $F_{2,3,L}$: $\Delta P_{\mathrm{qq}}^{(1)}$, $\Delta P_{\mathrm{qg}}^{(1)}$, $c_{g_1,\,\mathrm{q/g}}^{(2)}$

  Zijlstra, van Neerven (93) [Err. 97, 07]

  γ_5: Larin scheme $\Leftrightarrow$ 't Hooft, Veltman (72); Breitenlohner, Maison (77)

- All NLO splitting functions $\Delta P_{ij}^{(1)}$ using OPE / lightlike axial gauge

  Mertig, van Neerven (95) [not hep-ph version] / Vogelsang (95/6)

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Transformation from L/HVBM to $\overline{\rm MS}$ scheme at NNLO   Matiounine et al. (98)

$$Z_{ik}(\alpha_{\rm S}(\mu^2)) \;=\; \delta_{iq}\delta_{kq}\left(a_{\rm S}\, z^{(1)}_{\rm ns} + a_{\rm S}^2\,(z^{(2)}_{\rm ns} + z^{(2)}_{\rm ps}) + \ldots\right)$$

Non-singlet: $c_{g_1} \leftrightarrow c_{F_3}$. Pure singlet, $z^{(n)}_{\rm gq} = 0$: second calculation needed

Large-x limits of the splitting functions

$x \to 1$ (threshold): expect suppression of helicity flip by $(1-x)^2 \leftrightarrow 1/N^2$

cf. Brodsky, Burkhardt, Schmid (94)

E.g., leading-order (LO) splitting functions, with $\delta_{ik}^{(0)} \equiv P_{ik}^{(0)} - \Delta P_{ik}^{(0)}$

$$\delta_{\mathrm{qq}}^{(0)} = 0 \;\;, \quad \delta_{ik}^{(0)} = \text{const} \cdot (1-x)^2 + \ldots \quad \text{for} \quad ik = \mathrm{qg},\, \mathrm{gq},\, \mathrm{gg}$$

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NLO, in the standard version ('M') of $\overline{\text{MS}}$ Mertig & van Neerven; Vogelsang

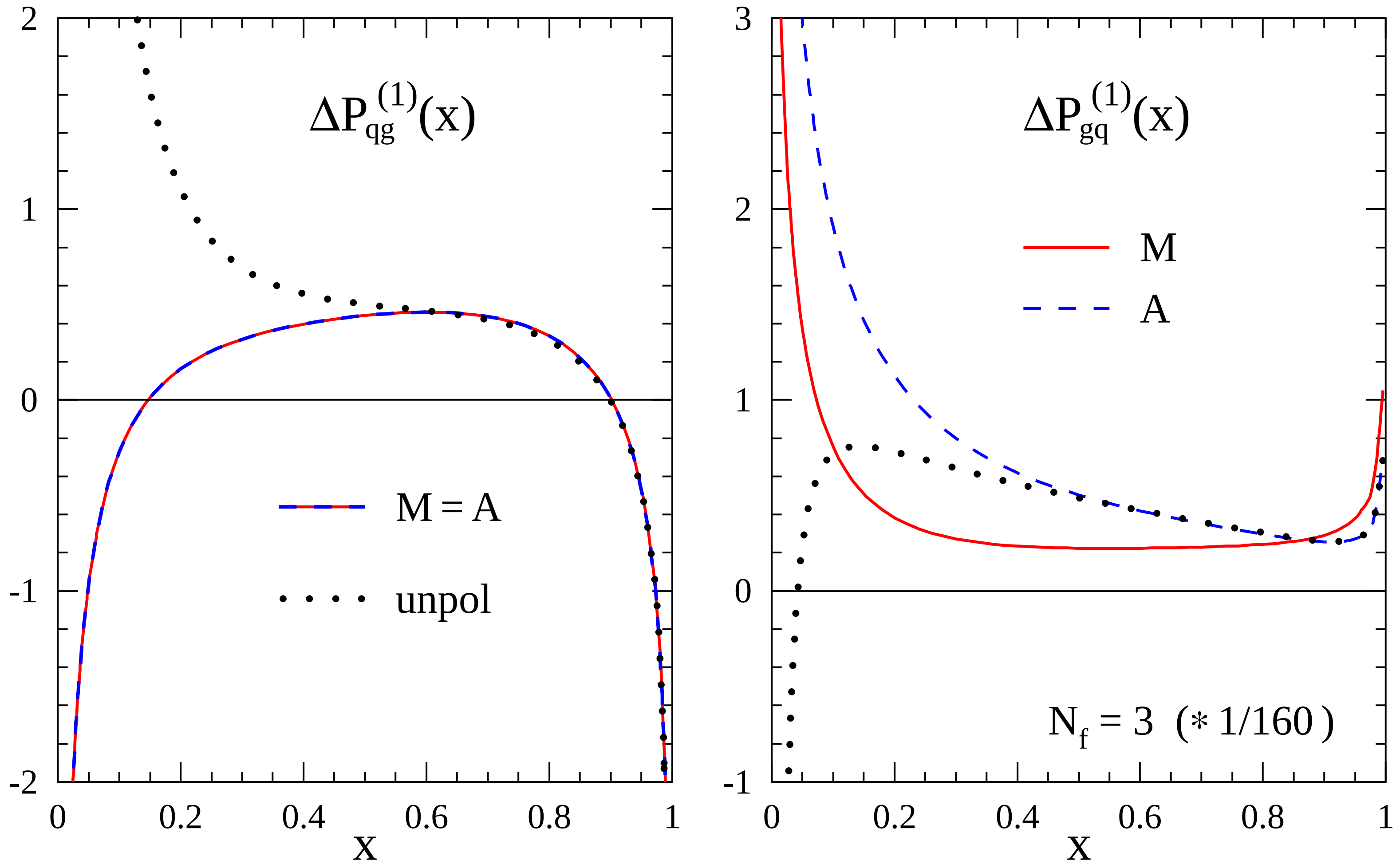
$$\delta_{ij}^{(1)} = \text{const} \cdot (1-x)^a \quad \text{for} \quad ik = \rm qq,\ gg\ (\mathit{a}=1),\ qg\ (\mathit{a}=2)$$

$$\delta_{\rm gq}^{(1)} = 8C_F(C_A-C_F)(2-x)\ln(1-x) + 4C_F\beta_0 - 6C_F^2 + (20/3C_FC_A + 2C_F^2 - 8/3C_Fn_f)(1-x) + \mathcal{O}(1-x)^2$$

Physics or scheme artifact? Flavour-singlet physical kernels, if available for corresponding quantities, can provide insight cf. Furmanski, Petronzio (81)

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2} f + CPf = \left(\beta(a_{\rm s})\frac{dC}{da_{\rm s}} + CP\right)C^{-1}F = KF$$

Off-diagonal NLO splitting functions

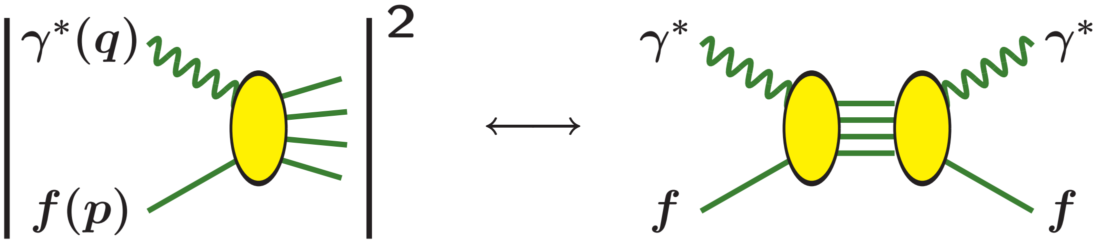


M: standard scheme, A: additional $z_{gq}^{(1)} = -\Delta P_{gq}^{(0)}$ in trf. from Larin scheme,

removes all $(1-x)^0$ and $(1-x)^1$ terms in $\delta_{gq}^{(1)}$ ...

Third order via forward Compton amplitudes

Optical theorem: probe-parton total cross sections $\longleftrightarrow$ forward amplitudes



Dispersion relation in x : coefficient of $(2p\cdot q)^N$ $\leftrightarrow$ N-th Mellin moment

$$A^N = \int_0^1 dx\, x^{N-1} A(x)$$

Unpol.: Larin, van Ritbergen, Vermaseren (94), [Mincer], . . . , MVV (04) [all-N]

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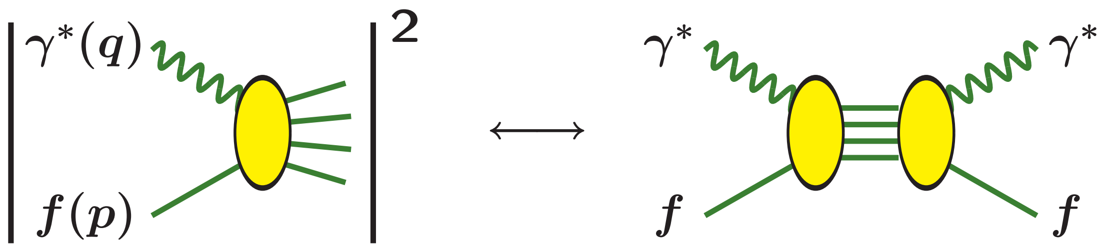

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Pol. case: projection of partonic tensor on g_1 in $D = 4 - 2\varepsilon$ dimensions

$$\hat{g}_1 = 2\,[(D-2)(D-3)(p\cdot q)]^{-1}\, \varepsilon_{\mu\nu pq}\, \widehat{W}_A^{\mu\nu}$$

ε^{-1}: $\Delta P_{\rm qq}^{(2)}(N)$, $\Delta P_{\rm qg}^{(2)}(N)$ MVV (Loops & Legs 2008)

ϵ^0 : N^{3}LO coefficient functions for g_1, mod. scheme transf. of pure singlet

One colour factor of $\Delta P_{\rm qg}^{(2)}(N)$

$$\frac{1}{8}\Delta P^{(2)}_{\rm qg}(N)\Big|_{C_F^2 n_f} = 2\Delta p_{\rm qg}(-S_{-4}+2S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}-S_{1,1,2}-5S_{1,2,1}$$
$$+4S_{1,3}+2S_{2,-2}-6S_{2,1,1}+6S_{2,2}+7S_{3,1}-3S_4)$$
$$-3\zeta_3(2D_0^2+4D_1^2-9D_0+12D_1)+4S_{-3}(D_0^2-2D_0+2D_1)+8S_{1,-2}(2D_1^2-D_0+D_1)$$
$$-2S_{2,1}(4D_0^2+2D_1^2-11D_0+11D_1)+S_{1,1,1}(5D_0^2-2D_1^2-21/2D_0+12D_1)$$
$$-2S_{1,2}(2D_0^2-2D_1^2-5D_0+5D_1)+2S_3(3D_0^2+6D_1^2-11D_0+11D_1)$$
$$+2S_{-2}(8D_1^3-5D_0^2-6D_1^2+10D_0-9D_1)-S_{1,1}(10D_0^3+6D_1^3-35/2D_0^2-5D_1^2$$
$$+29D_0-36D_1)+2S_2(4D_0^3+6D_1^3-10D_0^2-4D_1^2+17D_0-22D_1)\ -6D_2(S_{-2}+1)$$
$$+S_1(7D_0^4+4D_1^4-43/2D_0^3-15D_1^3+99/2D_0^2+18D_1^2-78D_0+329/4D_1)+32D_1^5$$
$$-15/2D_0^4-3D_1^4+59/8D_0^3+53/4D_1^3+77/8D_0^2+213/8D_1^2-1357/32D_0+777/16D_1$$

All harmonic sums with argument N, $D_k=(N+k)^{-1}$, $\Delta p_{\rm qg}=2D_1-D_0$

One colour factor of $\Delta P_{\rm qg}^{(2)}(N)$

$$\frac{1}{8}\Delta P_{\rm qg}^{(2)}(N)\Big|_{C_F^2 n_f} = 2\Delta p_{\rm qg}(-S_{-4}+2S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}-S_{1,1,2}-5S_{1,2,1} +4S_{1,3}+2S_{2,-2}-6S_{2,1,1}+6S_{2,2}+7S_{3,1}-3S_4)$$

$$-3\zeta_3(2D_0^2+4D_1^2-9D_0+12D_1)+4S_{-3}(D_0^2-2D_0+2D_1)+8S_{1,-2}(2D_1^2-D_0+D_1)$$
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All harmonic sums with argument N, $D_k=(N+k)^{-1}$, $\Delta p_{\rm qg}=2D_1-D_0$

- **Weight-four sums: as in unpol. case, up to replacement $p_{\rm qg}\to\Delta p_{\rm qg}$**
- **Very few terms with D_2, no corresponding primes in moment denom's**
- **No indices -1. Large-N pol.-unpol. suppression separately for each sum**
- **$x\to 0$ and $x\to 1$ knowledge: $D_{0,1}^5$, D_1^4 and $S_{1,1,1}$ terms predictable**

Accessing the lower row, $\Delta P^{(2)}_{\rm gq}$ and $\Delta P^{(2)}_{\rm gg}$

$\Delta P^{(2)}_{\rm gq,\,gg}$ enter $\gamma^* f$ amplitudes only at order α_s^4: need direct gluon coupling

Unpol.: $F_2^{\rm e.m.}$ complemented by scalar ϕ with $\phi G^{\mu\nu} G_{\mu\nu}$ coupling to gluons

$\Leftrightarrow$ Higgs-exchange DIS in heavy-top limit Furmanski, Petronzio (81)

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Polarized case: non-(pseudo)scalar probe required

- Extend to supersymmetric case, as done for NNLO antenna functions
  Gehrmann-de Ridder, Gehrmann, Glover (05)
- Consider graviton-exchange DIS
  Lam, Li (81), cf. Stirling, Vryonidou (11)

  Structure functions H_k, $k = 1-4, 6$: unpol. & pol. analogues of (F_2, F_ϕ)

Drawback: lots of higher tensor integrals, far beyond 2004 calculation of F_2, $F_\phi, \ldots$ and 2008 extension to g_1 $\Rightarrow$ fall back to fixed-N Mincer calculation

Mincer moments of $\Delta P^{(2)}_{\mathrm{gq}}$, coeff's of C_F^3

Odd moments $N \geq 3$ are accessible — **Lam, Li (1981)**

Results of the Mincer calculation, coefficient of C_F^3, Larin scheme

N = 3: 186505/7776

N = 5: 9473569/3037500

N = 7: -509428539731/193616640000

N = 9: -266884720969207/56710659600000

N = 11: -334956658917082 9651/608887229282640000

N = 13: -7517747672901480 22507/1304909471988 68256000

N = 15: -233668190199130264541 80147/40472269161987446784 00000

N = 17: -3052142278186280906801 74170947/5387328250831125958911 5520000

N = 19: -570679648684656807578199791973487/103793635967590259537308862400000

N = 21: -20443040920892357622791488 43319979/385456787045956248050132280576000

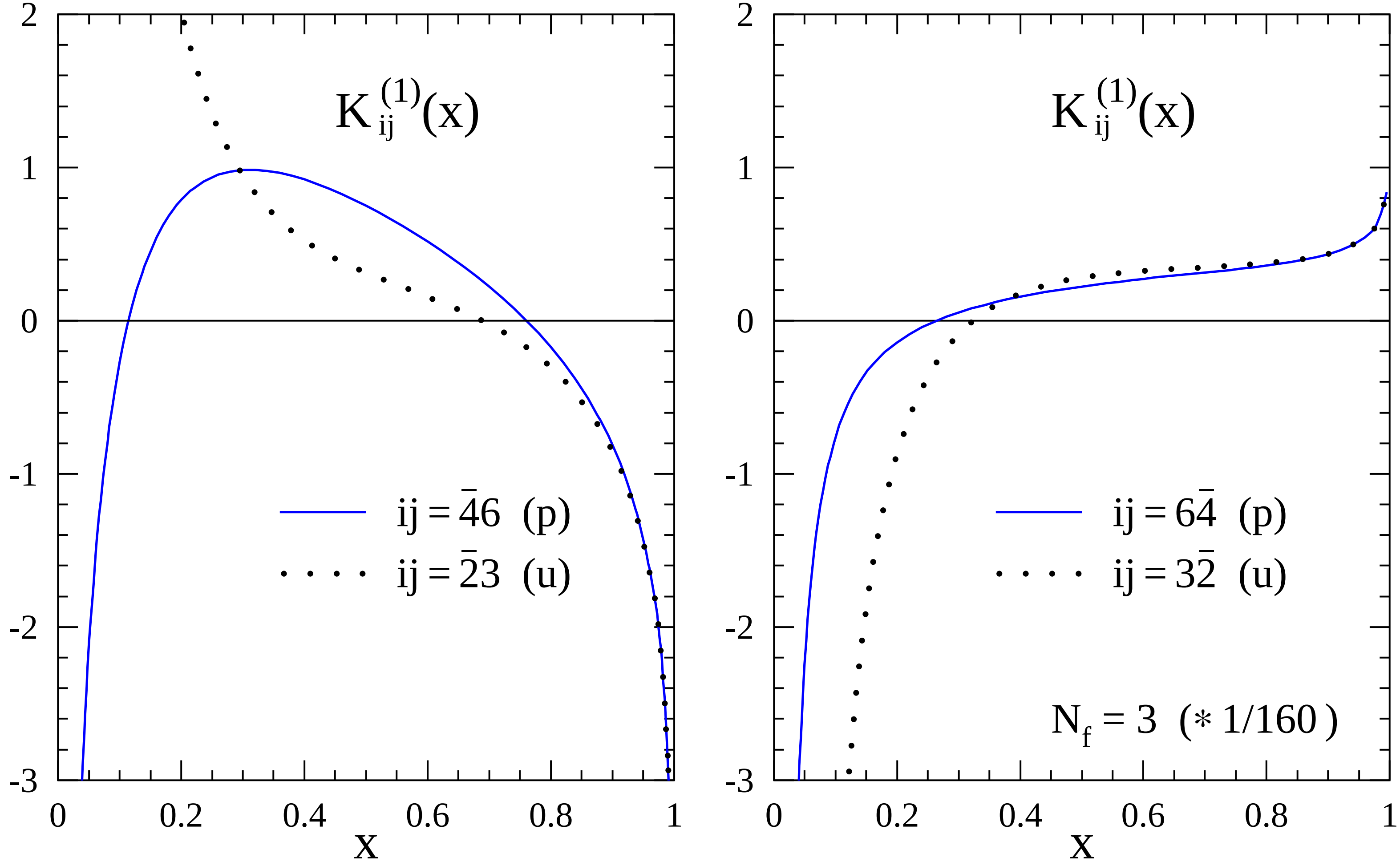
N = 23: -2891198401137614095302603332501398 23739/
5670701927098814115299960121507 1395840

N = 25: -18904732552838029376788307451029218 69938637/
38642690852856502186336030585116 0000000000

Machines: Zeuthen, NIKHEF (hardest cases), ulgqcd cluster Liverpool (bulk production)

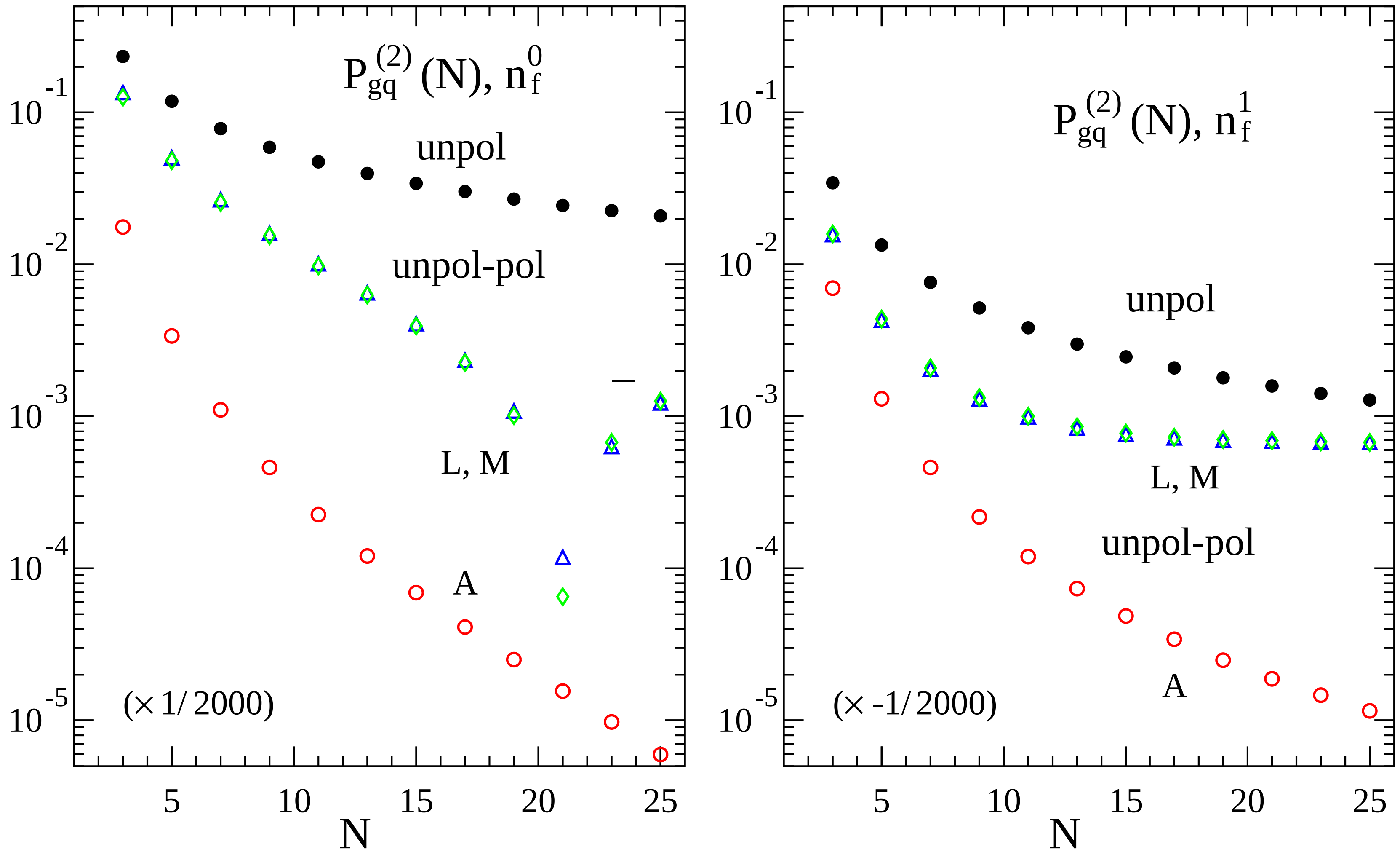
NLO physical kernels for graviton exchange

Unpol.: structure funct's $H_{\bar{2}}$ (LO: q) and H_3 (LO: g). Pol. analogues: $H_{\bar{4}}$, H_6



$\Rightarrow$ **Large-x behaviour of standard $\Delta P_{gq}^{(1)}$ is a factorization-scheme artifact**

Large-N (non-)suppression of $\Delta P_{\mathrm{gq}}^{(2)}$



Consistent with $\frac{1}{N^2}$ suppressed difference in A-scheme, $z_{\mathrm{gq}}^{(2)} = -\frac{1}{2}\Delta P_{\mathrm{gq}}^{(1)}$

Determination of $\Delta P^{(2)}_{\rm gq}$ at all N

Critical: n_f^0 parts. Coefficient of weight-4 sums fixed from unpolarized case

Weight $\leq$ 3 : 2$\times$32 coefficients with D_0 or D_1, plus up to 11 sums with D_{-1}

- 2×12 coefficients (of D_0^1 & D_1^1) fixed by $1/N^2$ A-scheme suppression
- 3 + 3 coefficient fixed by small-x & large-x (i.e., $S_{1,1,1}$) knowledge

$\Rightarrow$ Up to 45 unknown integer coefficients vs 12 odd moments $3 \leq N \leq 25$

In-house primes program (Jos): analysis of prime decomposition, derivation of relations ($\lesssim 10$) between coefficients via the Chinese remainder theorem

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Eliminate six (or more) 'unpleasant' coefficients: D_0^2, D_1^2, $D_0^2 S_1$, $D_1^2 S_1$, ...

Turn to the number-theory professionals (LLL algorithm), cf. Velizhanin (12)

www.numbertheory.org/php/axb.html (Keith Matthews, Queensland)

'Solves a system of linear Diophantine equations ... via the Havas-Majewski-Matthews LLL-based algorithm. ... . We find ... the solutions X with minimal length, using a modification of the Fincke-Pohst algorithm'

One colour factor of $\Delta P^{(2)}_{\rm gq}(N)$

$$\frac{1}{8}\Delta P^{(2)}_{\rm gq}(N)\Big|_{C_F^3} = 2\Delta p_{\rm qg}(-S_{-4} + 6S_{-2,-2} + 4S_{1,-3} + 2S_{1,1,1,1} + S_{1,1,2} + 3S_{1,2,1} - 3S_{1,3} + 2S_{2,-2} + 2S_{2,1,1} - 2S_{2,2})$$

$$+6\zeta_3\Delta p_{\rm qg}(2S_1 - 3) - 4S_{-3}(2D_0^2 - D_0 + D_1) - 8S_{1,-2}(D_1^2 - 2D_0 + 2D_1)$$

$$+S_{1,1,1}(2D_0^2 - 5D_1^2 - 6D_0 - 3/2D_1) - 2S_{1,2}(D_1^2 + 4D_0 - D_1)$$

$$-S_{2,1}(4D_0^2 + 4D_1^2 - 4D_0 + 7D_1) + S_3(2D_0^2 + D_1^2 + 6D_0 - 3/2D_1)$$

$$-S_{-2}(8D_1^3 + 4D_0^2 + 18D_1^2 - 26D_0 + 24D_1) + 2S_2(D_1^3 + 2D_1^2 + 10D_0 - 4D_1)$$

$$-S_{1,1}(6D_0^3 + 6D_1^3 + 4D_0^2 + 5D_1^2 + 2D_0 - 7/4D_1) - 6D_{-1}(S_{-2} + 1)$$

$$-S_1(6D_0^4 + 7D_1^4 + 4D_0^3 + 23/2D_1^3 - 27/2D_0^2 + 39/4D_1^2 - 8D_0 + 23/4D_1)$$

$$-8D_0^5 - 12D_1^5 + 23D_0^4 - 28D_1^4 - 39/4D_0^3 - 427/8D_1^3 - 341/8D_0^2 - 767/8D_1^2$$

$$+2427/16D_0 - 4547/32D_1$$

All harmonic sums with argument N, $D_k = (N+k)^{-1}$, $\Delta p_{\rm gq} = 2D_0 - D_1$

$C_F C_A^2$, $C_F^2 C_A$ parts somewhat longer, rest much simpler ($N=25$ not needed)

All-N formula for $\Delta P^{(2)}_{\rm gg}(N)$ analogous; overall most difficult: its C_A^3 part

Higher-N checks, first-moment results

Most difficult colour factors of both cases: all moments to $N = 25$ used for determining the coefficients $\Rightarrow$ validate results by computing $N = 27,\ 29$

$$-\Delta P_{\rm gq}^{(2)}(27) = 4609770383587605432813291530849726335264810727/ \\ 982934508627216318966565777854990940800000000\, C_F^3 + \ldots$$

Total execution time: 256 874 306.6 sec. Maximum disk space: 1 261 024 031 636 bytes

Plus Mincer check of $\Delta P_{\rm gq}^{(2)}(29)$ for $C_A - 2C_F \rightarrow 0$ and $\Delta P_{\rm gq}^{(2)}(27, 29)|_{C_A^3}$

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\end{aligned}
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Plus Mincer check of $\Delta P^{(2)}_{\rm gq}(29)$ for $C_A - 2C_F \rightarrow 0$ and $\Delta P^{(2)}_{\rm gq}(27, 29)|_{C_A^{\,3}}$

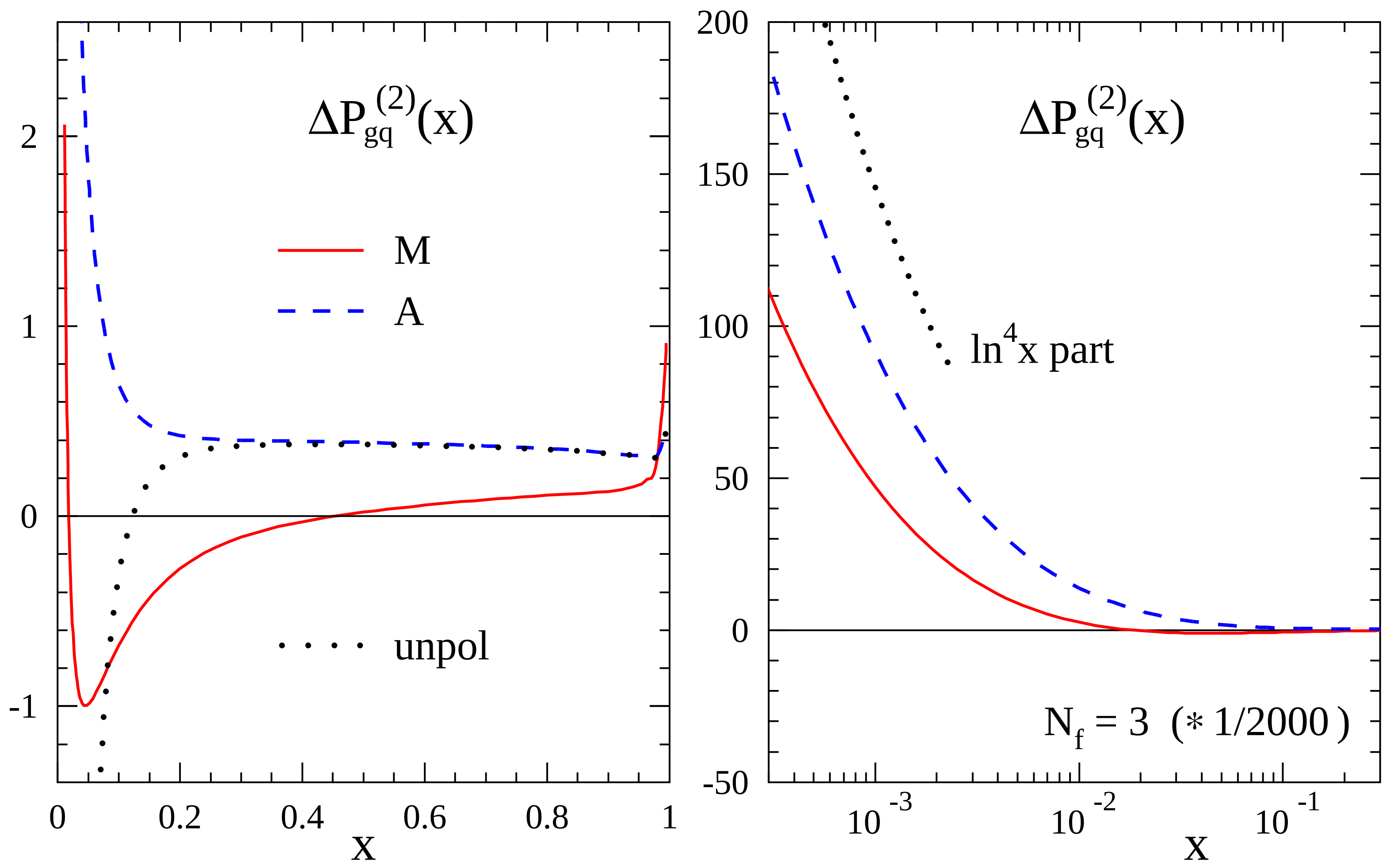
First moments: transform to x-space in terms of HPLs, then calculate $N = 1$

Remiddi, Vermaseren (99)

$$
\begin{aligned}
\Delta P^{(2)}_{\rm gq}(N=1) \;=\; & \frac{1607}{12} C_F C_A^2 - \frac{461}{4} C_F^2 C_A + \frac{63}{2} C_F^3 \\
& + \Big(\frac{41}{3} - 72\zeta_3\Big) C_F C_A n_f - \Big(\frac{107}{2} - 72\zeta_3\Big) C_F^2 n_f - \frac{13}{3} C_F n_f^2
\end{aligned}
$$

$$
\Delta P^{(2)}_{\rm gg}(N=1) \;=\; \frac{2857}{54} C_A^3 - \frac{1415}{54} C_A^2 n_f + \ldots = \beta_2
$$
– another check

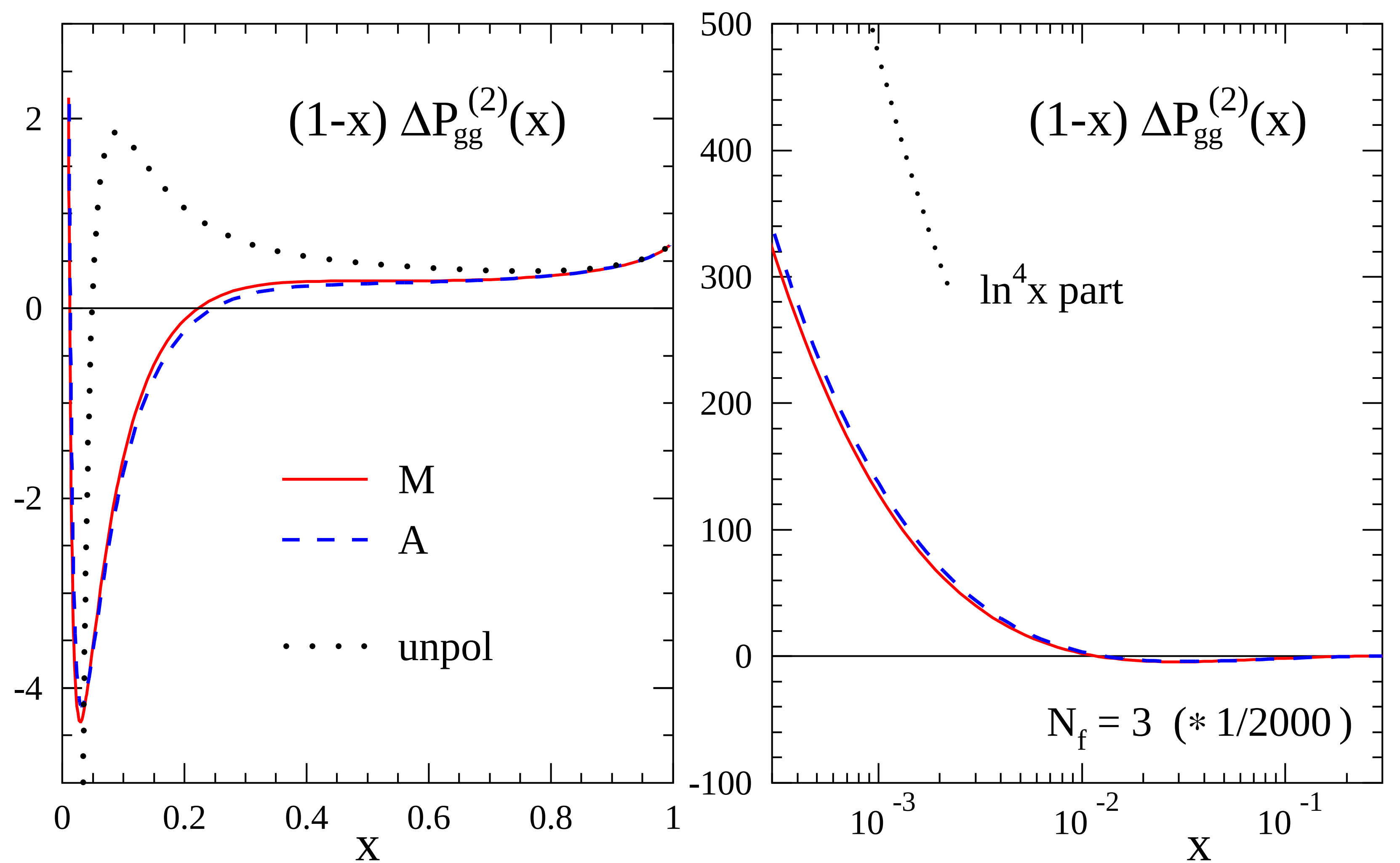
The third-order splitting function $\Delta P^{(2)}_{\rm gq}(x)$



$\ln^4 x$: 1996 Blümlein, A.V. coefficient = result for NNLO physical kernel $K_{6\bar{4}}$

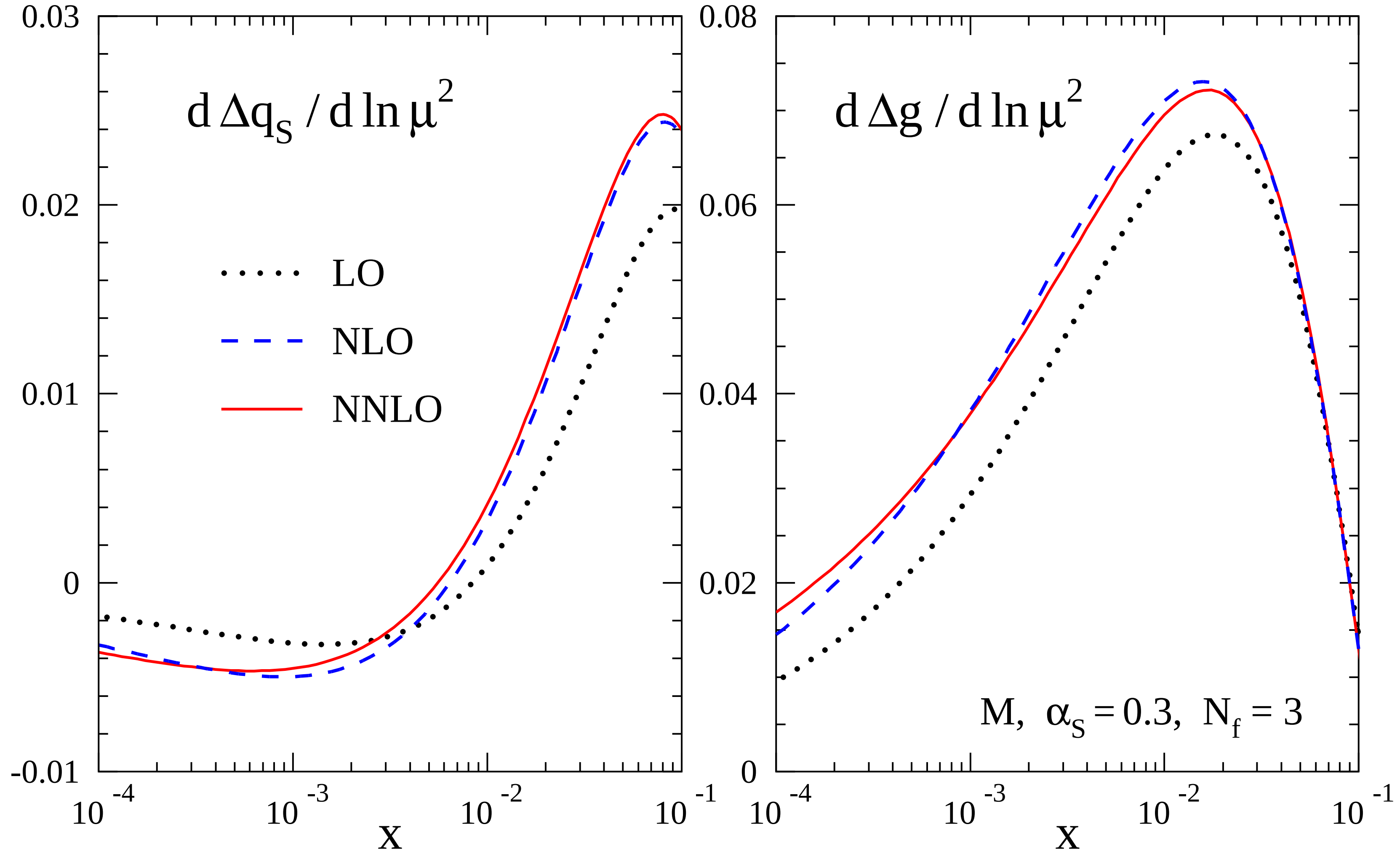
Same situation for $\Delta P^{(2)}_{\rm qg} \leftrightarrow K_{\bar{4}6}$ – clarifies 2008 query

The third-order splitting function $\Delta P^{(2)}_{gg}(x)$



Coefficient of $\ln^4 x$ identical to K_{66}: straight agreement with '96 prediction

Polarized singlet quark and gluon evolution



Initial q and g distributions of QCD-Pegasus manual, evolution benchmarks

A.V. (04); Salam, A.V. (HERA/LHC workshop 04/05)

Summary and outlook

'Spät kommt Ihr – doch Ihr kommt. Der weite Weg, . . . , entschuldigt Euer Säumen.'

'Late you come, yet you come. The long way, . . . , excuses your tarrying'

Schiller, Wallenstein (1799)

All NNLO spin splitting functions $\Delta P_{ij}^{(2)}(x)$ calculated, finally

- New part (lower matrix row) by brute force, insight and number theory

  3$^{\rm rd}$-order Mincer calculation of gravition-exchange DIS also performed for upper row and unpol. case: full agreement with previous results

- Agreement with all previous partial results (if interpreted properly) and expectations: $x \to 0, 1$, gg @ $N = 1$, leading n_f Bennett, Gracey (98)

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- Agreement with all previous partial results (if interpreted properly) and expectations: $x \rightarrow 0, 1$, gg @ $N = 1$, leading n_f Bennett, Gracey (98)

- Numerical effects small down to low x, as for unpolarized case

  Standard pol. $\overline{\rm MS}$ factorization (γ_5 treatment) a bit unphysical for $x \rightarrow 1$ – not a practical problem: no reason to change scheme after 19 years

- Re-calculation of transformation from Larin scheme $z_{\rm iq}^{(2)}$ worthwhile

  Knowledge of $z_{\rm ps}^{(3)}$ would fix N^{3}LO quark coefficient function for g_1